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Connections Between Various Subclasses of Planar Harmonic Mappings Involving Generalized Bessel Functions

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Abstract : The purpose of the present paper is to establish connections between various subclasses of harmonic univalent functions by applying certain convolution operator involving generalized Bessel functions of first kind. To be more precise, we investigate such connections with Goodman-Rønning-type harmonic univalent functions in the open unit disc U.

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1 Introduction

Let A denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : z \in C \text{ and } |z| < 1\}$ and satisfy the normalization condition $f(0) = f'(0) - 1 = 0$. Now, we recall that the generalized Bessel function of the first kind $w = w_{p,b,c}$ is defined as the particular solution of the second-order linear homogenous differential equation

$$z^2\omega''(z) + bz\omega'(z) + \left[cz^2 - p^2 + (1-b)p\right]\omega(z) = 0, \tag{1.2}$$

where $b, p, c \in C$, which is a natural generalization of Bessel's equation. This function has the familiar representation

$$\omega(z) = \omega_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n c^n}{n!\Gamma(p+n+\frac{b+1}{2})} \left(\frac{z}{2}\right)^{2n+p}, \quad z \in C. \tag{1.3}$$

The differential equation (1.2) permits the study of Bessel, modified Bessel, spherical Bessel function and modified spherical Bessel functions all together. Solutions of (1.2) are referred to as the generalized Bessel function of order p. The particular solution given by (1.3) is called the generalized Bessel function of the first kind of order p. Although the series defined above is convergent everywhere, the function $\omega_{p,b,c}$ is generally not univalent in U. It is worth mentioning that, in particular, when $b = c = 1$, we reobtain the Bessel function $\omega_{p,1,1} = J_p$, and for $c = -1, b = 1$ the function $\omega_{p,1,-1}$ becomes the modified Bessel function I_p. Now, consider the function $u_{p,b,c}$ defined by the transformation

$$u_{p,b,c}(z) = 2^p\Gamma\left(p + \frac{b+1}{2}\right) z^{-p/2}\omega_{p,b,c}(z^{1/2}).$$

By using the well-known Pochhammer (or Appell) symbol, defined in terms of the Euler Gamma function for $a \neq 0, -1, -2, ...$ by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & \text{if } n = 0 \\ a(a+1)...(a+n-1), & \text{if } n = 1, 2, 3, ..., \end{cases},$$

we obtain for the function $u_{p,b,c}$ the following representation

$$u_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-c/4)^n}{\left(p + \frac{(b+1)}{2}\right)_n} \frac{z^n}{n!}, \tag{1.4}$$

where $p + (b+1)/2 \neq 0, -1, -2, ...$ This function is analytic on C and satisfies the second-order linear differential equation

$$4z^2u''(z) + 2\,(2p+b+1)\,zu'(z) + czu(z) = 0.$$

For convenience throughout in the sequel, we use the following notations:

$$u_{p,b,c} = u_p, \quad k = p + \frac{b+1}{2}.$$

Let H be the family of all harmonic functions of the form $f = h + \overline{g}$, where

$$h(z) = z + \sum_{n=2}^{\infty} A_n z^n, \quad g(z) = \sum_{n=1}^{\infty} B_n z^n, \quad |B_1| < 1, \quad (z \in U), \tag{1.5}$$

are in the class A. For complex parameters $c_1, k_1, c_2, k_2 \ \ (k_1, k_2 \neq 0, -1, -2, ...)$, we define the functions $\phi_1(z) = zu_{p_1}(z)$ and $\phi_2(z) = zu_{p_2}(z)$.

Corresponding to these functions, we introduce the following convolution operator

$$\Omega \equiv \Omega \begin{pmatrix} k_1, & c_1 \\ k_2, & c_2 \end{pmatrix} : H \to H$$

defined by

$$\Omega \begin{pmatrix} k_1, & c_1 \\ k_2, & c_2 \end{pmatrix} f = f * (\phi_1 + \overline{\phi_2}) = h(z) * \phi_1(z) + \overline{g(z) * \phi_2(z)}$$

for any function $f = h + \overline{g}$ in H.

Letting

$$\Omega \begin{pmatrix} k_1, & c_1 \\ k_2, & c_2 \end{pmatrix} f(z) = H(z) + \overline{G(z)},$$

where

$$H(z) = z + \sum_{n=2}^{\infty} \frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!} A_n z^n, \quad G(z) = \sum_{n=1}^{\infty} \frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-1)!} B_n z^n. \quad (1.6)$$

Denote by S_H the subclass of H that are univalent and sense-preserving in U. Note that $\frac{f - \overline{B_1 f}}{1-|B_1|^2} \in S_H$ whenever $f \in S_H$. We also let the subclass S_H^0 of S_H

$$S_H^0 = \{ f = h + \overline{g} \in S_H : g'(0) = B_1 = 0 \}.$$

The classes S_H^0 and S_H were first studied in [1]. Also, we let K_H^0, $S_H^{*,0}$ and C_H^0 denote the subclasses of S_H^0 of harmonic functions which are, respectively, convex, starlike and close-to-convex in U. Also, let T_H^0 be the class of sense-preserving, typically real harmonic functions $f = h + \overline{g}$ in H. For definitions and properties of these classes, one may refer to ([2] - [18]).

For $0 \le \gamma < 1$, let

$$N_H(\gamma) = \left\{ f \in H : \operatorname{Re}\left(\frac{f'(z)}{z'} \right) \ge \gamma, \quad z = re^{i\theta} \in U \right\},$$

and

$$G_H(\gamma) = \left\{ f \in H : \operatorname{Re}\left\{ \left(1 + e^{i\alpha}\right) \frac{zf'(z)}{f(z)} - e^{i\alpha} \right\} \ge \gamma, \quad \alpha \in R, \ z \in U \right\},$$

where

$$z' = \frac{\partial}{\partial \theta}\left(z = re^{i\theta}\right), \ z'' = \frac{\partial}{\partial \theta}(z'), \ f'(z) = \frac{\partial}{\partial \theta} f\left(re^{i\theta}\right), f''(z) = \frac{\partial}{\partial \theta}\left(f'(z)\right).$$

Define

$$TN_H(\gamma) \equiv N_H(\gamma) \cap T \ \text{ and } TG_H(\gamma) \equiv G_H(\gamma) \cap T$$

where T consists of the functions $f = h + \overline{g}$ in S_H so that h and g are of the form

$$h(z) = z - \sum_{n=2}^{\infty} |A_n| z^n, \quad g(z) = \sum_{n=1}^{\infty} |B_n| z^n. \quad (1.7)$$

The classes $N_H(\gamma)$, $TN_H(\gamma)$, $G_H(\gamma)$ and $TG_H(\gamma)$ were initially introduced and studied, respectively, in ([5], [17]). A function f in $G_H(\gamma)$ is called Goodman-Rønning-type harmonic univalent functions in U.

Throughout this paper, we will frequently use the notation

$$\Omega\left(f\right)=\Omega\left(\begin{array}{cc} k_1, & c_1 \\ k_2, & c_2 \end{array}\right)f.$$

The generalized Bessel function is a recent topic of study in Geometric Function Theory (e.g. see the work of [6]- [9] and [13]). Motivated by results on connections between various subclasses of analytic and harmonic univalent functions by using hypergeometric functions (see [3], [4], [10], [12], [14]-[16] and [18] and by work of Baricz [6]-[9]), we establish a number of connections between the classes $G_H(\gamma)$, K_H^0, $S_H^{*,0}$, C_H^0 and $N_H(\beta)$ by applying the convolution operator Ω.

2 Main Results

In order to establish connections between harmonic convex functions and Goodman-Rønning-type harmonic univalent functions, we need following results in Lemma 2.1, Lemma 2.2 and Lemma 2.4.

Lemma 2.1 ([1], [11])**.** *If $f=h+\overline{g}\in K_H^0$ where h and g are given by* (1.5) *with $B_1=0$, then*

$$|A_n|\le\frac{n+1}{2},\quad |B_n|\le\frac{n-1}{2}.$$

Lemma 2.2 ([17])**.** *Let $f=h+\overline{g}$ be given by* (1.5)*. If $0\le\gamma<1$ and*

$$\sum_{n=2}^{\infty}\left(2n-1-\gamma\right)|A_n|+\sum_{n=1}^{\infty}\left(2n+1+\gamma\right)|B_n|\le 1-\gamma, \tag{2.1}$$

then f is sense-preserving, Goodman-Rønning-type harmonic univalent functions in U and $f\in G_H(\gamma)$.

Remark 2.3. *In [17], it is also shown that $f=h+\overline{g}$ given by* (1.4) *is in the family $TG_H(\gamma)$, if and only if the coefficient condition* (2.1) *holds. Moreover, if $f\in TG_H(\gamma)$, then*

$$|A_n|\le\frac{1-\gamma}{2n-1-\gamma},\qquad n\ge 2,$$

$$|B_n|\le\frac{1-\gamma}{2n+1+\gamma},\qquad n\ge 1.$$

Lemma 2.4 ([9])**.** *If $b,p,c\in C$ and $k\neq 0,-1,-2,...$ then the function u_p satisfies the recursive relation $4ku_p'(z)=-cu_{p+1}(z)$ for all $z\in C$.*

Theorem 2.5. *Let* $c_1, c_2 < 0,\ k_1, k_2 > 0,\ (k_1, k_2 \neq 0, -1, -2, ...)$. *If for some* $\gamma (0 \leq \gamma < 1)$ *and the inequality*

$$2u''_{p_1}(1) + (7-\gamma)u'_{p_1}(1) + 2(1-\gamma)u_{p_1}(1) + 2u''_{p_2}(1) + (5+\gamma)u'_{p_2} \leq 4(1-\gamma) \quad (2.2)$$

is satisfied then $\Omega\left(K_H^0\right) \subset G_H(\gamma)$.

Proof. Let $f = h + \overline{g} \in K_H^0$ where h and g are of the form (1.5) with $B_1 = 0$. We need to show that $\Omega(f) = H + \overline{G} \in G_H(\gamma)$, where H and G defined by (1.6) with $B_1 = 0$ are analytic functions in U.

In view of Lemma 2.2, we need to prove that

$$P_1 \leq 1 - \gamma,$$

where

$$P_1 = \sum_{n=2}^{\infty} (2n - 1 - \gamma) \left| \frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!} A_n \right| + \sum_{n=2}^{\infty} (2n + 1 + \gamma) \left| \frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-1)!} B_n \right|.$$

In view of Lemma 2.1, we have

$$\begin{aligned}
P_1 &\leq \frac{1}{2}\left[\sum_{n=2}^{\infty}(n+1)(2n-1-\gamma)\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!} + \sum_{n=2}^{\infty}(n-1)(2n+1+\gamma)\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-1)!}\right] \\
&= \frac{1}{2}\left[\sum_{n=2}^{\infty}\left\{2(n-1)(n-2) + (7-\gamma)(n-1) + 2(1-\gamma)\right\}\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!}\right. \\
&\quad \left. + \sum_{n=2}^{\infty}\left\{2(n-2) + (5+\gamma)\right\}\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-2)!}\right] \\
&= \frac{1}{2}\left[2\sum_{n=0}^{\infty}\frac{(-c_1/4)^{n+1}}{(k_1)_{n+1}(n-1)!} + (7-\gamma)\sum_{n=0}^{\infty}\frac{(-c_1/4)^{n+1}}{(k_1)_{n+1}n!} + 2(1-\gamma)\sum_{n=0}^{\infty}\frac{(-c_1/4)^{n+1}}{(k_1)_{n+1}(n+1)!}\right. \\
&\quad \left. +2\sum_{n=0}^{\infty}\frac{(-c_2/4)^{n+1}}{(k_2)_{n+1}(n-1)!} + (5+\gamma)\sum_{n=0}^{\infty}\frac{(-c_2/4)^{n+1}}{(k_2)_{n+1}n!}\right] \\
&= \frac{1}{2}\left[2\frac{(-c_1/4)^2}{k_1(k_1+1)}\sum_{n=0}^{\infty}\frac{(-c_1/4)^{n-1}}{(k_1+2)_{n-1}(n-1)!} + (7-\gamma)\frac{(-c_1/4)}{k_1}\sum_{n=0}^{\infty}\frac{(-c_1/4)^n}{(k_1+1)_n n!}\right. \\
&\quad + 2(1-\gamma)\left\{u_{p_1}(1) - 1\right\} + 2\frac{(-c_2/4)^2}{k_2(k_2+1)}\sum_{n=1}^{\infty}\frac{(-c_2/4)^{n-1}}{(k_2+2)_{n-1}(n-1)!} \\
&\quad \left. + (5+\gamma)\frac{(-c_2/4)}{k_2}\sum_{n=0}^{\infty}\frac{(-c_2/4)^n}{(k_2+1)_n n!}\right] \\
&= \frac{1}{2}\left[2\frac{(-c_1/4)^2}{k_1(k_1+1)}u_{p_1+2}(1) + (7-\gamma)\frac{(-c_1/4)}{k_1}u_{p_1+1}(1) + 2(1-\gamma)\left\{u_{p_1}(1) - 1\right\}\right.
\end{aligned}$$

$$2\frac{(-c_2/4)^2}{k_2(k_2+1)}u_{p_2+2}(1)+(5+\gamma)\frac{(-c_2/4)}{k_2}u_{p_2+1}(1)\Bigg]$$
$$=\frac{1}{2}\left[2u''_{p_1}(1)+(7-\gamma)u'_{p_1}(1)+2(1-\gamma)u_{p_1}(1)+2u''_{p_2}(1)+(5+\gamma)u'_{p_2}(1)-2(1-\gamma).\right]$$

Now $P_1 \leq 1-\gamma$ follows from the given condition This completes the proof. □

Analogous to Theorem 2.5, we next find conditions of the classes $S_H^{*,0}$, C_H^0 with $G_h(\gamma)$. However we first need the following result which may be found in [1], [11].

Lemma 2.6. *If $f = h + \overline{g} \in S_H^{*,0}$ or C_H^0 where h and g are given by* (1.5) *with $B_1 = 0$, then*

$$|A_n| \leq \frac{(2n+1)(n+1)}{6}, \quad |B_n| \leq \frac{(2n-1)(n-1)}{6}.$$

Theorem 2.7. *If $c_1, c_2 < 0$, $k_1, k_2 > 0, (k_1, k_2 \neq 0, -1, -2, ...)$. If for some $\gamma(0 \leq \gamma < 1)$ and the inequality*

$$4u'''_{p_1}(1)+2(14-\gamma)u''_{p_1}(1)+3(13-3\gamma)u'_{p_1}(1)+6(1-\gamma)u_{p_1}(1)+4u'''_{p_2}(1)$$
$$+2(10+\gamma)u''_{p_2}(1)+3(5+\gamma)u'_{p_2}(1)\leq 12(1-\gamma) \quad (2.3)$$

is satisfied, then

$$\Omega(S_H^{*,0}) \subset G_H(\gamma) \text{ and } \Omega(C_H^0) \subset G_H(\gamma).$$

Proof. Let $f = h + \overline{g} \in S_H^{*,0}$ (C_H^0 where h and g are given by (1.2) with $B_1 = 0$. We need to show that $\Omega(f) = H + \overline{G} \in G_H(\gamma)$, where H and G defined by (1.6) with $B_1 = 0$ are analytic functions in U. In view of Lemma 2.2, it is enough to show that $P_1 \leq 1-\gamma$, where

$$P_1=\sum_{n=2}^{\infty}(2n-1-\gamma)\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!}|A_n|+\sum_{n=2}^{\infty}(2n+1+\gamma)\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-1)!}|B_n|.$$

In view of Lemma 2.6, we have

$$P_1\leq\frac{1}{6}\left[\sum_{n=2}^{\infty}(2n+1)(n+1)(2n-1-\gamma)\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!}+\sum_{n=2}^{\infty}(2n-1)(n-1)(2n+1+\gamma)\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-1)!}\right]$$
$$=\frac{1}{6}\left[\sum_{n=2}^{\infty}\{4(n-1)(n-2)(n-3)+(28-2\gamma)(n-1)(n-2)+(39-9\gamma)(n-1)+6(1-\gamma)\}\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!}\right.$$
$$\left.+\sum_{n=2}^{\infty}\{4(n-2)(n-3)+(20+2\gamma)(n-2)+(15+3\gamma)\}\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-2)!}\right]$$
$$=\frac{1}{6}\left[\left\{4\sum_{n=0}^{\infty}\frac{(-c_1/4)^{n+1}}{(k_1)_{n+1}(n-2)!}+(28-2\gamma)\sum_{n=0}^{\infty}\frac{(-c_1/4)^{n+1}}{(k_1)_{n+1}(n-1)!}+(39-9\gamma)\sum_{n=0}^{\infty}\frac{(-c_1/4)^{n+1}}{(k_1)_{n+1}n!}\right.\right.$$

$$+ 6(1-\gamma) \sum_{n=0}^{\infty} \frac{(-c_1/4)^{n+1}}{(k_1)_{n+1}(n+1)!} \Big\} + \Big\{ 4 \sum_{n=0}^{\infty} \frac{(-c_2/4)^{n+1}}{(k_2)_{n+1}(n-2)!} + (20+2\gamma) \sum_{n=0}^{\infty} \frac{(-c_2/4)^{n+1}}{(k_2)_{n+1}(n-1)!}$$

$$+ (15+3\gamma) \sum_{n=0}^{\infty} \frac{(-c_2/4)^{n+1}}{(k_2)_{n+1} n!} \Big\} \Big]$$

$$= \frac{1}{6} \Big[\Big\{ 4u'''_{p_1}(1) + (28-2\gamma)u''_{p_1}(1) + (39-9\gamma)u'_{p_1}(1) + 6(1-\gamma)\left\{u_{p_1}(1) - 1\right\}$$

$$+ \left\{ 4u'''_{p_2}(1) + (20+2\gamma)u''_{p_2}(1) + (15+3\gamma)u'_{p_2}(1) \right\} - 6(1-\gamma) \Big].$$

Now $P_1 \leq 1-\gamma$ follows from the given condition.

□

In order to determine connection between $TN_H(\beta)$ and $G_H(\gamma)$, we need the following results in Lemma 2.8 and 2.10.

Lemma 2.8 ([5]). *Let $f = h + \overline{g}$ where h and g are given by* (1.5) *with $B_1 = 0$, and suppose that $0 \leq \beta < 1$. Then*

$$f \in TN_H(\beta) \Leftrightarrow \sum_{n=2}^{\infty} n\,|A_n| + \sum_{n=2}^{\infty} n\,|B_n| \leq 1-\beta.$$

Remark 2.9. *If $f \in TN_H(\beta)$, then $|A_n| \leq \frac{1-\beta}{n}$ and $|B_n| \leq \frac{1-\beta}{n}$, $\quad n \geq 2$.*

Lemma 2.10. *If $c < 0$ and $k > 1$, then*

$$\sum_{n=0}^{\infty} \frac{(-c/4)^n}{(k)_n (n+1)!} = \frac{-4(k-1)}{c} \left[u_{p-1}(1) - 1\right].$$

Proof. We can write

$$\sum_{n=0}^{\infty} \frac{(-c/4)^n}{(k)_n (n+1)!} = \frac{(k-1)}{(-c/4)} \sum_{n=0}^{\infty} \frac{(-c/4)^{n+1}}{(k-1)_{n+1}(n+1)!}$$

$$= \frac{-4(k-1)}{c} \left[u_{p-1}(1) - 1\right].$$

□

Theorem 2.11. *If $c_1, c_2 < 0$, $k_1, k_2 > 1$. If for some $\beta(0 \leq \beta < 1)$ and $\gamma(0 \leq \gamma < 1)$ and the inequality*

$$(1-\beta)\left[2\left\{u_{p_1}(1) - 1\right\} + (1+\gamma)\frac{4(k_1-1)}{c_1}\left[u_{p_1-1}(1) - 1 - \frac{(-c_1/4)}{k_1-1}\right] \right.$$

$$\left. + 2u_{p_2}(1) - (1+\gamma)\frac{4(k_2-1)}{c_2}\left[u_{p_2-1}(1) - 1\right]\right] \leq 1-\gamma$$

is satisfied then

$$\Omega\left(TN_H(\beta)\right) \subset G_H(\gamma).$$

Proof. Let $f = h + \overline{g} \in TN_H(\beta)$ where h and g are given by (1.5). In view of Lemma 2.2, it is enough to show that $P_2 \leq 1 - \gamma$, where

$$P_2 = \sum_{n=2}^{\infty} (2n-1-\gamma)\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!}|A_n| + \sum_{n=1}^{\infty} (2n+1+\gamma)\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-1)!}|B_n|.$$

Using Remark 2.9, we have

$$\begin{aligned}
P_2 &\leq (1-\beta)\left[\sum_{n=2}^{\infty}\left\{2-\frac{(1+\gamma)}{n}\right\}\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!} + \sum_{n=1}^{\infty}\left\{2+\frac{(1+\gamma)}{n}\right\}\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-1)!}\right] \\
&= (1-\beta)\left[2\sum_{n=0}^{\infty}\frac{(-c_1/4)^{n+1}}{(k_1)_{n+1}(n+1)!} - (1+\gamma)\sum_{n=0}^{\infty}\frac{(-c_1/4)^{n+1}}{(k_1)_{n+1}(n+2)!}\right. \\
&\left. + 2\sum_{n=0}^{\infty}\frac{(-c_2/4)^{n}}{(k_2)_{n}n!} + (1+\gamma)\sum_{n=0}^{\infty}\frac{(-c_2/4)^{n}}{(k_2)_{n}(n+1)!}\right] \\
&= (1-\beta)\left[2\left\{u_{p_1}(1)-1\right\} - (1+\gamma)\frac{(k_1-1)}{(-c_1/4)}\sum_{n=0}^{\infty}\frac{(-c_1/4)^{n+2}}{(k_1-1)_{n+2}(n+2)!}\right. \\
&\left. + 2u_{p_2}(1) + (1+\gamma)\frac{(k_2-1)}{(-c_2/4)}\sum_{n=0}^{\infty}\frac{(-c_2/4)^{n+1}}{(k_2-1)_{n+1}(n+1)!}\right] \\
&= (1-\beta)\left[2\left\{u_{p_1}(1)-1\right\} + (1+\gamma)\frac{4(k_1-1)}{c_1}\left[u_{p_1-1}(1) - 1 - \frac{(-c_1/4)}{k_1-1}\right]\right. \\
&\left. + 2u_{p_2}(1) - (1+\gamma)\frac{4(k_2-1)}{c_2}\left[u_{p_2-1}(1)-1\right]\right] \\
&\leq 1-\gamma,
\end{aligned}$$

by the given hypothesis. □

In next theorem, we establish connections between $TG_H(\gamma)$ and $G_H(\gamma)$.

Theorem 2.12. *Let $c_1, c_2 < 0$, $k_1, k_2 > 0$. If for some $\gamma (0 \leq \gamma < 1)$ the inequality*

$$u_{p_1} + u_{p_2} \leq 2$$

is satisfied, then $\Omega\left(TG_H(\gamma)\right) \subset G_H(\gamma)$.

Proof. Making use of Lemma 2.2 and the definition of P_2 in Theorem 2.3, we only need to prove that $P_2 \leq 1-\gamma$. Using Remark 2.3, it follows that

$$\begin{aligned} P_2 &= \sum_{n=2}^{\infty}(2n-1-\gamma)\,\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!}\,|A_n| + \sum_{n=1}^{\infty}(2n+1+\gamma)\,\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-1)!}\,|B_n| \\ &\leq (1-\gamma)\left[\sum_{n=2}^{\infty}\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!} + \sum_{n=1}^{\infty}\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-1)!}\right] \\ &= (1-\gamma)\left[\sum_{n=0}^{\infty}\frac{(-c_1/4)^{n+1}}{(k_1)_{n+1}(n+1)!} + \sum_{n=0}^{\infty}\frac{(-c_2/4)^{n}}{(k_2)_{n}n!}\right] \\ &= (1-\gamma)\left[u_{p_1}(1) - 1 + u_{p_2}(1)\right] \\ &\leq (1-\gamma), \end{aligned}$$

by the given condition and this completes the proof. □

In next theorem, we present conditions on the parameters k_1, k_2, c_1, c_2 and obtain a characterization for operator Ω which maps $TG_H(\gamma)$ on to itself.

Theorem 2.13. *If $c_1, c_2 < 0$, $k_1, k_2 > 0 (k_1, k_2 \neq 0, -1, -2, ...)$ and $\gamma (0 \leq \gamma < 1)$. Then*

$$\Omega\left(TG_H(\gamma)\right) \subset TG_H(\gamma),$$

if and only if,

$$u_{p_1}(1) + u_{p_2}(1) \leq 2$$

Proof. The proof of above theorem is similar to that of Theorem 2.4. Therefore we omits the details involved. □

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