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An Observation on Set-Valued Contraction Mappings in Modular Metric Spaces

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Abstract : We present in this paper the existence criteria of a fixed point of a set-valued contraction in modular metric space. Our results also fix the error in the preceding paper [1, P. Chaipunya, C. Mongkolkeha, W. Sintunavarat, and P. Kumam, Fixed-point theorems for multivalued mappings in modular metric spaces, *Abstract and Applied Analysis*, 2012:14, 2012.], with some extra hypotheses.

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1 Introduction

The theory of modular metric space was initially developed by Chistyakov [2], extending the earlier concept of modular spaces. This modular metric space has overcame the difficulties when a linear structure is absent. By the same author [3], some uses of this space is observed. He also paid great attention towards an extension of the renowned contraction principle and its further applications in modular metric spaces as one can see in [4, 5]. At about the same time, another approach of the contraction principle in this space was also proposed by Mongkolkeha *et al.* [6, 7]. Further generalizations can be found in [8, 9].

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On the other hand, the set-valued alternative of the contraction principle was given in [1, 10]. Unfortunately, the proofs of the main results contains a small, but defective gap (we shall discuss this matter precisely in the forthcoming section). This leaves the problem of the set-valued contraction principle open.

This research is conducted to properly give a sufficient conditions for a set-valued contraction to possess a fixed point. Our results also fix the slip found in [1, 10], under some additional assumptions. The organization of this manuscript is as follows: We recall, in Section 2, some preliminaries and consider a very fundamental topological results in modular metric spaces. Finally, in Section 3, we discuss the mentioned error, convey to our main results some auxiliary definitions and lemmas, proposed our main theorems concerning the fixed point of set-valued contractions.

2 Modular Metric Spaces

In the rest of this paper, we shall write N, R, R^+, R_+ to represent the set of all positive integers, all reals, all positive reals, and all non-negative reals, respectively.

We shall study, under this section, some elementary topological results in modular metric spaces. We recall first the definition of a modular metric space according to Chistyakov [2].

Definition 2.1 ([2]). *Let X be a nonempty set. A* metric modular *on X is a non-negative extended real-valued function w defined on $R^+ \times X \times X$ (we write $w_t(x, y)$ instead of $w(t, x, y)$) such that:*

(a) For any $x, y \in X$, $x = y$ if and only if $w_t(x, y) = 0$ for all $t > 0$.

(b) For any $x, y \in X$ and any $t > 0$, $w_t(x, y) = w_t(y, x)$.

(c) For any $x, y, z \in X$ and any $s, t > 0$, $w_{s+t}(x, y) \leq w_s(x, z) + w_t(z, y)$.

For a given $x' \in X$, the restriction $X_w(x') = \{x \in X, \lim_{t\longrightarrow\infty} w_t(x', x) = 0\}$ is called a modular metric space *around x'. If $X_w(x) = X$ for all $x \in X$, we write X_w in place of $X_w(x)$.*

Remark 2.2. *w is nonincreasing in t.*

Given a modular metric space X_w. Suppose that $x \in X_w$ and $r > 0$, we define an open ball of radius r around x by

$$B(x; r) := \left\{ z \in X, \sup_{t>0} w_t(x, z) < r \right\}.$$

Let $BM(F)$ be a set containing all open balls in X_w. We may easily see that $BM(F)$ actually acts as a base determining a unique topology on X, namely τ. Always assume that X_w is a given modular metric space equipped with the topology generated by $BM(F)$.

With the same elementrary proofs (and so omitted) as in a classical metric space, we may obtain the following results:

Proposition 2.3. *X_w is Hausdorff separable.*

Proposition 2.4. *In X_w, the compactness and sequential compactness characterizes each others.*

Proposition 2.5. *A sequence (x_n) in X_w converges to a point $x \in X$ if and only if for any given $\varepsilon > 0$, we have $\sup_{t>0} w_t(x, x_n) < \varepsilon$ for sufficiently large $n \in N$.*

We may now define a Cauchy sequence in parallel to the characterization in Proposition 2.5.

Definition 2.6. *A sequence (x_n) in X_w is Cauchy if for any $\varepsilon > 0$, there holds that $\sup_{t>0} w_t(x_m, x_n) < \varepsilon$ for sufficiently large $m, n \in N$.*

Naturally, each convergent sequence is Cauchy. If the converse is true for all sequence in X_w, we say that X_w is *complete.*

Definition 2.7. *A set $Z \subset X_w$ is said to be bounded if $\sup_{x,y\in Z} \sup_{t>0} w_t(x, y) < \infty$.*

We may note that a non-singleton finite set in a modular metric space is no need to be bounded (for instance, take any metric space (M, ρ), and the metric modular $(t, x, y) \in R^+ \times M \times M \mapsto \frac{\rho(x,y)}{t}$). This fact gives an example of a compact set which is not bounded, in contrast to metric spaces. However, a compact set is always closed by Proposition 2.3.

In accordance to Chaipunya *et al.* [1], for $x \in X_w$ and $Y, Z \subset X_w$, we write

$$\begin{cases} w_t(x, Z) := \inf_{z\in Z} w_t(x, z), \\ e_t(Y, Z) := \sup_{y\in Y} w_t(y, Z), \\ W_t(Y, Z) := \max\{e_t(Y, Z), e_t(Z, Y)\}. \end{cases}$$

A number of fundamental properties of these functions for closed bounded sets can be found in [1]. In fact, such properties also work, with the same proofs, for closed (and not necessarily bounded) sets. Also note that if $Z \subset X_w$ is closed and $z \in X_w$, we have $z \in Z$ if and only if $w_t(z, Z) = 0$ for all $t > 0$.

3 Set-Valued Contractions

3.1 A Remark on Set-Valued Contraction

Given a set-valued map $F : X_w \rightrightarrows X_w$, if there exists a constant $k \in (0, 1)$ such that

$$W_t(F(x), F(y)) \le k w_t(x, y), \tag{3.1}$$

for all $t > 0$ and all $x, y \in X_w$, we say that F is a set-valued contraction.

The existence of fixed points for a set-valued contraction in modular metric space is first considered in [1, Theorem 3.3]. The original proof exploited the existence of a sequence (x_n) such that, for each $n \in N$, $x_n \in F(x_n)$ and

$$w_s(x_n, x_{n+1}) \le k^n + W_s(F(x_{n-1}), F(x_n)), \tag{3.2}$$

where $s > 0$ is pre-given. Note that the property (3.2) is not preserved upon the change of s. Unfortunately, (3.2) is needed for all $s > 0$, and this leaves out a gap in this proof.

To fill this gap in, we need some additional definitions, lemmas, and assumptions. These materials will be discussed in the succeeding section.

3.2 Auxiliary Results

Definition 3.1. *A nonempty subset $Z \subset X_w$ is said to be* reachable *from a point $x \in X_w$ if*

$$\inf_{z\in Z}\sup_{t>0} w_t(x,z) = \sup_{t>0}\inf_{z\in Z} w_t(x,z) < \infty.$$

Remark 3.2. *To show the reachability, we only need to show that*

$$\inf_{z\in Z}\sup_{t>0} w_t(x,z) \le \sup_{t>0}\inf_{z\in Z} w_t(x,z) < \infty,$$

since the reverse is always true.

An advantage of the notion of reachability is illustrated in the following lemma:

Lemma 3.3. *Given two nonempty closed subsets $Y, Z \subset X_w$ and a point $z \in Z$. Suppose that Y is reachable from z. Then, to each $\varepsilon > 0$, there corresponds a point $y_\varepsilon \in Y$ such that $\sup_{t>0} w_t(z, y_\varepsilon) \le \varepsilon + \sup_{t>0} W_t(Y, Z)$.*

Proof. Let $\varepsilon > 0$ be given. It is clear that we can find a point $y_\varepsilon \in Y$ such that $\sup_{t>0} w_t(z, y_\epsilon) \le \varepsilon + \inf_{y\in Y}\sup_{t>0} w_t(z, y)$. By the reachability of Y from z, we have

$$\inf_{y\in Y}\sup_{t>0} w_t(z,y) = \sup_{t>0}\inf_{y\in Y} w_t(z,y) = \sup_{t>0} w_t(z,Y) \le \sup_{t>0} W_t(Y,Z).$$

The conclusion thus follows. □

On the other hand, let us turn to a simple sufficient condition for a subset $Z \subset X_w$ to be reachable from $x \in X_w$.

Lemma 3.4. *Given a point $x \in X_w$ and a nonempty compact subset $Z \subset X_w$. If the metric modular w is l.s.c. in X and either $\inf_{z\in Z}\sup_{t>0} w_t(x,z)$ or $\sup_{t>0}\inf_{z\in Z} w_t(x,z)$ is finite, then Z is reachable from x.*

Proof. For each $s > 0$, we can find a sequence (z_n^s) such that

$$w_s(x, z_n^s) \longrightarrow \inf_{z\in Z} w_s(x,z).$$

Since Z is compact, we may assume that (z_n^s) converges to some point $z^s \in Z$. Since w is l.s.c. in X, we have

$$w_s(x, z^s) \le \liminf_{n\longrightarrow\infty} w_s(x, z_n^s) = \inf_{z\in Z} w_s(x,z),$$

and therefore $w_s(x, z^s) = \inf_{z\in Z} w_s(x, z)$. Finally, we have

$$\inf_{z\in Z}\sup_{t>0} w_t(x, z) \le \sup_{t>0} w_t(x, z^s) = \sup_{t>0}\inf_{z\in Z} w_s(x, z).$$

This completes the proof. □

3.3 Existence Theorems

At this stage, we exploit the notion of reachability and its supplementary results to deduce some fixed point theorems for set-valued contractions. The obtained result also fix the error in [1]. Additionally assume through the rest of the paper that X_w is complete.

Theorem 3.5. *Suppose that F is a set-valued contraction (w.r.t. $k \in (0, 1)$) on X_w having compact values, and that the metric modular w is l.s.c. in X. If there exist two points $x_0 \in X_w$ and $x_1 \in F(x_0)$ such that the set $\{x_0, x_1\}$ is bounded and $F(x_1)$ is reachable from x_1, then F has a fixed point.*

Proof. Since $F(x_1)$ is reachable from x_1, by using Lemma 3.3, we may choose $x_2 \in F(x_1)$ such that

$$\sup_{t>0} w_t(x_1, x_2) \le \sup_{t>0} W_t(F(x_0), F(x_1)) + k.$$

From the above evidence and the hypothesis that $\{x_0, x_1\}$ is bounded, it comes to the following inequalities:

$$\begin{aligned}
\sup_{t>0} w_t(x_2, F(x_2)) &\le \sup_{t>0} W_t(F(x_1), F(x_2)) \\
&\le k \sup_{t>0} w_t(x_1, x_2) \\
&\le k[\sup_{t>0} W_t(F(x_0), F(x_1)) + k] \\
&\le k^2 \sup_{t>0} w_t(x_0, x_1) + k^2 \\
&< \infty.
\end{aligned}$$

Since F is compact valued, we apply Lemma 3.4 to guarantee that $F(x_2)$ is actually reachable from x_2. Inductively, by this procedure, we define a sequence (x_n) in X_w satisfying the following properties for all $n \in N$:

$$\begin{cases}
x_n \in F(x_{n-1}), \\
\sup_{t>0} w_t(x_n, x_{n+1}) \le \sup_{t>0} W_t(F(x_{n-1}), F(x_n)) + k^n, \\
F(x_n) \text{ is reachable from } x_n.
\end{cases}$$

Hence, by the contractivity of F, we have

$$\begin{aligned}\sup_{t>0} w_t(x_n, x_{n+1}) &\leq \sup_{t>0} W_t(F(x_{n-1}), F(x_n)) + k^n \\ &\leq k \sup_{t>0} w_t(x_{n-1}, x_n) + k^n \\ &\leq k[k \sup_{t>0} w_t(x_{n-2}, x_{n-1}) + k^{n-1}] + k^n \\ &\leq k^2 \sup_{t>0} w_t(x_{n-2}, x_{n-1}) + 2k^n.\end{aligned}$$

Thus, by induction, we have

$$\sup_{t>0} w_t(x_n, x_{n+1}) \leq k^n \sup_{t>0} w_t(x_0, x_1) + nk^n.$$

Moreover, it follows that

$$\sup_{t>0} \sum_{n\in N} w_t(x_n, x_{n+1}) \leq \sum_{n\in N} \sup_{t>0} w_t(x_n, x_{n+1}) \leq \sup_{t>0} w_t(x_0, x_1) \sum_{n\in N} k^n + \sum_{n\in N} nk^n < \infty.$$

Thus, for any $\varepsilon > 0$, we may find $n_* \in N$ such that for $m, n \in N$ and $m > n$, we have

$$\begin{aligned}\sup_{t>0} w_t(x_n, x_m) &\leq \sup_{t>0}[w_{\frac{t}{m-n}}(x_n, x_{n+1}) + w_{\frac{t}{m-n}}(x_{n+1}, x_{n+2}) + \cdots + w_{\frac{t}{m-n}}(x_{m-1}, x_m)] \\ &\leq \sup_{t>0} w_t(x_n, x_{n+1}) + \sup_{t>0} w_t(x_{n+1}, x_{n+2}) + \cdots + \sup_{t>0} w_t(x_{m-1}, x_m) \\ &\leq \sum_{n=n_*}^{\infty} \sup_{t>0} w_t(x_n, x_{n+1}) \\ &< \varepsilon.\end{aligned}$$

Hence, (x_n) is a Cauchy sequence and so the completeness of X_w implies that (x_n) converges to some point $x \in X_w$. Consequently, we may conclude from the contractivity of F that the sequence $(F(x_n))$ converges to $F(x)$. Since $x_n \in F(x_{n-1})$, we have for any $t > 0$,

$$0 \leq w_t(x, F(x)) \leq w_{\frac{t}{2}}(x, x_n) + w_{\frac{t}{2}}(x_n, F(x)) \leq w_{\frac{t}{2}}(x, x_n) + W_{\frac{t}{2}}(F(x_{n-1}), F(x)),$$

which implies that $w_t(x, F(x)) = 0$ for all $t > 0$. Since $F(x)$ is closed, it then follows that $x \in F(x)$. □

Along with the set-valued contraction (3.1), we may consider another class of maps: Let $F : X_w \rightrightarrows X_w$.If the inequality

$$W_t(F(x), F(y)) \leq k[w_t(x, F(x)) + w_t(y, F(y))]$$

is satisfied for all $t > 0$ and all $x, y \in X_w$, at some fixed $k \in (0, \frac{1}{2})$, we say that F is a *set-valued Kannan's contraction.* We close our paper with the following theorem which is similarly obtained to the preceding theorem.

Theorem 3.6. *Suppose that F is a set-valued Kannan's contraction (w.r.t. $k \in (0, \frac{1}{2})$) on X_w having compact values, and that the metric modular w is l.s.c. in X. If there exist two points $x_0 \in X_w$ and $x_1 \in F(x_0)$ such that the set $\{x_0, x_1\}$ is bounded and $F(x_1)$ is reachable from x_1, then F has a fixed point.*

Proof. Since $F(x_1)$ is reachable from x_1, by using Lemma 3.3, we may choose $x_2 \in F(x_1)$ such that

$$\sup_{t>0} w_t(x_1, x_2) \leq \sup_{t>0} W_t(F(x_0), F(x_1)) + k.$$

Now, observe that

$$\begin{aligned}
\sup_{t>0} w_t(x_2, F(x_2)) &\leq \sup_{t>0} W_t(F(x_1), F(x_2)) \\
&\leq k \sup_{t>0} w_t(x_1, F(x_1)) + k \sup_{t>0} w_t(x_2, F(x_2)) \\
&\leq k \sup_{t>0} W_t(F(x_0), F(x_1)) + k \sup_{t>0} w_t(x_2, F(x_2)) \\
&\leq k \sup_{t>0} w_t(x_0, F(x_0)) + k \sup_{t>0} w_t(x_1, F(x_1)) + k \sup_{t>0} w_t(x_2, F(x_2)) \\
&\leq k \sup_{t>0} w_t(x_0, x_1) + k \sup_{t>0} w_t(x_1, F(x_1)) + k \sup_{t>0} w_t(x_2, F(x_2)).
\end{aligned}$$

Writing $\xi := \frac{k}{1-k} < 1$, we obtain, from the boundedness of $\{x_0, x_1\}$ and the reachability of $F(x_1)$ from x_1, that

$$\sup_{t>0} w_t(x_2, F(x_2)) \leq \xi \sup_{t>0} w_t(x_0, x_1) + \xi \sup_{t>0} w_t(x_1, F(x_1)) < \infty.$$

Thus, $F(x_2)$ is reachable from x_2. Inductively, we can construct a sequence (x_n) in X_w with exactly the same properties appearing in the proof of Theorem 3.5.

Now, consider further that

$$\begin{aligned}
\sup_{t>0} w_t(x_n, x_{n+1}) &\leq \sup_{t>0} W_t(F(x_{n-1}), F(x_n)) + k^n \\
&\leq k \sup_{t>0} w_t(x_{n-1}, F(x_{n-1})) + k \sup_{t>0} w_t(x_n, F(x_n)) + k^n \\
&\leq k \sup_{t>0} w_t(x_{n-1}, F(x_{n-1})) + k \sup_{t>0} w_t(x_n, x_{n+1}) + k^n.
\end{aligned}$$

Moreover, we get

$$\begin{aligned}\sup_{t>0} w_t(x_n,x_{n+1}) &\leq \xi \sup_{t>0} w_t(x_{n-1},x_n) + \frac{k^n}{1-k}\\ &\leq \xi^2 \sup_{t>0} w_t(x_{n-2},x_{n-1}) + \frac{k^n}{(1-k)^2} + \frac{k^n}{(1-k)}\\ &\leq \xi^2 \sup_{t>0} w_t(x_{n-2},x_{n-1}) + 2\cdot\frac{k^n}{(1-k)^2}\\ &\vdots\\ &\leq \xi^n \sup_{t>0} w_t(x_0,x_1) + n\xi^n.\end{aligned}$$

As in the proof of Theorem 3.5, the sequence (x_n) converges to some $x \in X_w$. Observe that

$$\begin{aligned}\sup_{t>0} w_t(x,F(x)) &= \sup_{t>0} \delta_t(\{x\},F(x))\\ &\leq \sup_{t>0} \delta_t(\{x\},F(x_n)) + \sup_{t>0} \delta_t(F(x_n),F(x))\\ &= \sup_{t>0} w_t(x,F(x_n)) + \sup_{t>0} \delta_t(F(x_n),F(x))\\ &\leq \sup_{t>0} w_t(x,x_{n+1}) + \sup_{t>0} W_t(F(x_n),F(x))\\ &\leq \sup_{t>0} w_t(x,x_{n+1}) + k\sup_{t>0} w_t(x_n,F(x_n)) + k\sup_{t>0} w_t(x,F(x))\\ &= (1+k)\sup_{t>0} w_t(x,x_{n+1}) + k\sup_{t>0} w_t(x,F(x)).\end{aligned}$$

Thus, we have

$$\sup_{t>0} w_t(x,F(x)) \leq (\tfrac{1+k}{1-k}) \sup_{t>0} w_t(x,x_{n+1}).$$

Letting $n \longrightarrow \infty$ to conclude the theorem.

□

4 Concluding Remarks

The main theorems discussed in the previous section ensure the existence of a fixed point for set-valued contractions and Kannan's contraction, assuming some additional conditions and auxiliary results. Our results correct the faulty proofs in [1]. Finally, we shall conclude our paper with the following open problems:

Question 1. *Can we drop the l.s.c. of w in Theorems 3.5 and 3.6?*

Question 2. *Is it possible to obtain the result when F is not compact valued?*

Question 3. *Can we weaken the notion of reachability in Theorems 3.5 and 3.6?*

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