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Operators on Hilbert Spaces of Sequences

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Abstract : In this paper, we study bounded weighted composition operators on Hilbert spaces of sequences.

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1 Introduction

Let S denote the vector space of all sequences $x = (x_k)_{k=1}^{\infty}$ of complex numbers. Any topological vector subspace of S is called a sequence space. The well known Banach spaces of sequences are ℓ_p spaces, $1 \leq p \leq \infty$ defined as

$$\ell_p = \{x \in S : \text{ and } \sum_{n=0}^{\infty} |x_n|^p < \infty\}$$

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where N denotes the set of non-negative integers and norm on ℓ_p is given by

$$\| x \|_p = (\sum_{n=0}^{\infty} |x_n|^p)^{1/p}, \text{ for } 1 \leq p < \infty$$

and

$$\| x \|_\infty = \sup_{0 \leq n < \infty} |x_n| \text{ for } p = \infty, \quad \text{for all} \quad x \in \ell_p$$

For $p = 2$, ℓ_2 is the Hilbert space under the inner product defined as

$$\langle x, y \rangle = \sum_{n=0}^{\infty} x_n \bar{y}_n \quad \forall\ x, y \in \ell_2$$

This is the earliest known example of a Hilbert space founded by Hilbert himself. This space is widely studied by several mathematicians in connection with the study of unilateral shift, bilateral shift, multiplication operators, composition operators, cyclic, hyper cyclic operators and weighted composition operators (See Carlson [1], Singh and Komal [2]). The symbol $\sharp(E)$ denotes the cardinality of the set E.

Let $\lambda = \{\lambda_k\}_{k=0}^{\infty}$ be a strictly increasing sequence of positive reals tending to infinity, that is $0 < \lambda_0 < \lambda_1 < \ldots$ and $\lambda_k \to \infty$ as $k \to \infty$ we define a sequence space

$$\ell_p^\lambda = \{x \in S : \sum_{n=0}^{\infty} |\frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_k|^p < \infty\}$$

It is known that ℓ_p^λ is a Banach space under the norm

$$\| x \|_\Lambda = (\sum_{n=0}^{\infty} |\frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_k|^p)^{\frac{1}{p}} < \infty$$

For $p = 2$, ℓ_2^λ is a Hilbert space under the inner product

$$\langle x, y \rangle = \langle \Lambda x, \Lambda y \rangle$$

where $(\Lambda x)(n) = \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_k, \quad n \in N$

The set $\{e_\lambda^{(n)}\}_{n=0}^{\infty}$ where

$$e_\lambda^{(n)}(k) = \begin{cases} (-1)^{k-n} \frac{\lambda_n}{\lambda_k - \lambda_{k-1}}, & n \leq k \leq n+1 \\ 0, & \text{elsewhere} \end{cases}$$

is an orthonormal basis for ℓ_2^λ. The spaces ℓ_p and ℓ_p^λ do not include each other if $\frac{1}{\lambda} \notin \ell_p$ for $0 < p < \infty$. The inclusion $\ell_p \subset \ell_p^\lambda$ holds if and only if $\frac{1}{\lambda} \in \ell_p$ for

$1 \leq p < \infty$. The inclusion is proper if in addition $\lim_{n\to\infty} \inf \frac{\lambda_{n+1}}{\lambda_n} = 1$. The equality $\ell_p^\lambda = \ell_p$ holds if and only if $\lim_{n\to\infty} \inf \frac{\lambda_{n+1}}{\lambda_n} > 1$, where $1 \leq p < \infty$.
In this paper, we initiated the study of weighted composition operators on ℓ_p^λ. We assume that $\lim_{n\to\infty} \inf \frac{\lambda_{n+1}}{\lambda_n} = 1$. For detailed study of Hilbert spaces of sequences we refer to Mursaleen and Noman [3], [4] and [5].

2 Bounded Composition Operators on Hilbert spaces of Sequences

In this section we obtain a condition for bounded composition operators.

Theorem 2.1. *Let $T : N \to N$ be a mapping. Then $C_T : \ell_p \to \ell_p^\lambda$, $1 \leq p < \infty$ is bounded if there exist $M > 0$ such that*

$$\sharp(T^{-1}(\{n\})) \leq M \quad \forall \quad n \in N.$$

Proof. For $x \in \ell_p$, consider

$$\begin{aligned}
\| C_T x \|_\Lambda^p &= \sum_{n=0}^{\infty} |\sum_{k=0}^{n} \frac{\lambda_k - \lambda_{k-1}}{\lambda_n} x_{T(k)}|^p \\
&\leq \sum_{n=0}^{\infty} \Big[\sum_{k=0}^{n} (\frac{\lambda_k - \lambda_{k-1}}{\lambda_n})^{1/p} |x_{T(k)}| (\frac{\lambda_k - \lambda_{k-1}}{\lambda_n})^{1/q}\Big]^p \\
&\leq \sum_{n=0}^{\infty} \Big[\sum_{k=0}^{n} (\frac{\lambda_k - \lambda_{k-1}}{\lambda_n} |x_{T(k)}|^p (\sum_{k=0}^{n} (\frac{\lambda_k - \lambda_{-1}}{\lambda_n}))^{p/q}\Big]
\end{aligned}$$

(by Holder's inequality)

$$\begin{aligned}
&\leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} (\frac{\lambda_k - \lambda_{k-1}}{\lambda_n}) |x_{T(k)}|^p \\
&\leq \sum_{k=0}^{\infty} (\lambda_k - \lambda_{k-1}) |x_{T(k)}|^p \sum_{n=k}^{\infty} \frac{1}{\lambda_n} \\
&\leq L \sum_{k=0}^{\infty} |x_{T(k)}|^p, \qquad \text{where } L = \sup_k (\lambda_k - \lambda_{k-1}) \sum_{n=k}^{\infty} \frac{1}{\lambda_n} < \infty \\
&= L \sum_{k=0}^{\infty} \sum_{m\in T^{-1}(k)} |x_{T(m)}|^p \\
&= L \sum_{k=0}^{\infty} \sum_{m\in T^{-1}(k)} |x_k|^p \\
&\leq LM \sum_{k=0}^{\infty} |x_k|^p \\
&= LM \| x \|_p^p .
\end{aligned}$$

Therefore we conclude that C_T is a bounded operator. □

Corollary 2.2. *If $T : N \to N$ is a constant function, then C_T is not a bounded operator on ℓ_p^λ.*

Proof. Suppose $T : N \to N$ is a constant function. Then $T(n) = n_0 \;\; \forall \;\; n \in N$. Take $x \in \ell_p^\lambda$ such that $x_{n_0} \neq 0$. Then from the equality

$$\begin{aligned}
\| C_T x \|_\lambda^p &= \sum_{n=0}^{\infty} |\frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_{T(k)}|^p \\
&= \sum_{n=0}^{\infty} \frac{1}{\lambda_n^p} |\sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_{n_0}|^p \\
&= \sum_{n=0}^{\infty} \frac{1}{\lambda_n^p} |\lambda_n x_{n_0}|^p \\
&= \sum_{n=0}^{\infty} |x_{n_0}|^p \\
&= \infty
\end{aligned}$$

Thus C_T is not bounded operator. □

Example 2.3. *Let $T : N \to N$ be defined by $T(n) = n + 1$ and $\lambda_n = n^2$. For any $x \in \ell_2$,*

$$\begin{aligned}
\| C_T x \|_\Lambda^2 &= \sum_{n=0}^{\infty} |\sum_{k=0}^{n} \frac{\lambda_k - \lambda_{k-1}}{\lambda_n} x_{T(k)}|^2 \\
&= \sum_{n=0}^{\infty} |\sum_{k=0}^{n} (\frac{\lambda_k - \lambda_{k-1}}{\lambda_n})^{1/2} (\frac{\lambda_k - \lambda_{k-1}}{\lambda_n})^{1/2} x_{k+1}|^2 \\
&\leq \sum_{n=0}^{\infty} [\sum_{k=0}^{n} \frac{\lambda_k - \lambda_{k-1}}{\lambda_n} \sum_{k=0}^{n} \frac{\lambda_k - \lambda_{k-1}}{\lambda_n} |x_{k+1}|^2] \\
&= \sum_{n=0}^{\infty} \frac{1}{\lambda_n} \sum_{k=0}^{n} (\frac{\lambda_k - \lambda_{k-1}}{\lambda_n}) |x_{k+1}|^2 \\
&= \sum_{k=0}^{\infty} (\lambda_k - \lambda_{k-1}) |x_{k+1}|^2 \sum_{n=k}^{\infty} \frac{1}{\lambda_n^2}
\end{aligned}$$

$$= \sum_{k=0}^{\infty} |x_{k+1}|^2((k^2-(k-1)^2)\sum_{n=k}^{\infty}\frac{1}{n^4}$$
$$= \sum_{k=0}^{\infty} |x_{k+1}|^2(2k-1)\sum_{n=k}^{\infty}\frac{1}{n^4}$$
$$= M\sum_{k=0}^{\infty} |x_{k+1}|^2, \qquad \textit{where } M = \sup_k(2k-1)\sum_{n=k}^{\infty}\frac{1}{n^4} < \infty$$
$$= M \parallel x \parallel^2 .$$

Hence $C_T : \ell_2 \to \ell_2^\lambda$ is a bounded operator.

Remark 2.4. *The above example shows that the unilateral shift operator is a bounded operator on ℓ_2^λ.*

3 Bounded Multiplication Operators on Hilbert spaces of Sequences

Theorem 3.1. *Let $\theta : N \to C$ be a bounded function . Then $M_\theta : \ell_p \to \ell_p^\lambda$ is a bounded operator.*

Proof. Suppose θ is a bounded function. Then $\exists\ M > 0$ such that

$$|\theta(n)| \leq M \quad \forall \quad n \in N.$$

For $x \in \ell_p$, consider

$$\begin{aligned}
\parallel M_\theta x \parallel_\lambda^p &= \sum_{n=0}^{\infty}|\sum_{k=0}^{n}\frac{(\lambda_k-\lambda_{k-1})}{\lambda_n}\theta(k)x_k|^p \\
&\leq \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(\lambda_k-\lambda_{k-1})}{\lambda_n}|\theta(k)|^p|x_k|^p \\
&= \sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})|\theta(k)|^p|x_k|^p\sum_{n=k}^{\infty}\frac{1}{\lambda_n} \\
&\leq M^pL\sum_{k=0}^{\infty}|x_k|^p, \qquad \text{where } L = \sup_k(\lambda_k-\lambda_{k-1})\sum_{n=k}^{\infty}\frac{1}{\lambda_n} < \infty \\
&= M^pL \parallel x \parallel^p .
\end{aligned}$$

Hence $\parallel M_\theta x \parallel \leq t \parallel x \parallel$ where $t = ML^{\frac{1}{p}}$ This proves that M_θ is a bounded operator. □

4 Bounded Weighted Composition Operators on Hilbert spaces of Sequences

Theorem 4.1. *Let $w : N \to C$ and $T : N \to N$ be two mappings. If there exists $M > 0$ such that*

$$\sum_{m\in T^{-1}(k)} |w(m)|^p \leq M \qquad \forall \;\; k \in N,$$

then $M_{w,T} : \ell_p \to \ell_p^\lambda$ is a bounded operator.

Proof. For $x \in \ell_p$, consider

$$\begin{aligned}
\| \quad M_{w,T}x \|_\lambda^p &= \sum_{n=0}^{\infty} |\sum_{k=0}^{n} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n} w(k) x_{T(k)}|^p \\
&\leq \sum_{n=0}^{\infty} [\sum_{k=0}^{n} \frac{\lambda_k - \lambda_{k-1}}{\lambda_n} |(w(k)|.|x_{T(k)}|]^p \\
&= \sum_{k=0}^{\infty} (\lambda_k - \lambda_{k-1}) |w(k)|^p |x_{T(k)}|^p \sum_{n=k}^{\infty} \frac{1}{\lambda_n} \\
&= \sum_{m=0}^{\infty} (\lambda_m - \lambda_{m-1}) |w(m)|^p |x_{T(m)}|^p \sum_{n=m}^{\infty} \frac{1}{\lambda_n} \\
&\leq L \sum_{m=0}^{\infty} |w(m)|^p |x_{T(m)}|^p \text{ where } L = \sup_m (\lambda_m - \lambda_{m-1}) \sum_{n=m}^{\infty} \frac{1}{\lambda_n} < \infty \\
&= L \sum_{m=0}^{\infty} (\sum_{k\in T^{-1}(m)} |w(k)|^p) |x_m|^p \\
&\leq ML \sum_{m=0}^{\infty} |f(m)|^p \\
&= LM \| x \| _p^p.
\end{aligned}$$

Hence $\| M_{w,T}x \| \leq t \| x \|_p \quad \forall \;\; x \in \ell_p$, where $t = (LM)^{\frac{1}{p}}$.
This proves that $M_{w,T}$ is a bounded operator. ☐

Example 4.2. *Let $T : N \to N$ be defined by*
$T(k) = 3n + 3$ if $k \in \{3n, 3n + 1, 3n + 2\}$ for each $n \in N$. Clearly,

$$T^{-1}(k) = \begin{cases} \{k-3, k-2, k-1\}, & \text{if } k = 3(m+1) \quad for \;\; m \in N \\ \phi, & elsewhere \end{cases}$$

Define $w : N \to C$ by

$$w(n) = \frac{i}{n+1} \quad \forall \; n \in N.$$

Now

$$\sum_{m\in T^{-1}(k)} |(w(m)|^p = \begin{cases} |w(k-3)|^p + |w(k-2)|^p + |w(k-1)|^p, & if \; k = 3(m+1)(m \in N) \\ 0, & elsewhere \end{cases}$$

$$= \begin{cases} \frac{1}{(k-2)^p} + \frac{1}{(k-1)^p} + \frac{1}{k^p}, & \text{if } k = 3(m+1) \ \ for \ \ each \ \ m \in N \\ 0, & elsewhere \end{cases}$$

Hence $\sum_{m \in T^{-1}(k)} |w(m)|^p \leq 3 \quad \forall \ \ k \in N$

Thus in view of above theorem, $M_{w,T}$ *is a bounded operator.*

5 A Characterization of Reducing subspace of Multiplication operators on Hilbert spaces of Sequences

Theorem 5.1. *Let* $\theta : N \to C$ *be a map having distinct values. Let* $M_\theta : \ell_p \to \ell_p^\lambda$ *be a bounded operator. Then a closed subspace* E *of* ℓ_2 *is a reducing subspace of* M_θ *if and only if there exists a non-empty subset* M *of* N *such that*

$$E = \{x \in \ell_2 : x_n = 0 \ \ \forall \ \ n \notin M\}.$$

Proof. Suppose $E = \{x \in \ell_2 : x_n = 0 \ \ \forall \ \ n \notin M\}$, where M is a non-empty subset of N. Then clearly $E^\perp = \{x \in \ell_2 : x_m = 0 \ \ \forall \ \ m \in M\}$. Clearly E and $E^\perp$ are invariant under M_θ. Hence E is a reducing subspace of M_θ.

Conversely, suppose M_θ has a reducing subspace say E. Let P be the projection on E. Then

$$PM_\theta = M_\theta P.$$

For any $n \in N$, we have

$$PM_\theta e_n = M_\theta P e_n$$

or $\qquad \theta(n) P e_n = \theta P e_n$

or $\qquad (\theta - \theta(n) I) P e_n = 0 \ \ \forall \, n \in N.$

This implies that $(Pe_n)(m) = 0$ for every $m \neq n$. Hence $Pe_n = \alpha_n e_n$ for some $\alpha_n \in C$.

Now $\alpha_n e_n = P e_n = P^2 e_n = \alpha_n^2 e_n$. Therefore $\alpha_n = 0$ or $\alpha_n = 1$. If $\alpha_n = 0$, then $Pe_n = 0$ in which case $e_n \in E^\perp$. Further, if $\alpha_n = 1$, then $e_n \in E$. Let $M = \{n \in N : Pe_n = e_n\}$. Then clearly $E = \{x \in \ell_2 : x_n = 0 \ \ \forall \ \ n \notin M\}$, which proves the result. □

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