



# Common Fixed Point Theorems for Hybrid Pairs of Maps in Fuzzy Metric Spaces

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**Abstract :** The purpose of this paper is to introduce the notion of tangential mappings for a hybrid pair of single-valued and multi-valued mappings in fuzzy metric spaces and utilize the same to prove common fixed point theorems for such mappings in fuzzy metric spaces which generalize several previously known results.

**Keywords :** fuzzy metric space; hybrid map; common fixed point.

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## 1 Introduction

The evolution of fuzzy mathematics solely banks on the notion of fuzzy set which was introduced by Zadeh [1] in 1965 with a view to represent the vagueness in everyday life. In mathematical programming, various problems are often expressed

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as optimization of suitable goal functions equipped with specific constraints suggested by some concrete practical problem owing to its concrete situation. There exist many real life problems that consider multiple objectives and generally it is very difficult to get a feasible solution wherein an optimum of all the objective functions can be realized. The feasible method of resolving such problems is the use of fuzzy sets (e.g. [2]). In fact, the richness of applications has engineered the all round development of fuzzy mathematics. Like many other concepts, the study of fuzzy metric space has also been carried out in several ways (e.g., [3, 4]). George and Veeramani [5] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [6] with a view to obtain a Hausdorff topology on fuzzy metric spaces and this has recently found very fruitful applications in quantum particle physics particularly in string theory and  $\varepsilon^\infty$  theory (e.g. [7] and references cited therein). In recent years, many authors have proved fixed and common fixed point theorems in metric and fuzzy metric spaces. To mention a few, we cite [2, 8–17].

The concept of hybrid tangential mappings in metric spaces was introduced by Kamran [18], which is an improvement over (E.A) property and by now there exist numerous results of this kind (e.g. [19, 20]). In this paper, we define this concept in fuzzy metric spaces and utilize the same to prove common fixed point theorems in fuzzy spaces. Our results are improvement over some relevant results contained in [21–25] besides some other ones.

In what follows, we state some definitions and results which are required in our subsequent discussion.

**Definition 1.1** ([26]). A binary operation  $*$  :  $[0, 1] \times [0, 1] \rightarrow [0, 1]$  is said to be continuous t-norm if

- (I)  $*$  is commutative and associative;
- (II)  $*$  is continuous;
- (III)  $a * 1 = a$  for all  $a \in [0, 1]$ ;
- (IV)  $a * b \leq c * d$  whenever  $a \leq c$  and  $b \leq d$  for all  $a, b, c, d \in [0, 1]$ .

The two classical examples of t-norms are  $a * b = ab$  and  $a * b = \min\{a, b\}$ .

**Definition 1.2** ([6]). The 3-tuple  $(X, M, *)$  is said to be a fuzzy metric space if  $X$  is an arbitrary set,  $*$  is continuous t-norm and  $M$  is a fuzzy set on  $X^2 \times [0, \infty)$  satisfying the following conditions (for all  $x, y, z \in X$  and  $t, s > 0$ ):

- (KM<sub>1</sub>)  $M(x, y, 0) = 0$ ;
- (KM<sub>2</sub>)  $M(x, y, t) = 1 \forall t > 0$  iff  $x = y$ ;
- (KM<sub>3</sub>)  $M(x, y, t) = M(y, x, t)$ ;
- (KM<sub>4</sub>)  $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$ ;
- (KM<sub>5</sub>)  $M(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]$  is left continuous.

Such fuzzy metric spaces are often referred as KM-fuzzy metric spaces.

**Remark 1.3** ([27]). *The function  $M(x, y, t)$  is often interpreted as the nearness between  $x$  and  $y$  with respect to  $t$ .*

**Lemma 1.4** ([28]). *For every  $x, y \in X$ , the mapping  $M(x, y, \cdot)$  is nondecreasing on  $(0, \infty)$ .*

**Definition 1.5** ([5]). The 3-tuple  $(X, M, *)$  is said to be a GV-fuzzy metric space if  $X$  is an arbitrary set,  $*$  is continuous t-norm and  $M$  is a fuzzy set on  $X^2 \times (0, \infty)$  satisfying the following conditions (for all  $x, y, z \in X$  and  $t, s > 0$ ):

$$(GV_1) \quad M(x, y, 0) > 0;$$

$$(GV_2) \quad M(x, y, t) = 1 \text{ iff } x = y;$$

$$(GV_3) \quad M(x, y, t) = M(y, x, t);$$

$$(GV_4) \quad M(x, z, t + s) \geq M(x, y, t) * M(y, z, s);$$

$$(GV_5) \quad M(x, y, \cdot) : (0, \infty) \rightarrow [0, 1] \text{ is continuous.}$$

As mentioned earlier, such fuzzy metric spaces are often referred as GV-fuzzy metric spaces.

**Example 1.6** ([5]). *Let  $(X, d)$  be a metric space wherein  $a * b = ab$  for  $a, b \in [0, 1]$ . Then, one can define a fuzzy metric  $M_d(x, y, t)$  by*

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}, \quad x, y \in X.$$

**Definition 1.7** ([29]). Let  $CB(X)$  be the set of all nonempty closed bounded subsets of  $X$ . Then for every  $A, B, C \in CB(X)$  and  $t > 0$ ,

$$M(A, B, t) = \min\{\min M(a, B, t), \min M(A, b, t)\}$$

where  $M(C, y, t) = \max\{M(z, y, t) : z \in C\}$ .

**Remark 1.8** ([22]). *Obviously  $M(A, B, t) \leq M(a, B, t)$  whenever  $a \in A$  and  $M(A, B, t) = 1$  iff  $A = B$ . Obviously,  $1 = M(A, B, t) \leq M(a, B, t)$  for all  $a \in A$ .*

**Definition 1.9** ([21]). A sequence  $\{x_n\}$  in a KM or GV-fuzzy metric space  $(X, M, *)$  is said to be convergent to some  $x \in X$  if for all  $t > 0$ , there is some  $n_0 \in \mathbb{N}$  such that  $\lim_{n \rightarrow \infty} M(x_n, x, t) = 1$  for all  $n \geq n_0$ .

**Definition 1.10** ([23]). Let  $CL(X)$  be the set of all nonempty closed subsets of a metric space  $(X, d)$  and  $F : Y \subseteq X \rightarrow CL(X)$ . Then the map  $f : Y \rightarrow X$  is said to be F-weakly commuting at  $x \in X$  if  $ffx \in Ffx$  provided that  $fx \in Y$  for all  $x \in Y$ .

**Definition 1.11** ([24]). Two pairs  $(f, F)$  and  $(g, G)$  of self mappings of a  $KM$  (or  $GV$ )-fuzzy metric space  $(X, M, *)$  are said to satisfy the common property (E.A) if there exist two sequences  $\{x_n\}$  and  $\{y_n\}$  in  $X$  such that for all  $t > 0$

$$\lim_{n \rightarrow \infty} M(fx_n, Fx_n, t) = \lim_{n \rightarrow \infty} M(gy_n, Gy_n, t) = 1.$$

**Definition 1.12** ([22]). Let  $f, g : X \rightarrow X$  and  $F, G : X \rightarrow CB(X)$  of a  $KM$  (or  $GV$ )-fuzzy metric space  $(X, M, *)$ . Then the hybrid pair of mappings  $(f, F)$  and  $(g, G)$  are said to satisfy the common property (E.A) if there exist two sequences  $\{x_n\}$  and  $\{y_n\}$  in  $X$ , some  $u \in X$  and  $A, B \in CB(X)$  such that

$$\lim_{n \rightarrow \infty} Fx_n = A, \lim_{n \rightarrow \infty} Gy_n = B, \lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gy_n = u \in A \cap B.$$

**Definition 1.13** ([18]). Let  $(X, d)$  be metric space,  $f, g : X \rightarrow X$  and  $F, G : X \rightarrow CL(X)$ . Then the hybrid pair  $(f, F)$  is said to be  $g$ -tangential at  $u \in X$  if there exist two sequences  $\{x_n\}, \{y_n\}$  in  $X$  such that  $\lim_{n \rightarrow \infty} Gy_n \in CL(X)$  and

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gy_n = u \in A = \lim_{n \rightarrow \infty} Fx_n.$$

**Remark 1.14** ([18]). *If the hybrid pair of mappings  $(f, F)$  and  $(g, G)$  satisfies the common property (E.A), then  $(f, F)$  is  $g$ -tangential whereas  $(g, G)$  is  $f$ -tangential but not conversely (in general).*

**Definition 1.15** ([23]). Let  $(X, d)$  be metric space. If  $f, g : Y \subseteq X \rightarrow X$  and  $F, G : Y \rightarrow CL(X)$ , then the hybrid pair  $(f, F)$  is said to be  $g$ -tangential at  $u \in Y$  with respect to  $G$  if there exist two sequences  $\{x_n\}, \{y_n\}$  and  $A \in CL(X)$  in  $Y$  such that  $\lim_{n \rightarrow \infty} Gy_n \in CL(X)$  and

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gy_n = u \in A = \lim_{n \rightarrow \infty} Fx_n.$$

**Remark 1.16** ([23]). *The hybrid pairs of mappings  $(f, F)$  and  $(g, G)$  satisfy the common property (E.A) if and only if  $(f, F)$  is  $g$ -tangential with respect to  $G$  and  $(g, G)$  is  $f$ -tangential with respect to  $F$  but the converse is not necessarily true. Notice that the common (E.A) property reduces to E.A property (cf. [30]) if we restrict to a single pair.*

**Definition 1.17** ([25]). A map  $f : Y \subseteq X \rightarrow X$  is said to be coincidentally idempotent w.r.t. a mapping  $F : Y \rightarrow CL(X)$  if  $f$  is idempotent at the coincidence points of  $(f, F)$ , i.e.,  $ffx = fx$  for all  $x \in X$  with  $fx \in Fx$  provided that  $fx \in Y$ . The following theorem is proved via common property (E.A).

**Theorem 1.18** ([22, Theorem 3.3]). *Let  $(X, M, *)$  be fuzzy metric space. If  $f, g : X \rightarrow X$  and  $F, G : X \rightarrow CB(X)$  are mappings which satisfy*

- (a)  $(f, F)$  and  $(g, G)$  satisfy the property (E.A);

(b) for all  $x \neq y$  in  $X$ ,

$$M(Fx, Gy, t) > \psi \{ \min \{ M(fx, gy, t), M(fx, Fx, t), M(gy, Gy, t), \\ M(fx, Gy, t), M(gy, Fx, t) \} \}$$

where  $\psi$  is a continuous function  $\psi : [0, 1] \rightarrow [0, 1]$  such that  $\psi$  is non increasing on  $[0, 1]$  and  $\psi(t) > t \forall t \in [0, 1]$ .

If  $f(X)$  and  $g(X)$  are closed subsets of  $X$ , then  $(f, F)$  and  $(g, G)$  have coincidence point. Moreover,  $(f, F)$  and  $(g, G)$  have fixed point provided that  $f$  is  $F$ -weakly commuting at  $v \in X$  and  $g$  is  $G$ -weakly commuting at  $w \in X$ .

## 2 Main Results

Firstly, we rewrite Definition 1.13, 1.15 and 1.17.

**Definition 2.1.** Let  $(X, M, *)$  be fuzzy metric space,  $f, g : X \rightarrow X$  and  $F, G : X \rightarrow CL(X)$ . Then the hybrid pair  $(f, F)$  is said to be  $g$ -tangential at  $u \in X$  if there exist two sequences  $\{x_n\}, \{y_n\}$  in  $X$  such that  $\lim_{n \rightarrow \infty} Gy_n \in CL(X)$  and

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gy_n = u \in A = \lim_{n \rightarrow \infty} Fx_n.$$

**Definition 2.2.** Let  $(X, M, *)$  be fuzzy metric space. If  $f, g : Y \subseteq X \rightarrow X$  and  $F, G : Y \rightarrow CL(X)$ , then the hybrid pair of mappings  $(f, F)$  is said to be  $g$ -tangential at  $u \in Y$  with respect to  $G$  if there exist two sequences  $\{x_n\}, \{y_n\}$  and  $A \in CL(X)$  in  $Y$  such that  $\lim_{n \rightarrow \infty} Gy_n \in CL(X)$  and

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gy_n = u \in A = \lim_{n \rightarrow \infty} Fx_n.$$

**Definition 2.3.** Let  $(X, M, *)$  be fuzzy metric space. A map  $f : Y \subseteq X \rightarrow X$  is said to be coincidentally idempotent w.r.t. a mapping  $F : Y \rightarrow CL(X)$  if  $f$  is idempotent at the coincidence points of  $(f, F)$ , i.e.,  $ffx = fx$  for all  $x \in X$  with  $fx \in Fx$  provided that  $fx \in Y$ .

**Remark 2.4.** If the hybrid pair of mappings  $(f, F)$  and  $(g, G)$  satisfy the common property (E.A), then  $(f, F)$  is  $g$ -tangential with respect to  $G$  whereas  $(g, G)$  is  $f$ -tangential with respect to  $F$  but the converse is not necessary true.

Let  $\Phi$  be the family of all mappings  $\phi : [0, 1]^6 \rightarrow [0, 1]$  satisfying the following properties:

- ( $\phi_1$ )  $\phi$  is non increasing in 3<sup>rd</sup>, 4<sup>th</sup>, 5<sup>th</sup>, 6<sup>th</sup>,
- ( $\phi_{21}$ ) if  $\phi(u, 1, 1, u, u, 1) \geq 0$  or
- ( $\phi_{22}$ )  $\phi(u, 1, u, 1, 1, u) \geq 0 \forall u \in [0, 1]$  implies  $u = 1$ .

**Example 2.5.** Define  $\phi : [0, 1]^6 \rightarrow [0, 1]$  as

$$\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \min \left\{ t_2, \frac{t_3 + t_4}{2}, \frac{t_5 + t_6}{2} \right\}.$$

( $\phi_1$ ) Obvious.

( $\phi_2$ ) Let  $0 \leq u \leq 1$  and  $\phi(u, 1, 1, u, u, 1) = u - \min\{1, \frac{1+u}{2}, \frac{1+u}{2}\} = u - \frac{1+u}{2} = \frac{u-1}{2} \geq 0$ . Then  $u \geq 1$  but  $u \leq 1$ . Then  $u = 1$ .

Several other examples satisfying the requirements of preceding implicit function can easily be constructed.

Now, we prove our main theorem as follows.

**Theorem 2.6.** Let  $f, g : Y \subseteq X \rightarrow X$  be two mappings from a subset  $Y$  of a fuzzy metric space  $(X, M, *)$  into  $X$  and  $F, G : Y \rightarrow CL(X)$  which satisfy the following conditions:

- (a) the hybrid pair  $(f, F)$  is  $g$ -tangential at  $u \in X$  with respect to  $G$  (or the hybrid pair  $(g, G)$  is  $f$ -tangential at  $u \in X$  with respect to  $F$ ),
- (b) there exists  $\phi \in \Phi$  such that

$$\phi(M(Fx, Gy, t), M(fx, gy, t), M(fx, Fx, t), M(gy, Gy, t), M(fx, Gy, t), M(gy, Fx, t)) \geq 0,$$

for all  $x, y \in X$ .

Then

- (1) the hybrid pair  $(f, F)$  have a coincidence point  $v \in Y$  provided that  $f(Y)$  is a closed subset of  $X$ .
- (2) the hybrid pair  $(g, G)$  have a coincidence point  $w \in Y$  provided that  $g(Y)$  is a closed subset of  $X$ .
- (3) the hybrid pair  $(f, F)$  have a common fixed point provided that  $f$  is  $F$ -weakly commuting at  $v \in X$ ,  $ffv = fv$  and  $fv \in Y$ .
- (4) the hybrid pair  $(g, G)$  have a common fixed point provided that  $g$  is  $G$ -weakly commuting at  $w \in Y$ ,  $ggw = gw$  and  $gw \in Y$ .
- (5)  $f, g, F, G$  have a common fixed point provided that both (3) and (4) are true.

*Proof.* Since the hybrid pair  $(f, F)$  is  $g$ -tangential at  $u \in Y$  with respect to  $G$ , there exist two sequences  $\{x_n\}, \{y_n\}$  in  $Y$  and  $A, B \in CL(X)$  such that  $\lim_{n \rightarrow \infty} Gy_n = B$  and

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gy_n = u \in A = \lim_{n \rightarrow \infty} Fx_n.$$

Now, we proceed to show that  $A = B$ . To do this, consider

$$\phi(M(Fx_n, Gy_n, t), M(fx_n, gy_n, t), M(fx_n, Fx_n, t), M(gy_n, Gy_n, t), M(fx_n, Gy_n, t), M(gy_n, Fx_n, t)) \geq 0$$

which on letting  $n \rightarrow \infty$  gives rise

$$\phi(M(A, B, t), M(u, u, t), M(u, A, t), M(u, B, t), M(u, B, t), M(u, A, t)) \geq 0$$

so that

$$\begin{aligned} & \phi(M(A, B, t), 1, 1, M(A, B, t), M(A, B, t), 1) \\ & \geq \phi(M(A, B, t), 1, M(u, A, t), M(u, B, t), M(u, B, t), M(u, A, t)) \\ & \geq 0. \end{aligned}$$

Owing to  $(\phi_{21})$ , we have  $M(A, B, t) = 1$  so that  $A = B$ .

To prove (1), let  $f(Y)$  is closed, then there exists some  $v \in Y$  such that  $u = fv$ . Now, we show that  $A = Fv$ . To accomplish this, consider

$$\begin{aligned} & \phi(M(Fv, Gy_n, t), M(fv, gy_n, t), M(fv, Fv, t), M(gy_n, Gy_n, t), M(fv, Gy_n, t), \\ & \qquad \qquad \qquad M(gy_n, Fv, t)) \geq 0 \end{aligned}$$

which on letting  $n \rightarrow \infty$  gives rise

$$\begin{aligned} & \phi(M(Fv, A, t), M(fv, u, t), M(fv, Fv, t), M(u, A, t), M(fv, A, t), \\ & \qquad \qquad \qquad M(u, Fv, t)) \geq 0 \end{aligned}$$

so that

$$\begin{aligned} & \phi(M(Fv, A, t), 1, M(A, Fv, t), 1, 1, M(A, Fv, t)) \\ & \geq \phi(M(Fv, A, t), 1, M(u, Fv, t), M(u, A, t), M(u, A, t), M(u, Fv, t)) \\ & \geq 0. \end{aligned}$$

Owing to  $(\phi_{22})$ , this gets us  $M(A, Fv, t) = 1$  which implies  $A = Fv$ . Then  $fv \in Fv$  this proves (1). The proof of (2) is similar to that of (1). In order to prove (3), using the conditions given in (3), we have  $ffv = fv$  and  $ffv \in Ffv$  so that  $u = fu \in Fu$ . The proof of (4) is similar to that of (3) while (5) follows immediately.

In case the hybrid pair  $(g, G)$  is  $f$ -tangential at  $u \in X$  with respect to  $F$ , a proof on the lines of the preceding case can be outlined. This concludes the proof.  $\square$

Now, we furnish an example to illustrate Theorem 2.6.

**Example 2.7.** Let  $(X, M, *)$  be a fuzzy metric space where  $X = [0, 1]$ ,  $a * b = \min\{a, b\}$  for all  $a, b \in [0, 1]$  and

$$M(x, y, t) = \frac{t}{t + |x - y|}$$

for all  $t > 0, x, y \in X$ . Define  $\phi : [0, 1]^6 \rightarrow [0, 1]$  as

$$\phi\{t_1, t_2, t_3, t_4, t_5, t_6\} = t_1 - t_2$$

and define the maps  $F, G, f, g$  on  $X$  as  $Fx = [\frac{2x}{3}, 1]$ ,  $Gx = [x^2, 1]$  and  $fx = \frac{2x}{3}$ ,  $gx = x^2$  for all  $x, y \in X$ . Define two sequences  $\{x_n\} = \{\frac{1}{n}\}$ ,  $\{y_n\} = \{\frac{1}{2n}\}$ ,  $n \in \mathbb{N}$  in  $X$ . As,

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gy_n = 0 \in [0, 1] = \lim_{n \rightarrow \infty} Fx_n,$$

the hybrid pair  $(f, F)$  is  $g$ -tangential at  $0 \in X$  with respect to  $G$  besides

$$\phi\{M(Fx, Gy, t), M(fx, gy, t), M(fx, Fx, t), M(gy, Gy, t), M(fx, Gy, t), M(gy, Fx, t)\} = 0.$$

Thus, all the conditions of Theorem 2.6 are satisfied and  $0$  remains fixed under all the four involed maps.

One can derive the following corollary from Theorem 2.6 involving a hybrid pair of mappings  $(f, F)$  satisfying the property (E.A).

**Corollary 2.8.** Let  $(X, M, *)$  be fuzzy metric space. If  $f : Y \subseteq X \rightarrow X$  and  $F : Y \rightarrow CL(X)$  be a pair of hybrid mappings satisfying the following conditions:

- (a) the pair  $(f, F)$  satisfy the property (E.A),
- (b) for all  $x, y \in Y$ ,

$$M(fx, Fy, t) \geq \min \left\{ M(fx, fy, t), \frac{M(fx, Fx, t) + M(fy, Fy, t)}{2}, \frac{M(fx, Fy, t) + M(fy, Fx, t)}{2} \right\}.$$

If  $f(Y)$  is a closed subset of  $Y$ , then  $(f, F)$  have a common fixed point provided that  $f$  is  $F$ -weakly commuting at  $v \in X$  and  $ffv = fv$  for  $v \in C(f, F)$ .

**Remark 2.9.**

- (1) Theorem 2.6 is a generalization of Theorem 2.8 in [23].
- (2) Corollary 2.8 is a generalization of Theorem 3.10 in [8].

Our next theorem involves a sequence of multivalued mappings.

**Theorem 2.10.** Let  $\{F_n\}, n \in \mathbb{N}$  be a sequence of multi-valued mappings from a subset  $Y$  of a fuzzy metric space  $(X, M, *)$  into  $CL(X)$  and  $f, g : Y \rightarrow X$  which satisfy the following conditions:

- (a) either the pair  $(f, F_k)$  is  $g$ -tangential at  $u_k \in Y$  with respect to  $F_l$  (or the hybrid pair  $(g, F_l)$  is  $f$ -tangential at  $u_l \in Y$  with respect to  $F_k$  where  $k = 2n - 1$  and  $l = 2n$  for all  $n \in \mathbb{N}$ ),
- (b)  $\overline{\bigcup F_k(Y)} \subseteq g(Y)$  and  $\overline{\bigcup F_l(Y)} \subseteq f(Y)$ ,



(c) *there exists  $\phi \in \Phi$  such that*

$$\phi(M(F_kx, F_l y, t), M(fx, gy, t), M(fx, F_kx, t), M(gy, F_l y, t), M(fx, F_l y, t), M(gy, F_kx, t)) \geq 0$$

*for all  $x, y \in X$ .*

*Then*

- (1)  *$(f, F_k)$  have a coincidence point  $u_k \in Y$ ;*
- (2)  *$(g, F_l)$  have a coincidence point  $u_l \in Y$ ;*
- (3)  *$(f, F_k)$  have a common fixed point provided that  $f$  is  $F_k$  -weakly commuting at  $u_k$  and  $f$  is coincidentally idempotent w.r.t.  $F_k$ ;*
- (4)  *$(g, F_l)$  have a common fixed point provided that  $g$  is  $F_l$  -weakly commuting at  $u_l$  and  $g$  is coincidentally idempotent w.r.t.  $F_l$ .*

*Proof.* Since the hybrid pair  $(f, F_k)$  is  $g$ -tangential at  $u_k \in Y$  with respect to  $F_l$ , there exist two sequences  $\{x_{kn}\}, \{y_{kn}\}$  in  $Y$  and  $A_k, B_k \in CL(X)$  such that  $\lim_{n \rightarrow \infty} F_l y_{kn} = B_k$  and

$$\lim_{n \rightarrow \infty} f x_{kn} = \lim_{n \rightarrow \infty} g y_{kn} = u_k \in A_k = \lim_{n \rightarrow \infty} F_k x_{kn}.$$

Now, we show that  $A_k = B_k$ . As

$$\phi(M(F_k x_{kn}, F_l y_{kn}, t), M(f x_{kn}, g y_{kn}, t), M(f x_{kn}, F_k x_{kn}, t), M(g y_{kn}, F_l y_{kn}, t), M(f x_{kn}, F_l y_{kn}, t), M(g y_{kn}, F_k x_{kn}, t)) \geq 0$$

which on making  $n \rightarrow \infty$  gives rise

$$\phi(M(A_k, B_k, t), 1, 1, M(u_k, B_k, t), M(u_k, B_k, t), 1) \geq 0$$

so that

$$\begin{aligned} &\phi(M(A_k, B_k, t), 1, 1, M(A_k, B_k, t), M(A_k, B_k, t), 1) \\ &\geq \phi(M(A_k, B_k, t), 1, M(u_k, A_k, t), M(u_k, B_k, t), M(u_k, B_k, t), M(u_k, A_k, t)) \\ &\geq 0. \end{aligned}$$

Owing to  $(\phi_{21})$ , we have  $M(A_k, B_k, t) = 1$  yielding thereby  $A_k = B_k$ .

As  $u_k \in \bigcup F_l(Y)$  and  $\bigcup F_l(Y) \subset f(Y)$ , there exist  $z_k \in Y$  such that  $u_k = f z_k$ . Now, we show that  $F_k z_k = A_k$ . As

$$\phi(M(F_k z_k, F_l y_{kn}, t), M(f z_k, g y_{kn}, t), M(f z_k, F_k z_k, t), M(g y_{kn}, F_l y_{kn}, t), M(f z_k, F_l y_{kn}, t), M(g y_{kn}, F_k z_k, t)) \geq 0$$

which on making  $n \rightarrow \infty$  reduces to

$$\phi(M(F_k z_k, A_k, t), 1, 1, M(u_k, A_k, t), M(u_k, A_k, t), 1) \geq 0$$

so that  $F_k z_k = A_k$  which proves (1).

The remaining parts are easy to prove. This concludes the proof. □

**Remark 2.11.** *Theorem 2.10 is a generalization of Theorem 2 in [25].*

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