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Common Fixed Points via Weakly $(\psi, \mathcal{S}, \mathcal{C})$-Contraction Mappings on Ordered Metric Spaces and Application to Integral Equations

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Abstract : In this paper, two new concept of weakly $(\psi, \mathcal{S}, \mathcal{C})$-contractive mappings and generalized weakly $(\psi, \mathcal{S}, \mathcal{C})$-contractive mappings for a pair of maps in ordered metric space are designed and then establish common fixed point results on ordered complete metric spaces. To demonstrate our results, an example is given. At the same time as applications of the presented theorems, we get hold of common fixed point results for generalized contraction of integral type and we prove an existence theorem for solutions of a system of integral equations.

Keywords : partially ordered set; asymptotically regular map; orbitally complete metric space; orbital continuity; weakly increasing maps; weakly C-contractive map.
2010 Mathematics Subject Classification : 47H10; 54H25.

1 Introduction and Preliminaries

A multitude of generalizations of the Classical Banach Contraction Principle [1] are available in the existing literature of metric fixed point theory. The majority of these generalizations are obtained by improving the underlying contraction condition (e.g. [2]). In this connection, Chatterjea [3] introduced the notion of $\mathcal{C}$-contraction as:

Definition 1.1. A mapping $\mathcal{T}:\mathcal{X}\to\mathcal{X}$ where $(\mathcal{X},d)$ is a metric space is said to be a $\mathcal{C}$-contraction if there exists $\alpha\in(0,\frac{1}{2})$ such that for all $x,y\in\mathcal{X}$ the following inequality holds:

$$d(\mathcal{T}x,\mathcal{T}y)\le\alpha(d(x,\mathcal{T}y)+d(y,\mathcal{T}x)).$$

Choudhury [4] generalized $\mathcal{C}$-contraction and introduced a notion of weakly $\mathcal{C}$-contraction.

Definition 1.2. A mapping $\mathcal{T}:\mathcal{X}\to\mathcal{X}$, where $(\mathcal{X},d)$ is a metric space is said to be weakly $\mathcal{C}$-contractive (or a weak $\mathcal{C}$-contraction) if for all $x,y\in\mathcal{X}$,

$$d(\mathcal{T}x,\mathcal{T}y)\le\frac{1}{2}[d(x,\mathcal{T}y)+d(y,\mathcal{T}x)]-\varphi(d(x,\mathcal{T}y),d(y,\mathcal{T}x)),$$

where $\varphi:[0,+\infty)^2\to[0,+\infty)$ is a continuous function such that $\varphi(x,y)=0$ if and only if $x=y=0$.

Generalization of the above Banach contraction principle has been a heavily investigated branch research. (see, e.g., [2]).

Browder and Petryshyn introduced the concept of asymptotic regularity of a self-map at a point in a metric space.

Definition 1.3 ([5])**.** A self-map $\mathcal{T}$ on a metric space $(\mathcal{X},d)$ is said to be asymptotically regular at a point $x\in\mathcal{X}$ if $\lim_{n\to\infty}d(\mathcal{T}^nx,\mathcal{T}^{n+1}x)=0$.

Recall that the set $\mathcal{O}(x_0;\mathcal{T})=\{\mathcal{T}^nx_0:n=0,1,2,\dots\}$ is called the orbit of the self-map $\mathcal{T}$ at the point $x_0\in\mathcal{X}$.

Definition 1.4 ([6])**.** A metric space $(\mathcal{X},d)$ is said to be $\mathcal{T}$-orbitally complete if every Cauchy sequence contained in $\mathcal{O}(x;\mathcal{T})$ (for some x in $\mathcal{X}$) converges in $\mathcal{X}$.

Here, it can be pointed out that every complete metric space is $\mathcal{T}$-orbitally complete for any $\mathcal{T}$, but a $\mathcal{T}$-orbitally complete metric space need not be complete.

Definition 1.5 ([5])**.** A self-map $\mathcal{T}$ defined on a metric space $(\mathcal{X},d)$ is said to be orbitally continuous at a point z in $\mathcal{X}$ if for any sequence $\{x_n\}\subset\mathcal{O}(x;\mathcal{T})$ (for some $x\in\mathcal{X}$), $x_n\to z$ as $n\to\infty$ implies $\mathcal{T}x_n\to\mathcal{T}z$ as $n\to\infty$.

Clearly, every continuous self-mapping of a metric space is orbitally continuous, but not conversely.

Sastry et al. [7] extended the above concepts to two and three mappings and employed them to prove common fixed point results for commuting mappings. In what follows, we collect such definitions for two maps.

Definition 1.6. Let $\mathcal{S},\mathcal{T}$ be three self-mappings defined on a metric space $(\mathcal{X},d)$.

1. If for a point $x_0 \in \mathcal{X}$, there exits a sequence $\{x_n\}$ in $\mathcal{X}$ such that $x_{2n+1} = \mathcal{S}x_{2n}$, $x_{2n+2} = \mathcal{T}x_{2n+1}$, $n = 0, 1, 2, \dots$, then the set $\mathcal{O}(x_0; \mathcal{S}, \mathcal{T}) = \{x_n : n = 1, 2, \dots\}$ is called the orbit of $(\mathcal{S}, \mathcal{T})$ at x_0.
2. The space $(\mathcal{X}, d)$ is said to be $(\mathcal{S}, \mathcal{T})$-orbitally complete at x_0 if every Cauchy sequence in $\mathcal{O}(x_0; \mathcal{S}, \mathcal{T})$ converges in $\mathcal{X}$.
3. The map $\mathcal{T}$ is said to be orbitally continuous at x_0 if it is continuous on $\mathcal{O}(x_0; \mathcal{T})$.
4. The pair $(\mathcal{S}, \mathcal{T})$ is said to be asymptotically regular (in short a.r.) at x_0 if there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that $x_{2n+1} = \mathcal{S}x_{2n}$, $x_{2n+2} = \mathcal{T}x_{2n+1}$, $n = 0, 1, 2, \dots$, and $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$.

Existence of fixed point in ordered metric spaces was first investigated by Ran and Reurings [8] who presented its applications to matrix equations. Subsequently, Nieto and López [9] extended this result for nondecreasing mappings and applied it to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions. Thereafter, several authors obtained many fixed point theorems in ordered metric spaces. For more details see [10–37] and the references cited therein.

In this paper we generalize the results of Harjani and Sadarangani [21, 38] (and, hence, some other related common fixed point results) in two directions. First we introduce the notion of weakly $(\psi, \mathcal{S}, \mathcal{C})$-contractive condition in metric space and then the existence and (under additional assumptions) uniqueness of their common fixed point where mapping $\mathcal{S}$ is $\mathcal{T}$-strictly weakly isotone increasing is obtained. Further, we introduce the notion of generalized weakly $(\psi, \mathcal{S}, \mathcal{C})$-contractive condition in metric space and then establish the existence and uniqueness of their common fixed point where mapping $\mathcal{S}$ is $\mathcal{T}$-strictly weakly isotone increasing, a pair $(\mathcal{S}, \mathcal{T})$ is asymptotically regular in orbitally complete metric space. We furnish suitable example to demonstrate the validity of the hypotheses of our results. At the end, as applications of the presented theorems, we get hold of common fixed point results for generalized contraction of integral type and we prove an existence theorem for solutions of a system of integral equations.

2 Notation and Definitions

First, we introduce some further notation and definitions that will be used later.

If $(\mathcal{X}, \preceq)$ is a partially ordered set then $x, y \in \mathcal{X}$ are called comparable if $x \preceq y$ or $y \preceq x$ holds. A subset $\mathcal{K}$ of $\mathcal{X}$ is said to be totally ordered if every two elements of $\mathcal{K}$ are comparable. If $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is such that, for $x, y \in \mathcal{X}$, $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$, then the mapping $\mathcal{T}$ is said to be non-decreasing.

Definition 2.1. Let $(\mathcal{X}, \preceq)$ be a partially ordered set and $\mathcal{S}, \mathcal{T} : \mathcal{X} \to \mathcal{X}$.

1. [39, 40] The pair $(\mathcal{S}, \mathcal{T})$ is called weakly increasing if $\mathcal{S}x \preceq \mathcal{T}\mathcal{S}x$ and $\mathcal{T}x \preceq \mathcal{S}\mathcal{T}x$ for all $x \in \mathcal{X}$.

2. [38–40] The mapping $\mathcal{S}$ is said to be $\mathcal{T}$-weakly isotone increasing if for all $x \in \mathcal{X}$ we have $\mathcal{S}x \preceq \mathcal{T}\mathcal{S}x \preceq \mathcal{S}\mathcal{T}\mathcal{S}x$.
3. [42] The mapping $\mathcal{S}$ is said to be $\mathcal{T}$-strictly weakly isotone increasing if, for all $x \in \mathcal{X}$ such that $x \prec \mathcal{S}x$, we have $\mathcal{S}x \prec \mathcal{T}\mathcal{S}x \prec \mathcal{S}\mathcal{T}\mathcal{S}x$.

Remark 2.2.

(1) None of two weakly increasing mappings need be non-decreasing. There exist some examples to illustrate this fact in [43].

(2) If $\mathcal{S}, \mathcal{T} : \mathcal{X} \to \mathcal{X}$ are weakly increasing, then $\mathcal{S}$ is $\mathcal{T}$-weakly isotone increasing and hence $\mathcal{T}$-strictly weakly isotone increasing.

(3) $\mathcal{S}$ can be $\mathcal{T}$-strictly weakly isotone increasing, while some of these two mappings can be not strictly increasing (see the following example).

Example 2.3. Let $\mathcal{X} = [0, +\infty)$ be endowed with the usual ordering and define $\mathcal{S}, \mathcal{T} : \mathcal{X} \to \mathcal{X}$ as

$$\mathcal{S}x = \begin{cases} 2x, & \text{if } x \in [0,1], \\ 3x, & \text{if } x > 1; \end{cases} \qquad \mathcal{T}x = \begin{cases} 2, & \text{if } x \in [0,1], \\ 2x, & \text{if } x > 1. \end{cases}$$

Clearly, we have $x \prec \mathcal{S}x \prec \mathcal{T}\mathcal{S}x \prec \mathcal{S}\mathcal{T}\mathcal{S}x$ for all $x \in \mathcal{X}$, and so, $\mathcal{S}$ is $\mathcal{T}$-strictly weakly isotone increasing; $\mathcal{T}$ is not strictly increasing.

Definition 2.4. Let $\mathcal{X}$ be a nonempty set. Then $(\mathcal{X}, d, \preceq)$ is called an ordered metric space if

(i) $(\mathcal{X}, d)$ is a metric space,

(ii) $(\mathcal{X}, \preceq)$ is a partially ordered set.

The space $(\mathcal{X}, d, \preceq)$ is called regular if the following hypothesis holds: if $\{z_n\}$ is a non-decreasing sequence in $\mathcal{X}$ with respect to $\preceq$ such that $z_n \to z \in \mathcal{X}$ as $n \to \infty$, then $z_n \preceq z$.

3 Common Fixed Points Theorems For Weakly $(\psi, \mathcal{S}, \mathcal{C})$-Contraction Mappings

We will prove some fixed point theorems for self-mappings defined on a ordered complete metric space and satisfying certain weakly $(\psi, \mathcal{S}, \mathcal{C})$-contraction mappings. To achieve our goal, as in [44], we fixed the set of functions and denote by

1. Ψ_1 the class of functions $\psi : [0, +\infty) \to [0, +\infty)$ a strictly increasing, continuous function with $\psi(t) \leq \frac{1}{2}t$ for all $t > 0$ and $\psi(0) = 0$;

2. Φ_1 the class of functions $\varphi : [0, +\infty)^2 \to [0, +\infty)$ a strictly decreasing, continuous in each coordinate such that $\varphi(x, y) = 0$ if and only if $x = y = 0$ and $\varphi(x, y) \leq x + y$ for all $x, y \in [0, +\infty)$.

Definition 3.1. Let $(\mathcal{X}, d, \preceq)$ be a ordered metric space. Two mappings $\mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$ are called a weakly $(\psi, \mathcal{S}, \mathcal{C})$-contraction if

$$d(\mathcal{T}x, \mathcal{S}y) \quad \leq \quad \psi([d(x, \mathcal{S}y) + d(y, \mathcal{T}x)] - \varphi(d(x, \mathcal{S}y), d(y, \mathcal{T}x))),\ x \succeq y \quad (3.1)$$

for any $x, y \in \mathcal{X}$, $\psi \in \Psi_1$ and $\varphi \in \Phi_1$.

It is note that the weakly $(\psi, \mathcal{S}, \mathcal{C})$-contractions constitute a strictly larger class of mappings than weakly $\mathcal{C}$-contractions.

Now, we state and prove our first result.

Theorem 3.2. *Let $(\mathcal{X}, \preceq)$ be a partially ordered set and suppose that there exists a metric d in $\mathcal{X}$ such that $(\mathcal{X}, d)$ is a complete metric space. Suppose $\mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$ are two mappings satisfying weakly $(\psi, \mathcal{S}, \mathcal{C})$-contractions conditions for all comparable $x, y \in \mathcal{X}$.*

We assume the following hypotheses:

(i) $\mathcal{S}$ is $\mathcal{T}$-strictly weakly isotone increasing;

(ii) there exists an $x_0 \in \mathcal{X}$ such that $x_0 \prec \mathcal{S}x_0$;

(iii) $\mathcal{S}$ or $\mathcal{T}$ is continuous at x_0;

Then $\mathcal{S}$ and $\mathcal{T}$ have a common fixed point. Moreover, the set of common fixed points of $\mathcal{S}, \mathcal{T}$ is totally ordered if and only if $\mathcal{S}$ and $\mathcal{T}$ have one and only one common fixed point.

Proof. First of all we show that, if $\mathcal{S}$ or $\mathcal{T}$ has a fixed point, then it is a common fixed point of $\mathcal{S}$ and $\mathcal{T}$. Indeed, let z be a fixed point of $\mathcal{S}$. Now assume $d(z, \mathcal{T}z) > 0$. Put $x = y = z$ in the (3.1) condition, we have

$$\begin{aligned} d(\mathcal{T}z, z) = d(\mathcal{T}z, \mathcal{S}z)) &\leq \psi([d(z, \mathcal{S}z) + d(z, \mathcal{T}z)] - \varphi(d(x, \mathcal{S}z), d(z, \mathcal{T}z))) \\ &= d(z, \mathcal{T}z) - \frac{1}{2}\varphi(0, d(z, \mathcal{T}z)) \end{aligned}$$

wherefrom $\varphi(0, d(z, \mathcal{T}z)) \leq 0$, which is a contradiction. Thus by the property of φ, we have $d(z, \mathcal{T}z) = 0$ and so z is a common fixed point of $\mathcal{S}$ and $\mathcal{T}$. Analogously, one can observe that if z is a fixed point of $\mathcal{T}$, then it is a common fixed point of $\mathcal{S}$ and $\mathcal{T}$.

Let x_0 be such that $x_0 \prec \mathcal{S}x_0$. We can define a sequence $\{x_n\}$ in $\mathcal{X}$ as follows:

$$x_{2n+1} = \mathcal{S}x_{2n} \text{ and } x_{2n+2} = \mathcal{T}x_{2n+1} \text{ for } n \in \{0, 1, \ldots\}. \quad (3.2)$$

Since $\mathcal{S}$ is $\mathcal{T}$-strictly weakly isotone increasing, we have

$$\begin{aligned} x_1 = \mathcal{S}x_0 \prec \mathcal{T}\mathcal{S}x_0 = \mathcal{T}x_1 = x_2 \prec \mathcal{S}\mathcal{T}\mathcal{S}x_0 = \mathcal{S}\mathcal{T}x_1 = \mathcal{S}x_2 = x_3, \\ x_3 = \mathcal{S}x_2 \prec \mathcal{T}\mathcal{S}x_2 = \mathcal{T}x_3 = x_4 \prec \mathcal{S}\mathcal{T}\mathcal{S}x_2 = \mathcal{S}\mathcal{T}x_3 = \mathcal{S}x_4 = x_5, \end{aligned}$$

and continuing this process we get

$$x_1 \prec x_2 \prec \cdots \prec x_n \prec x_{n+1} \prec \cdots . \tag{3.3}$$

Now we claim that for all $n \in \mathbb{N}$, we have

$$d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1}). \tag{3.4}$$

From (3.3) we have that $x_n \prec x_{n+1}$ for all $n \in \mathbb{N}$. Then from (3.1) with $x = x_{2n-1}$ and $y = x_{2n}$, we get

$$\begin{aligned} &d(x_{2n}, x_{2n+1}) \\ &\quad = d(\mathcal{T}x_{2n-1}, \mathcal{S}x_{2n}) \\ &\quad \leq \psi([d(x_{2n-1}, \mathcal{S}x_{2n}) + d(x_{2n}, \mathcal{T}x_{2n-1})] - \varphi(d(x_{2n-1}, \mathcal{S}x_{2n}), d(x_{2n}, \mathcal{T}x_{2n-1}))) \\ &\quad = \psi([d(x_{2n-1}, x_{2n+1}) + d(x_{2n}, x_{2n})] - \varphi(d(x_{2n-1}, x_{2n+1}), 0)) \\ &\quad \leq \psi(d(x_{2n-1}, x_{2n+1}) - \varphi(d(x_{2n-1}, x_{2n+1}), 0)) \\ &\quad \leq \psi(d(x_{2n-1}, x_{2n+1})) \\ &\quad \leq \psi([d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n+1})]) \\ &\quad \leq \frac{1}{2}[d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n+1})]. \end{aligned} \tag{3.5}$$

Therefore,

$$d(x_{2n}, x_{2n+1}) \leq d(x_{2n-1}, x_{2n}) \text{ for any } n \in \mathbb{N}.$$

Similarly, we can prove that $d(x_{2n-1}, x_{2n}) \leq d(x_{2n-2}, x_{2n-1})$ for all $n \geq 1$. Therefore, we conclude that (3.4) holds.

Thus $\{d(x_n, x_{n+1})\}$ is a nondecreasing sequence of nonnegative real numbers. Consequently, there exists $\gamma \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \gamma. \tag{3.6}$$

Next, we prove that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0.$$

Passing to the limit as $n \to \infty$ in (3.5) we have

$$\gamma \leq \lim_{n\to\infty} \frac{1}{2} d(x_{n-1}, x_{n+1})) \leq \frac{1}{2}(\gamma + \gamma),$$

or

$$\lim_{n\to\infty} d(x_{n-1}, x_{n+1}) = 2\gamma \tag{3.7}$$

Passing to the limit as $n \to \infty$ in (3.5) and using (3.6), (3.7) and the continuity of ψ and φ, we have

$$\gamma \leq \psi(2\gamma - \varphi(2\gamma, 0)) \leq \gamma - \frac{1}{2}\varphi(2\gamma, 0) \leq \gamma.$$

Hence, we have $\varphi(2\gamma, 0) = 0$, that is, $\gamma = 0$, that is,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.8}$$

Now we prove that $\{x_n\}$ is a Cauchy sequence. To this end, it is sufficient to verify that $\{x_{2n}\}$ is a Cauchy sequence. Suppose, on the contrary, that $\{x_{2n}\}$ is not a Cauchy sequence. Then there is an $\varepsilon > 0$ such that for an integer $2k$ there exist integers $2m(k) > 2n(k) > 2k$ such that

$$d(x_{2n(k)}, x_{2m(k)}) > \varepsilon. \tag{3.9}$$

For every integer $2k$, let $2m(k)$ be the least positive integer exceeding $2n(k)$ satisfying (3.9) and such that

$$d(x_{2n(k)}, x_{2m(k)-1}) \le \varepsilon. \tag{3.10}$$

Now

$$\varepsilon < d(x_{2n(k)}, x_{2m(k)}) \le d(x_{2n(k)}, x_{2m(k)-1}) + d(x_{2m(k)-1}, x_{2m(k)}).$$

Then by (3.9) and (3.10) it follows that

$$\lim_{k\to\infty} d(x_{2n(k)}, x_{2m(k)}) = \varepsilon. \tag{3.11}$$

Also, by the triangle inequality, we have

$$\left| d(x_{2n(k)}, x_{2m(k)-1}) - d(x_{2n(k)}, x_{2m(k)}) \right| \le d(x_{2m(k)-1}, x_{2m(k)}).$$

By using (3.11) we get

$$\lim_{k\to\infty} d(x_{2n(k)}, x_{2m(k)-1}) = \varepsilon. \tag{3.12}$$

Now by (3.1) we get

$$\begin{aligned}
&d(x_{2n(k)}, x_{2m(k)}) \\
&\quad \le d(x_{2n(k)}, x_{2n(k)+1}) + d(x_{2n(k)+1}, x_{2m(k)}) \\
&\quad \le d(x_{2n(k)}, x_{2n(k)+1}) + d(\mathcal{T}x_{2n(k)}, \mathcal{S}x_{2m(k)-1}) \\
&\quad \le d(x_{2n(k)}, x_{2n(k)+1}) + \psi([d(x_{2m(k)-1}, \mathcal{S}x_{2n(k)}) + d(x_{2n(k)}, \mathcal{T}x_{2m(k)-1})] \\
&\qquad - \varphi(d(x_{2m(k)-1}, \mathcal{S}x_{2n(k)}), d(x_{2n(k)}, \mathcal{T}x_{2m(k)-1}))) \\
&\quad = d(x_{2n(k)}, x_{2n(k)+1}) + \psi([d(x_{2m(k)-1}, x_{2n(k)+1}) + d(x_{2n(k)}, x_{2m(k)})]) \\
&\qquad - \varphi(d(x_{2m(k)-1}, x_{2n(k)+1}), d(x_{2n(k)}, x_{2m(k)})).
\end{aligned} \tag{3.13}$$

Taking into account (3.8) and (3.11) and the continuity of ψ and φ, passing to the limit as $n \to \infty$ in the last inequality, we obtain

$$\varepsilon \le \psi([\varepsilon + 0] - \varphi(\varepsilon, 0)) \le \frac{1}{2}\varepsilon$$

and from the last inequality, $\varphi(\varepsilon, 0) \leq 0$. Therefore $\varphi(\varepsilon, 0) = 0$. From the fact that $\varphi(x, y) = 0 \Leftrightarrow x = y = 0$, we have $\varepsilon = 0$, a contradiction. Thus, assumption (3.9) is wrong. Therefore, $\{x_n\}$ is a Cauchy sequence.

From the completeness of $\mathcal{X}$ there exists $z \in \mathcal{X}$ such that $x_n \to z$ as $n \to \infty$. If $\mathcal{T}$ or $\mathcal{S}$ is continuous then it is clear that $\mathcal{T}z = z$ or $\mathcal{S}z = z$. Thus, it is immediate to conclude that $\mathcal{T}$ and $\mathcal{S}$ have a common fixed point.

Now, suppose that the set of common fixed points of $\mathcal{T}$ and $\mathcal{S}$ is totally ordered. We claim that there is a unique common fixed point of $\mathcal{T}$ and $\mathcal{S}$. Assume to the contrary that $\mathcal{S}u = \mathcal{T}u = u$ and $\mathcal{S}v = \mathcal{T}v = v$ but $u \neq v$. By supposition, we can replace x by u and y by v in (3.1) and the property of ψ, we obtain

$$\begin{aligned} d(u, v) = d(\mathcal{T}u, \mathcal{S}v) &\leq \psi([d(u, \mathcal{S}v) + d(v, \mathcal{T}u)] - \varphi(d(u, \mathcal{S}v), d(v, \mathcal{T}u))) \\ &= \psi(2d(u, v) - \varphi(d(u, v), d(v, u))), \end{aligned}$$

that is,

$$d(u, v) \leq d(u, v) - \frac{1}{2}\varphi(d(u, v), d(v, u)) \leq d(u, v).$$

This gives us $\varphi(d(u, v), d(v, u)) = 0$, and, by definition of φ, $d(u, v) = 0$, that is, $u = v$. This finishes the proof. □

Now, we are also able to prove the existence of a common fixed point of two mappings without using the continuity of $\mathcal{S}$ or $\mathcal{T}$. More precisely, we have the following theorem.

Theorem 3.3. *Let* $(\mathcal{X}, d, \preceq)$ *and* $\mathcal{S}, \mathcal{T} : \mathcal{X} \to \mathcal{X}$ *satisfy all the conditions of Theorem 3.2, except that condition* (iii) *is substituted by*

(iii') $\mathcal{X}$ *is regular.*

Then the same conclusions as in Theorem 3.2 hold.

Proof. Following the proof of Theorem 3.2, we have that $\{x_n\}$ is a Cauchy sequence in $(\mathcal{X}, d)$ which is orbitally complete at x_0. Then, there exists $z \in \mathcal{X}$ such that

$$\lim_{n \to \infty} x_n = z.$$

Now suppose that $d(z, \mathcal{S}z) > 0$. From regularity of $\mathcal{X}$, we have $x_{2n} \preceq z$ for all $n \in \mathbb{N}$. Hence, we can apply the considered contractive condition. Then, setting $x = x_{2n}$ and $y = z$ in (3.1), we obtain:

$$\begin{aligned} d(x_{2n+2}, \mathcal{S}z) &= d(\mathcal{T}x_{2n+1}, \mathcal{S}z) \\ &\leq \psi(d(x_{2n+1}, \mathcal{S}z) + d(z, \mathcal{T}x_{2n+1})] - \varphi(d(x_{2n+1}, \mathcal{S}z), d(z, \mathcal{T}x_{2n+1}))) \\ &= \psi(d(x_{2n+1}, \mathcal{S}z) + d(z, x_{2n+2})] - \varphi(d(x_{2n+1}, \mathcal{S}z), d(z, x_{2n+z}))). \end{aligned}$$

Passing to the limit as $n \to \infty$ and using $x_n \to z$, continuities of ψ and φ, we have

$$d(z, \mathcal{S}z) \leq \frac{1}{2}d(z, \mathcal{S}z) - \frac{1}{2}\varphi(d(z, \mathcal{S}z), 0) \leq \frac{1}{2}d(z, \mathcal{S}z)$$

a contradiction. Therefore $d(z, \mathcal{S}z) = 0$ and thus $z = \mathcal{S}z$. Analogously, for $x = z$ and $y = x_{2n}$, one can prove that $\mathcal{T}z = z$. It follows that $z = \mathcal{S}z = \mathcal{T}z$, that is, $\mathcal{T}$ and $\mathcal{S}$ have a common fixed point. □

Corollary 3.4. *Let $(\mathcal{X}, \preceq)$ be a partially ordered set and suppose that there exists a metric d in $\mathcal{X}$ such that $(\mathcal{X}, d)$ is a complete metric space. Suppose $\mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$ are two mappings satisfying weakly $(\psi, \mathcal{S}, \mathcal{C})$-contractive conditions for all comparable $x, y \in \mathcal{X}$.*

We assume the following hypotheses:

(i') $\mathcal{S}$ and $\mathcal{T}$ are weakly increasing;

(iii') $\mathcal{X}$ is regular.

Then $\mathcal{S}$ and $\mathcal{T}$ have a common fixed point. Moreover, the set of common fixed points of $\mathcal{S}, \mathcal{T}$ is totally ordered if and only if $\mathcal{S}$ and $\mathcal{T}$ have one and only one common fixed point.

4 Common Fixed Points for Generalized Weakly $(\psi, \mathcal{S}, \mathcal{C})$-Contraction Mappings

In this section, we prove results for generalized weakly $(\psi, \mathcal{S}, \mathcal{C})$-contraction in ordered metric space. To complete the results, we need the following notion of a generalized weakly $(\psi, \mathcal{S}, \mathcal{C})$-contraction.

For convenience, we denote by

1. Ψ_2 the class of functions $\psi : [0, +\infty) \to [0, +\infty)$ a strictly increasing, continuous function with $\psi(t) \leq \frac{1}{4}t$ for all $t > 0$ and $\psi(0) = 0$;
2. Φ_2 the class of functions $\varphi : [0, +\infty)^4 \to [0, +\infty)$ a strictly decreasing, lower semi-continuous in each coordinate such that $\varphi(x, y, z, t) = 0$ if and only if $x = y = z = t = 0$ and $\varphi(x, y, z, t) \leq x + y + z + t$ for all $x, y, z, t \in [0, +\infty)$.

Definition 4.1. Let $(\mathcal{X}, d, \preceq)$ be a ordered metric space. Two mappings $\mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$ are called a generalized weakly $(\psi, \mathcal{S}, \mathcal{C})$-contraction if

$$d(\mathcal{T}x, \mathcal{S}y) \leq \psi([d(x, \mathcal{T}x) + d(y, \mathcal{S}y) + d(x, \mathcal{S}y) + d(y, \mathcal{T}x)] - \varphi(d(x, \mathcal{T}x), d(y, \mathcal{S}y), d(x, \mathcal{S}y), d(y, \mathcal{T}x))), \ for \ x \succeq y \tag{4.1}$$

for any $x, y \in \mathcal{X}$, $\varphi \in \Phi_2$ and $\psi \in \Psi_2$.

It is note that the generalized weakly $(\psi, \mathcal{S}, \mathcal{C})$-contractions constitute a strictly larger class of mappings than weakly $\mathcal{C}$-contractions.

Now, we state and prove our first result.

Theorem 4.2. *Let $(\mathcal{X}, \preceq)$ be a partially ordered set and suppose that there exists a metric d in $\mathcal{X}$ such that $(\mathcal{X}, d)$ is a complete metric space. Suppose $\mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$ are two mappings satisfying generalized weakly $(\psi, \mathcal{S}, \mathcal{C})$-contractions conditions for all comparable $x, y \in \mathcal{X}$.*

We assume the following hypotheses:

(i) $(\mathcal{S},\mathcal{T})$ *is a.r. at* $x_0 \in \mathcal{X}$*;*

(ii) $\mathcal{X}$ *is* $(\mathcal{S},\mathcal{T})$*-orbitally complete at* x_0*;*

(iii) $\mathcal{S}$ *is* $\mathcal{T}$*-strictly weakly isotone increasing;*

(iv) there exists an $x_0 \in \mathcal{X}$ *such that* $x_0 \prec \mathcal{S}x_0$*;*

(v) $\mathcal{S}$ *or* $\mathcal{T}$ *is orbitally continuous at* x_0*.*

Then $\mathcal{S}$ *and* $\mathcal{T}$ *have a common fixed point. Moreover, the set of common fixed points of* $\mathcal{S},\mathcal{T}$ *is totally ordered if and only if* $\mathcal{S}$ *and* $\mathcal{T}$ *have one and only one common fixed point.*

Proof. First of all we show that, if $\mathcal{S}$ or $\mathcal{T}$ has a fixed point, then it is a common fixed point of $\mathcal{S}$ and $\mathcal{T}$. Indeed, let z be a fixed point of $\mathcal{S}$. Now assume $d(z,\mathcal{T}z) > 0$. If we use the inequality (4.1), for $x = y = z$, we have

$$\begin{aligned} d(\mathcal{T}z,z) = d(\mathcal{T}z,\mathcal{S}z)) &\le \psi([d(z,\mathcal{T}z)+d(z,\mathcal{S}z)+d(z,\mathcal{S}z)+d(z,\mathcal{T}z)] \\ &\qquad - \varphi(d(z,\mathcal{T}z),d(z,\mathcal{S}z),d(z,\mathcal{S}z),d(z,\mathcal{T}z))) \\ &= \frac{1}{2}d(z,\mathcal{T}z) - \frac{1}{4}\varphi(d(z,\mathcal{T}z),0,0,d(z,\mathcal{T}z)) \end{aligned}$$

wherefrom $\varphi(d(z,\mathcal{T}z),0,0,d(z,\mathcal{T}z)) \le 0$, which is a contradiction. Thus by the property of φ, we have $d(z,\mathcal{T}z) = 0$ and so z is a common fixed point of $\mathcal{S}$ and $\mathcal{T}$. Analogously, one can observe that if z is a fixed point of $\mathcal{T}$, then it is a common fixed point of $\mathcal{S}$ and $\mathcal{T}$.

Since $(\mathcal{T},\mathcal{S})$ is a.r. at x_0 in $\mathcal{X}$, there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that

$$x_{2n+1} = \mathcal{S}x_{2n} \text{ and } x_{2n+2} = \mathcal{T}x_{2n+1} \text{ for } n \in \{0,1,\ldots\} \tag{4.2}$$

and

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \tag{4.3}$$

If $x_{n_0} = \mathcal{S}x_{n_0}$ or $x_{n_0} = \mathcal{T}x_{n_0}$ for some n_0, then the proof is finished. So assume $x_n \ne x_{n+1}$ for all n.

Since $\mathcal{S}$ is $\mathcal{T}$-strictly weakly isotone increasing, we have

$$\begin{aligned} x_1 = \mathcal{S}x_0 \prec \mathcal{T}\mathcal{S}x_0 = \mathcal{T}x_1 = x_2 \prec \mathcal{S}\mathcal{T}\mathcal{S}x_0 = \mathcal{S}\mathcal{T}x_1 = \mathcal{S}x_2 = x_3, \\ x_3 = \mathcal{S}x_2 \prec \mathcal{T}\mathcal{S}x_2 = \mathcal{T}x_3 = x_4 \prec \mathcal{S}\mathcal{T}\mathcal{S}x_2 = \mathcal{S}\mathcal{T}x_3 = \mathcal{S}x_4 = x_5, \end{aligned}$$

and continuing this process we get

$$x_1 \prec x_2 \prec \cdots \prec x_n \prec x_{n+1} \prec \cdots. \tag{4.4}$$

Now we prove that $\{x_n\}$ is a Cauchy sequence in the metric space $\mathcal{O}(x_0;\mathcal{S},\mathcal{T})$. To this end, it is sufficient to verify that $\{x_{2n}\}$ is a Cauchy sequence. Suppose, on the contrary, that $\{x_{2n}\}$ is not a Cauchy sequence. Then, there exists an

$\varepsilon > 0$ such that for each even integer $2k$ there are even integers $2n(k)$, $2m(k)$ with $2m(k) > 2n(k) > 2k$ such that

$$r_k = d(x_{2n(k)}, x_{2m(k)}) \geq \varepsilon \ \ for \ \ k \in \{1, 2, 3, \ldots\}. \tag{4.5}$$

For every even integer $2k$, let $2m(k)$ be the smallest number exceeding $2n(k)$ satisfying condition (4.5) for which

$$d(x_{2n(k)}, x_{2m(k)-2}) < \varepsilon. \tag{4.6}$$

From (4.5), (4.6) and the triangular inequality, we have

$$\begin{aligned}\varepsilon \leq r_k &\leq d(x_{2n(k)}, x_{2m(k)-2}) + d(x_{2m(k)-2}, x_{2m(k)-1}) + d(x_{2m(k)-1}, x_{2m(k)}) \\ &\leq \varepsilon + d(x_{2m(k)-2}, x_{2m(k)-1}) + d(x_{2m(k)-1}, x_{2m(k)}).\end{aligned}$$

Hence by (4.3), it follows that

$$\lim_{k \to +\infty} r_k = \varepsilon. \tag{4.7}$$

Now, from the triangular inequality, we have

$$| \, d(x_{2n(k)}, x_{2m(k)-1}) - d(x_{2n(k)}, x_{2m(k)}) \, | \leq d(x_{2m(k)-1}, x_{2m(k)}).$$

Passing to the limit as $k \to +\infty$ and using (4.3) and (4.7), we get

$$\lim_{k \to +\infty} d(x_{2n(k)}, x_{2m(k)-1}) = \varepsilon. \tag{4.8}$$

On the other hand, we have

$$\begin{aligned} & d(x_{2n(k)}, x_{2m(k)}) \qquad\qquad (4.9)\\ &\quad \leq d(x_{2n(k)}, x_{2n(k)+1}) + d(x_{2n(k)+1}, x_{2m(k)}) \\ &\quad \leq d(x_{2n(k)}, x_{2n(k)+1}) + d(\mathcal{S}x_{2n(k)}, \mathcal{T}x_{2m(k)-1}) \\ &\quad \leq d(x_{2n(k)}, x_{2n(k)+1}) + \psi([d(x_{2m(k)-1}, \mathcal{T}x_{2m(k)-1}) + d(x_{2n(k)}, \mathcal{T}x_{2n(k)}) \\ &\qquad\quad + d(x_{2m(k)-1}, \mathcal{T}x_{2n(k)}) + d(x_{2n(k)}, \mathcal{T}x_{2m(k)-1})] \\ &\qquad\quad - \varphi(d(x_{2m(k)-1}, \mathcal{T}x_{2m(k)-1}), d(x_{2n(k)}, \mathcal{T}x_{2n(k)}), d(x_{2m(k)-1}, \mathcal{T}x_{2n(k)}), \\ &\qquad\quad d(x_{2n(k)}, \mathcal{T}x_{2m(k)-1}))) \\ &\quad = d(x_{2n(k)}, x_{2n(k)+1}) + \psi([d(x_{2m(k)-1}, x_{2m(k)}) + d(x_{2n(k)}, x_{2n(k)+1}) \\ &\qquad\quad + d(x_{2m(k)-1}, x_{2n(k)+1}) + d(x_{2n(k)}, x_{2m(k)})] \\ &\qquad\quad - \varphi(d(x_{2m(k)-1}, x_{2m(k)}), d(x_{2n(k)}, x_{2n(k)+1}), \\ &\qquad\quad d(x_{2m(k)-1}, x_{2n(k)+1}), d(x_{2n(k)}, x_{2m(k)}))) \\ &\quad < d(x_{2n(k)}, x_{2n(k)+1}) + \frac{1}{4}[d(x_{2m(k)-1}, x_{2m(k)}) + d(x_{2n(k)}, x_{2n(k)+1}) \\ &\qquad\quad + d(x_{2m(k)-1}, x_{2n(k)+1}) + d(x_{2n(k)}, x_{2m(k)})] \\ &\qquad\quad - \frac{1}{4}\varphi(d(x_{2m(k)-1}, x_{2m(k)}), d(x_{2n(k)}, x_{2n(k)+1}), d(x_{2m(k)-1}, x_{2n(k)+1}), \\ &\qquad\quad d(x_{2n(k)}, x_{2m(k)})). \end{aligned}$$

Taking into account (4.3) and (4.7) and the continuity of ψ and φ, passing to the limit as $n \to \infty$ in the last inequality, we obtain

$$\varepsilon \leq \psi([\varepsilon + 0 + \varepsilon + 0] - \varphi(\varepsilon, 0, \varepsilon, 0)) \leq \frac{1}{2}\varepsilon - \frac{1}{4}\varphi(\varepsilon, 0, \varepsilon, 0) < \frac{1}{2}\varepsilon$$

and from the last inequality, $\varphi(\varepsilon, 0, \varepsilon, 0) \leq -\frac{1}{2}\varepsilon \leq 0$. Therefore $\varphi(\varepsilon, 0, \varepsilon, 0) = 0$. From the fact that $\varphi(x, y, z, t) = 0 \Leftrightarrow x = y = z = t = 0$, we have $\varepsilon = 0$, a contradiction. Thus, assumption (4.5) is wrong. Therefore, $\{x_n\}$ is a Cauchy sequence in the metric space $\mathcal{O}(x_0; \mathcal{S}, \mathcal{T})$.

Since $\mathcal{X}$ is $(\mathcal{S}, \mathcal{T}, \mathcal{R})$-orbitally complete at x_0, there exists some $z \in \mathcal{X}$ such that $x_n \to z$ as $n \to +\infty$. Now we show that z is a common fixed point of $\mathcal{T}$ and $\mathcal{S}$. Clearly, if $\mathcal{S}$ or $\mathcal{T}$ is orbitally continuous then $z = \mathcal{S}z$ or $z = \mathcal{T}z$. Thus, it is immediate to conclude that $\mathcal{T}$ and $\mathcal{S}$ have a common fixed point.

Now, suppose that the set of common fixed points of $\mathcal{T}$ and $\mathcal{S}$ is totally ordered. We claim that there is a unique common fixed point of $\mathcal{T}$ and $\mathcal{S}$. Assume to the contrary that $\mathcal{S}u = \mathcal{T}u = u$ and $\mathcal{S}v = \mathcal{T}v = v$ but $u \neq v$. By supposition, we can replace x by u and y by v in (4.1) and the property of ψ, we obtain

$$\begin{aligned} d(u, v) = d(\mathcal{S}u, \mathcal{T}v) &\leq \psi([d(v, \mathcal{T}v) + d(u, \mathcal{S}u) + d(v, \mathcal{S}u) + d(u, \mathcal{T}v)] \\ &\quad - \varphi(d(v, \mathcal{T}v), d(u, \mathcal{S}u), d(v, \mathcal{S}u), d(u, \mathcal{T}v))) \\ &< \frac{1}{2}d(v, u) - \frac{1}{4}\varphi(d(v, \mathcal{T}v), d(u, \mathcal{S}u), d(v, \mathcal{S}u), d(u, \mathcal{T}v)), \end{aligned}$$

a contradiction. Hence, $u = v$. The converse is trivial. □

Now, we are also able to prove the existence of a common fixed point of two mappings without using the continuity of $\mathcal{S}$ or $\mathcal{T}$. More precisely, we have the following theorem.

Theorem 4.3. *Let $(\mathcal{X}, d, \preceq)$ and $\mathcal{S}, \mathcal{T} : \mathcal{X} \to \mathcal{X}$ satisfy all the conditions of Theorem 4.2, except that condition* (iii) *is substituted by*

(iii') *$\mathcal{X}$ is regular.*

Then the same conclusions as in Theorem 4.2 hold.

Proof. Following the proof of Theorem 4.2, we have that $\{x_n\}$ is a Cauchy sequence in $(\mathcal{X}, d)$ which is orbitally complete at x_0. Then, there exists $z \in \mathcal{X}$ such that

$$\lim_{n\to\infty} x_n = z.$$

Now suppose that $d(z, \mathcal{S}z) > 0$. From regularity of $\mathcal{X}$, we have $x_{2n} \preceq z$ for all $n \in \mathbb{N}$. Hence, we can apply the considered contractive condition. Then, setting $x = x_{2n}$ and $y = z$ in (4.1), we obtain:

$$\begin{aligned} d(x_{2n+2}, \mathcal{S}z) &= d(\mathcal{T}x_{2n+1}, \mathcal{S}z) \\ &\leq \psi([d(x_{2n+1}, \mathcal{T}x_{2n+1}) + d(z, \mathcal{S}z) + d(x_{2n+1}, \mathcal{S}z) + d(z, \mathcal{T}x_{2n+1})] \\ &\qquad - \varphi(d(x_{2n+1}, \mathcal{T}x_{2n+1}), d(x_{2n+1}, \mathcal{S}z), d(x_{2n+1}, \mathcal{S}z), d(z, \mathcal{T}x_{2n+1}))) \\ &= \psi([d(x_{2n+1}, x_{2n+2}) + d(z, \mathcal{S}y) + d(x_{2n+1}, \mathcal{S}z) + d(z, x_{2n+2})] \\ &\qquad - \varphi(d(x_{2n+1}, x_{2n+z}), d(z, \mathcal{S}z), d(x_{2n+1}, \mathcal{S}z), d(z, x_{2n+z}))). \end{aligned}$$

Passing to the limit as $n \to \infty$ and using $x_n \to z$, properties of ψ and φ, we have

$$d(z, \mathcal{S}z) \leq \frac{1}{2}d(z, \mathcal{S}z) - \frac{1}{4}\varphi(0, d(z, \mathcal{S}z), d(z, \mathcal{S}z), 0) \leq \frac{1}{2}d(z, \mathcal{S}z)$$

a contradiction. Therefore $d(z, \mathcal{S}z) = 0$ and thus $z = \mathcal{S}z$. Analogously, for $x = z$ and $y = x_{2n}$, one can prove that $\mathcal{T}z = z$. It follows that $z = \mathcal{S}z = \mathcal{T}z$, that is, $\mathcal{T}$ and $\mathcal{S}$ have a common fixed point. □

Putting $\mathcal{S} = \mathcal{T}$ in Theorem 4.3, we obtain easily the following result.

Corollary 4.4. *Let $(\mathcal{X}, \preceq)$ be a partially ordered set and suppose that there exists a metric d in $\mathcal{X}$ such that $(\mathcal{X}, d)$ is a complete metric space. Suppose $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is mapping satisfying generalized weakly $(\psi, \mathcal{C})$-contraction conditions, that is,*

$$\begin{aligned} d(\mathcal{T}x, \mathcal{T}y) \leq \psi([&d(x, \mathcal{T}x) + d(y, \mathcal{T}y) + d(x, \mathcal{T}y) + d(y, \mathcal{T}x)] \\ &- \varphi(d(x, \mathcal{T}x), d(y, \mathcal{T}y), d(x, \mathcal{T}y), d(y, \mathcal{T}x))), \end{aligned}$$

for all comparable $x, y \in \mathcal{X}$, where $\varphi \in \Phi_2$ and $\psi \in \Psi_2$.

We assume the following hypotheses:

(i) $\mathcal{T}$ is a.r. at some point x_0;

(ii) $\mathcal{X}$ is $\mathcal{T}$-orbitally complete at x_0;

(iii) $\mathcal{T}$ is orbitally continuous at x_0 or $\mathcal{X}$ is regular.

Also suppose that $\mathcal{T}x \prec \mathcal{T}(\mathcal{T}x)$ for all $x \in \mathcal{X}$ such that $x \prec \mathcal{T}x$. If there exists an $x_0 \in \mathcal{X}$ such that $x_0 \prec \mathcal{T}x_0$ and the condition

$$\begin{cases} \{x_n\} \subset \mathcal{X} \text{ is a non-decreasing sequence with } x_n \to z \text{ in } \mathcal{X}, \\ \text{then } x_n \preceq z \text{ for all } n \end{cases}$$

holds, then $\mathcal{T}$ has a fixed point. Moreover, the set of fixed points of $\mathcal{T}$ is totally ordered if and only if it is singleton.

We illustrate Theorem 4.2 by another example which is obtained by modifying the one from [45].

Example 4.5. Let the set $\mathcal{X} = [0, +\infty)$ be equipped with the usual metric d and the order defined by

$$x \preceq y \iff x \geq y.$$

Consider the following self-mappings on $\mathcal{X}$:

$$\mathcal{T}x = \begin{cases} \frac{1}{2}x, & 0 \leq x \leq \frac{1}{2}, \\ 2x, & x > \frac{1}{2}, \end{cases} \qquad \mathcal{S}x = \begin{cases} \frac{1}{3}x, & 0 \leq x \leq \frac{1}{3}, \\ 3x, & x > \frac{1}{3}. \end{cases}$$

Take $x_0 = \frac{1}{3}$. Then it is easy to show that all the conditions (i)–(iii) of Theorem 4.2 are fulfilled on $O(x_0; \mathcal{S}, \mathcal{T})$. Take $\psi(t) = \frac{t}{4}$ and $\varphi(x, y, z, t) = \frac{x+y+z+t}{3}$ with $\varphi \in \Phi_2$ and $\psi \in \Psi_2$. Then contractive condition (4.1) takes the form

$$\left|\frac{1}{2}x - \frac{1}{3}y\right| \leq \frac{1}{6}\left[\frac{1}{2}x + \frac{2}{3}y + \left|x - \frac{1}{3}y\right| + \left|y - \frac{1}{2}x\right|\right],$$

for $x, y \in O(x_0; \mathcal{S}, \mathcal{T})$. Using substitution $y = tx$, $t > 0$, the last inequality reduces to

$$|3 - 2t| \leq \frac{1}{6}[3 + 4t + 2|3 - t| + 3|2t - 1|],$$

and can be checked by discussion on possible values for $t > 0$. Hence, all the conditions of Theorem 4.2 are satisfied and $\mathcal{S}, \mathcal{T}$ have a common fixed point (which is 0).

5 Existence of a Common Solution of Integral Equations

Consider the system of integral equations:

$$\begin{cases} u(t) = \int_0^T K_1(t, s, u(s))\ ds, \ t \in [0, T], \\ u(t) = \int_0^T K_2(t, s, u(s))\ ds, \ t \in [0, T], \end{cases} \tag{5.1}$$

where $T > 0$ and $K_1, K_2 : [0; T] \times [0; T] \times \mathbb{R} \to \mathbb{R}$ are continuous functions. The purpose of this section is to give an existence theorem for a solution of (5.1).

Previously, we consider the space $C(I, \mathbb{R})(I = [0, T])$ of continuous functions defined on I. Obviously, this space with the metric given by:

$$d(u, v) = \max_{t \in I} |u(t) - v(t)|, \ \forall\, u, v \in C(I, \mathbb{R}),$$

is a complete metric space. $C(I, \mathbb{R})$ can also be equipped with the partial order $\preceq$ given by:

$$u, v \in C(I, \mathbb{R}), \qquad u \preceq v \Leftrightarrow u(t) \leq v(t), \ \forall\, t \in I.$$

Moreover, in [9], it is proved that $(C(I, \mathbb{R}), \preceq)$ is regular.

Consider the mappings $\mathcal{T}, \mathcal{S} : C(I, \mathbb{R}) \to C(I, \mathbb{R})$ defined by

$$\mathcal{T}u(t) = \int_0^T K_1(t, s, u(s))\ ds, \ \text{for all } C(I, \mathbb{R}), \ t \in I$$

$$\mathcal{S}u(t) = \int_0^T K_2(t, s, u(s))\ ds, \ \text{for all } C(I, \mathbb{R}), \ t \in I.$$

Clearly, u is a solution of (5.1) if and only if u is a common fixed point of $\mathcal{T}$ and $\mathcal{S}$.

We shall prove the existence of a common fixed point of $\mathcal{T}$ and $\mathcal{S}$ under the certain conditions.

Theorem 5.1. *Suppose that the following hypotheses hold:*
(H1) for all $t, s \in I$, $u \in C(I, \mathbb{R})$, we have:

$$K_1(t,s,u(t)) \leq K_2\left(t, s, \int_0^T K_1(s,\tau,u(\tau))\, d\tau\right)$$

(H2) for all $t, s \in I$, $u \in C(I, \mathbb{R})$, we have:

$$K_2(t,s,u(t)) \leq K_1\left(t, s, \int_0^T K_2(s,\tau,u(\tau))\, d\tau\right);$$

(H3) there exists a continuous function $\alpha : I \times I \to \mathbb{R}_+$, and $\psi : \mathbb{R} \to \mathbb{R}^+$ and $\psi : {\mathbb{R}^+}^2 \to \mathbb{R}^+$ belong to the classes Ψ_1 and Φ_1 respectively such that

$$|K_1(t,s,x) - K_2(t,s,y)| \leq \alpha(t,s)\psi([d(x,\mathcal{S}y) + d(y,\mathcal{T}x)] - \varphi(d(x,\mathcal{S}y), d(y,\mathcal{T}x)))$$

for all $t, s \in I$ and $x, y \in \mathbb{R}$ such that $x \geq y$;
(H4) $\sup_{t \in I} \int_0^T \alpha(t,s)\, ds \leq 1$.
Then, the integral equations (5.1) have a solution $u^* \in C(I, \mathbb{R})$.

Proof. Let $u \in C(I, \mathbb{R})$. Using $(H1)$, for all $t \in I$, we have

$$\begin{aligned} \mathcal{T}u(t) &= \int_0^T K_1(t,s,u(s))\, ds \\ &\leq \int_0^T K_2\left(t, s, \int_0^T K_1(s,\tau,u(\tau))\, d\tau\right) ds \\ &= \int_0^T K_2(t,s,\mathcal{T}u(s))\, ds \\ &= \mathcal{S}\mathcal{T}u(t). \end{aligned}$$

Similarly, using $(H2)$, for all $t \in I$, we have

$$\begin{aligned} \mathcal{S}u(t) &= \int_0^T K_2(t,s,u(s))\, ds \\ &\leq \int_0^T K_1\left(t, s, \int_0^T K_2(s,\tau,u(\tau))\, d\tau\right) ds \\ &= \int_0^T K_1(t,s,\mathcal{S}u(s))\, ds \\ &= \mathcal{T}\mathcal{S}u(t). \end{aligned}$$

Then, we have $\mathcal{T}u \preceq \mathcal{S}\mathcal{T}u$ and $\mathcal{S}u \preceq \mathcal{T}\mathcal{S}u$ for all $u \in C(I, \mathbb{R})$. This implies that $\mathcal{T}$ and $\mathcal{S}$ are weakly increasing.

Now, for all $u, v \in C(I, \mathbb{R})$ such that $v \preceq u$, by $(H3)$ and $(H4)$, we have:

$$\begin{aligned}
|\mathcal{T}u(t) - \mathcal{S}v(t)| &\leq \int_0^T |K_1(t,s,u(s)) - K_2(t,s,v(s))| \, ds \\
&\leq \int_0^T \alpha(t,s)\psi([|u(s) - \mathcal{S}v(s)| + |v(s) - \mathcal{T}u(s)|] \\
&\quad - \varphi(|u(s) - \mathcal{S}v(s)|, |v(s) - \mathcal{T}v(s)|)) \, ds \\
&\leq \int_0^T \alpha(t,s)\psi([d(u,\mathcal{S}v), d(v,\mathcal{T}u)] - \varphi(d(u,\mathcal{S}v), d(v,\mathcal{T}v))) \, ds \\
&= \left(\int_0^T \alpha(t,s) \, ds\right)\psi([d(u,\mathcal{S}v), d(v,\mathcal{T}u)] - \varphi(d(u,\mathcal{S}v), d(v,\mathcal{T}v))) \\
&\leq \psi([d(u,\mathcal{S}v), d(v,\mathcal{T}u)] - \varphi(d(u,\mathcal{S}v), d(v,\mathcal{T}v))).
\end{aligned}$$

Hence, we proved that for all $u, v \in C(I, \mathbb{R})$ such that $u \succeq v$, we have

$$d(\mathcal{T}u, \mathcal{S}v) \leq \psi([d(u,\mathcal{S}v), d(v,\mathcal{T}u)] - \varphi(d(u,\mathcal{S}v), d(v,\mathcal{T}v))).$$

Now, all the hypotheses of Corollary 3.4 are satisfied. Then, $\mathcal{T}$ and $\mathcal{S}$ have a common fixed point $u^* \in C(I, \mathbb{R})$, that is, u^* is a solution to the integral equations (5.1). □

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