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Iterative Algorithm for Finite Family of k_i-Strictly Pseudo-Contractive Mappings for a General Hierarchical Problem in Hilbert Spaces[1]

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Abstract : In this work, we introduced the iterative scheme for finite family of k-strictly pseudo-contractive mappings. Then we prove strong convergence of algorithm (1.6) and solving a common solution of a general hierarchical problem and fixed point problems of finite family of k-strictly pseudo-contractive mappings.

Keywords : k-strictly pseudo-contractive mappings; hierarchical problem; variational inequality; fixed point; Hilbert spaces.
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1 Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H. The hierarchical problem is of finding $\tilde{x} \in Fix(T)$ such that

$$\langle S\tilde{x} - \tilde{x}, x - \tilde{x} \rangle \leq 0, \quad \forall x \in Fix(T), \tag{1.1}$$

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where S, T are two nonexpansive mappings and $Fix(T)$ to denote the fixed points set of T, that is $Fix(T) = \{x \in C : Tx = x\}$. Recently, this problem has been studied by many authors (see,[1]-[16]).

Now, we briefly recall some historic results which relate to the problem (1.1).

For solving the problem (1.1), in 2006, Moudafi and Mainge [3] first introduced an implicit iterative algorithm:

$$x_{t,s} = sQ(x_{t,s}) + (1-s)[tS(x_{t,s}) + (1-t)T(x_{t,s})] \tag{1.2}$$

and proved that the net $\{x_{t,s}\}$ defined by (1.2) strongly converges to x_t as $s \to 0$, where x_t satisfies $x_t{=}proj_{Fix(P_t)}Q(x_t)$, where $P_t{:}C \to C$ is a mapping defined by

$$P_t(x) = tS(x) + (1-t)T(x), \forall x \in C, t \in (0,1),$$

or, equivalently, x_t is the unique solution of the quasivariational inequality:

$$0 \in (I-Q)x_t + N_{Fix(P_t)}(x_t),$$

where the normal cone to $Fix(P_t), N_{Fix(P_t)}$ is defined as follows:

$$N_{Fix(P_t)} : x \to \begin{cases} \{u \in H : \langle y - x, u\rangle \leq 0\}, & \text{if } x \in Fix(P_t), \\ \emptyset, & \text{otherwise.} \end{cases}$$

Moreover, as $t \to 0$, the net $\{x_t\}$ in turn weakly converges to the unique solution x_∞ of the fixed point equation $x_\infty{=}proj_\Omega Q(x_\infty)$ or, equivalently, x_∞ is the unique solution of the variational inequality:

$$0 \in (I-Q)x_\infty + N_\Omega(x_\infty).$$

Recall that a mapping $f : C \longrightarrow C$ is said to be contractive if there exists a constant $\gamma \in (0,1)$ such that

$$\|fx - fy\| \leq \gamma\|x - y\|, \quad \forall x, y \in C.$$

A mapping $T : C \longrightarrow C$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

A mapping T is said to be k-strict pseudo-contractive if there exists $k \in [0,1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I-T)x - (I-T)y\|^2, \forall x, y \in D(T). \tag{1.3}$$

Note that the class of k-strict pseudo-contraction strictly includes the class of nonexpansive mappings. We see that, if $S_k : C \to C$ defined by $S_k x = kx + (1-k)Tx$ for all $x \in C$ where T is k-strict pseudo-contractive then S_k is nonexpansive mapping [17].

In this paper, motivate by Kangtunkarn and Suantai [18], we introduce a mapping for finding a common fixed point of T is a λ-strict pseudo-contractive mapping and $\{T_i\}_{i=1}^N$ a finite family of k_i-strict pseudo-contractive mappings of C into itself. For each $n \in \mathbb{N}$, and $j = 1, 2, ..., N$, let $\alpha_j^n = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}) \in [0,1] \times [0,1] \times [0,1]$ with $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$. We define the mapping $S_n : C \to C$ as follows:

$$\begin{aligned} U_{n,0} &= I; \\ U_{n,1} &= \alpha_1^{n,1} T_1 U_{n,0} + \alpha_2^{n,1} U_{n,0} + \alpha_3^{n,1} I; \\ U_{n,2} &= \alpha_1^{n,2} T_2 U_{n,1} + \alpha_2^{n,2} U_{n,1} + \alpha_3^{n,2} I; \\ U_{n,3} &= \alpha_1^{n,3} T_3 U_{n,2} + \alpha_2^{n,3} U_{n,2} + \alpha_3^{n,3} I; \\ &\vdots \quad ; \\ U_{n,N-1} &= \alpha_1^{n,N-1} T_{N-1} U_{n,N-2} + \alpha_2^{n,N-1} U_{n,N-2} + \alpha_3^{n,N-1} I; \\ S_n &= U_{n,N} = \alpha_1^{n,N} T_N U_{n,N-1} + \alpha_2^{n,N} U_{n,N-1} + \alpha_3^{n,N} I. \end{aligned} \tag{1.4}$$

Motivated and inspired by the results in the literature, in this paper, we consider a general hierarchical problem of finding $x^* \in F(T)$ such that, for any $n \geq 1$,

$$\langle S_n x^* - x^*, x - x^* \rangle \leq 0, \forall x \in F(S_\lambda), \tag{1.5}$$

where S_n is the S-mapping defined by (1.4) and S_λ is a nonexpansive mapping defined in Lemma 2.1.

Algorithm 1.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and let T is a λ-strict pseudo-contractive mapping with $S_\lambda x = \lambda x + (1-\lambda)Tx$ and $\{T_i\}_{i=1}^N$ be a finite family of k_i-strictly pseudo-contractive mapping of C into itself. Let $f : C \to C$ be a contraction with coefficient $\gamma \in (0,1)$. For any $x_0 \in C$, let $\{x_n\}$ be the sequence generated by*

$$x_{n+1} = \alpha_n S_n x_n + (1-\alpha_n) S_\lambda(\beta_n f(x_n) + (1-\beta_n)x_n), \quad \forall n \geq 0, \tag{1.6}$$

where $\{\alpha_n\}, \{\beta_n\}$ are two real numbers in $(0,1)$ and S_n is the $S-$mapping defined by (1.4).

We show that an explicit iterative algorithm which converges strongly to a solution x^* of the general hierarchical problem (1.5).

2 Preliminaries

In this section, we collect and give some definition and useful lemmas that will be used for our main results in the next section.

Lemma 2.1. *[17] Let $T : C \to C$ be a k-strictly pseudo-contraction. Defined $S_\lambda : C \to C$ by $S_\lambda x = \lambda x + (1-\lambda)Tx$ for each $x \in C$. Then, as $\lambda \in [k,1]$, S_λ is nonexpansive mapping and $F(T) = F(S_\lambda)$.*

Lemma 2.2. *In a real Hilbert space H, there holds the inequality*

1. $\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle$ *and* $\|x-y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2, \forall x, y \in H$.
2. $\|tx+(1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2, \forall t \in [0,1], \forall x, y \in H$.
3. $\|\sum_{i=0}^m \alpha_i x_i\|^2 = \sum_{i=0}^m \alpha_i \|x_i\|^2 - \sum_{i=0}^m \alpha_i\alpha_j\|x_i - x_j\|^2$ *for* $\sum_{i=0}^m \alpha_i = 1, \alpha_i \in [0,1], \forall i \in \{0,1,2,\dots,m\}$.

Definition 2.3. *[18] Let C be nonempty convex subset of real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of k_i-strictly pseudo-contractive mapping of C into itself. For each $j = 1,2,\dots,N$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $\alpha_1^j, \alpha_2^j, \alpha_3^j \in I \equiv [0,1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. We define the mapping $S : C \to C$ as follows:*

$$\begin{aligned}
U_0 &= I \\
U_1 &= \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I \\
U_2 &= \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I \\
U_3 &= \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I \\
&\vdots \\
U_{N-1} &= \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I \\
S &= U_N = \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I.
\end{aligned}$$

This mapping is called $S-$mapping generated by $T_1, \dots, T_N$ and $\alpha_1, \alpha_2, \dots, \alpha_N$.

Lemma 2.4. *[9] Let C be a nonempty closed convex subset of a real Hilbert space H and $S : C \to C$ be a self-mapping of C. If S is a k-strict pseudo-contraction mapping, then S satisfies the Lipschitz condition*

$$\|S_x - S_y\| \le \frac{1+k}{1-k}\|x-y\|, \quad \forall x, y \in C.$$

Lemma 2.5. *[19] Let $\{s_n\}$ be a sequence of nonnegative real number satisfying*

$$s_{n+1} = (1-\alpha_n)s_n + \delta_n + \eta_n, \quad \forall n \ge 0$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

1. $\sum_{n=1}^\infty \alpha_n = \infty$,
2. $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \le 0$ *or* $\sum_{n=1}^\infty |\delta_n| < \infty$,
3. $\sum_{n=1}^\infty |\eta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.6. *[18] Let C be a nonempty closed convex subset of real Hilbert space. Let $\{T_i\}_{i=1}^N$ be a finite family of k_i-strictly pseudo-contractive mapping of C into C with $\bigcap_{i=1}^N F(T_i) \ne \emptyset$ and $k = \max\{k_i : i = 1,2,\dots,N\}$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in$*

$I\times I\times I, j=1,2,3,\dots,N$, where $I=[0,1], \alpha_1^j+\alpha_2^j+\alpha_3^j=1, \alpha_1^j, \alpha_3^j\in(k,1)$ for all $j=1,2,\dots,N-1$ and $\alpha_1^N\in(k,1], \alpha_3^N\in(k,1], \alpha_2^j\in(k,1]$ for all $j=1,2,\dots,N$. Let S be the mapping generated by $T_1,\dots,T_N$ and $\alpha_1,\alpha_2,\dots,\alpha_N$. Then $F(S)=\bigcap_{i=1}^N F(T_i)$ and S is a nonexpansive mapping.

Lemma 2.7. *[20] A real Hilbert space H satisfies Opial's condition, i.e, for any sequence $\{x_n\}\subset H$ with $x_n\rightharpoonup x$, the inequality*

$$\liminf_{n\to\infty}\|x_n-x\|<\liminf_{n\to\infty}\|x_n-y\|,$$

holds for each $y\in H$ with $x\neq y$.

Lemma 2.8. *[21] Let C be a nonempty closed convex subset of a real Hilbert and $T:C\to C$ be a nonexpansive mapping. Then T is demi-closed on C,i.e.,if $x_n\rightharpoonup x\in C$ and $x_n-Tx_n\to 0$, then $x = Tx$.*

3 Main Results

In this section, we prove strong convergence of algorithm (1.6) and solving a common solution of a general hierarchical problems and fixed point problems of finite family of strict pseudo-contractive mappings. First, we can prove the lemmas that will be used in the main theorem.

Lemma 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H, let $\{T_i\}_{i=1}^N$ be a finite family of k_i-strictly pseudo-contraction of C into itself for some $k_i\in[0,1)$ and $k=\max\{k_i: i=1,2,\dots,N\}$ with $\bigcap_{i=1}^N F(T_i)\neq\emptyset$. Let S_n be the S-mapping generated by $T_1,T_2,\dots,T_N$ and $\alpha_1^{(n)},\alpha_2^{(n)},\dots,\alpha_N^{(n)}$, where $\alpha_j^{(n)}=(\alpha_1^{n,j},\alpha_2^{n,j},\alpha_3^{n,j})\in I\times I\times I, I=[0,1], \alpha_1^{n,j}+\alpha_2^{n,j}+\alpha_3^{n,j}=1$ and $k<a\leq \alpha_1^{n,j},\alpha_3^{n,j}\leq b<1$ for all $k<c\leq\alpha_1^{n,N}\leq 1, k\leq\alpha_3^{n,N}\leq d<1, k\leq\alpha_2^{n,j}\leq e<1$ for all $j=1,2,\dots,N$ and $\sum_{n=1}^\infty|\alpha_1^{n+1,j}-\alpha_1^{n,j}|<\infty, \sum_{n=1}^\infty|\alpha_3^{n+1,j}-\alpha_3^{n,j}|<\infty$ for all $j=\{1,2,3,\dots,N\}$. Then for all $x\in H, \sum_{n=1}^\infty\|S_{n+1}x-S_nx\|<\infty$.*

Proof. For each $x\in C$ and $n\in\mathbb{N}$, we have

$$\begin{aligned}\|U_{n+1,1}x-U_{n,1}x\| &= \|\alpha_1^{n+1,1}T_1x+(1-\alpha_1^{n+1,1}x)-\alpha_1^{n,1}T_1x+(1-\alpha_1^{n,1})x\|\\ &= \|\alpha_1^{n+1,1}T_1x-\alpha_1^{n+1,1}x-\alpha_1^{n,1}T_1x+\alpha_1^{n,1}x\|\\ &= \|(\alpha_1^{n+1,1}-\alpha_1^{n,1})T_1x-(\alpha_1^{n+1,1}-\alpha_1^{n,1})x\|\\ &= |\alpha_1^{n+1,1}-\alpha_1^{n,1}|\|T_1x-x\| \qquad (3.1)\end{aligned}$$

and for $n \in \mathbb{N}$, and for $k \in \{2, 3, \ldots, N\}$, we have

$$
\begin{aligned}
& \|U_{n+1,k}x - U_{n,k}x\| \\
= \ & \|\alpha_1^{n+1,k} T_k U_{n+1,k-1}x + \alpha_2^{n+1,k} U_{n+1,k-1}x + \alpha_3^{n+1,k}x \\
& -\alpha_1^{n,k} T_k U_{n,k-1}x + \alpha_2^{n,k} U_{n,k-1}x + \alpha_3^{n,k}x\| \\
= \ & \|\alpha_1^{n+1,k} T_k U_{n+1,k-1}x + \alpha_3^{n+1,k}x - \alpha_1^{n,k} T_k U_{n,k-1}x - \alpha_3^{n,k}x \\
& +\alpha_2^{n+1,k} U_{n+1,k-1}x - \alpha_2^{n,k} U_{n,k-1}x\| \\
= \ & \|\alpha_1^{n+1,k} T_k U_{n+1,k-1}x - \alpha_1^{n+1,k} T_k U_{n,k-1}x + \alpha_1^{n+1,k} T_k U_{n,k-1}x \\
& -\alpha_1^{n,k} T_k U_{n,k-1}x + (\alpha_3^{n+1,k} - \alpha_3^{n,k})x + \alpha_2^{n+1,k} U_{n+1,k-1}x - \alpha_2^{n,k} U_{k-1}x\| \\
= \ & \|\alpha_1^{n+1,k}(T_k U_{n+1,k-1}x - T_k U_{n,k-1}x) + (\alpha_1^{n+1,k} - \alpha_1^{n,k}) T_k U_{n,k-1}x \\
& +(\alpha_3^{n+1,k} - \alpha_3^{n,k})x + \alpha_2^{n+1,k} U_{n+1,k-1}x - \alpha_2^{n,k} U_{n,k-1}x\| \\
= \ & \|\alpha_1^{n+1,k}(T_k U_{n+1,k-1}x - T_k U_{n,k-1}x) + (\alpha_1^{n+1,k} - \alpha_1^{n,k}) \\
& \times T_k U_{n,k-1}x + (\alpha_3^{n+1,k} - \alpha_3^{n,k})x + \alpha_2^{n+1,k} U_{n+1,k-1}x \\
& -\alpha_2^{n+1,k} U_{n,k-1}x + \alpha_2^{n+1,k} U_{n,k-1}x - \alpha_2^{n,k} U_{n,k-1}x\| \\
= \ & \|\alpha_1^{n+1,k}(T_k U_{n+1,k-1}x - T_k U_{n,k-1}x) + (\alpha_1^{n+1,k} - \alpha_1^{n,k}) \\
& \times T_k U_{n,k-1}x + (\alpha_3^{n+1,k} - \alpha_3^{n,k})x + \alpha_2^{n+1,k}(U_{n+1,k-1}x \\
& -U_{n,k-1}x) + (\alpha_2^{n+1,k} - \alpha_2^{n,k}) U_{n,k-1}x\| \\
\leq \ & \alpha_1^{n+1,k}\|T_k U_{n+1,k-1}x - T_k U_{n,k-1}x\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_k U_{n,k-1}x\| \\
& +|\alpha_3^{n+1,k} - \alpha_3^{n,k}|\|x\| \\
& +\alpha_2^{n+1,k}\|U_{n+1,k-1}x - U_{n,k-1}x\| + |\alpha_2^{n+1,k} - \alpha_2^{n,k}|\|U_{n,k-1}x\| \\
= \ & \alpha_1^{n+1,k}\|T_k U_{n+1,k-1}x - T_k U_{n,k-1}x\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_k U_{n,k-1}x\| \\
& +\alpha_2^{n+1,k}\|U_{n+1,k-1}x - U_{n,k-1}x\| + |1 - \alpha_1^{n+1,k} \\
& -\alpha_3^{n+1,k} - 1 + \alpha_1^{n,k} + \alpha_3^{n,k}|\|U_{n,k-1}x\| + |\alpha_3^{n+1,k} - \alpha_3^{n,k}|\|x\| \\
\leq \ & \alpha_1^{n+1,k}\frac{1+k}{1-k}\|U_{n+1,k-1}x - U_{n,k-1}x\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_k U_{n,k-1}x\| \\
& +\alpha_2^{n+1,k}\|U_{n+1,k-1}x - U_{n,k-1}x\| + (|\alpha_1^{n,k} \\
& -\alpha_1^{n+1,k}| + |\alpha_3^{n,k} - \alpha_3^{n+1,k}|)\|U_{n,k-1}x\| + |\alpha_3^{n+1,k} - \alpha_3^{n,k}|\|x\| \\
\leq \ & \frac{1+k}{1-k}\|U_{n+1,k-1}x - U_{n,k-1}x\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_k U_{n,k-1}x\| \\
& +\frac{1+k}{1-k}\|U_{n+1,k-1}x - U_{n,k-1}x\| + (|\alpha_1^{n,k} - \alpha_1^{n+1,k}| \\
& +|\alpha_3^{n,k} - \alpha_3^{n+1,k}|)\|U_{n,k-1}x\| + |\alpha_3^{n+1,k} - \alpha_3^{n,k}|\|x\| \\
= \ & \frac{2}{1-k}\|U_{n+1,k-1}x - U_{n,k-1}x\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|(\|T_k U_{n,k-1}x \\
& +\|U_{n,k-1}x\|) + |\alpha_3^{n,k} - \alpha_3^{n+1,k}|(\|U_{n,k-1}x\| + \|x\|). \qquad (3.2)
\end{aligned}
$$

By (3.1) and (3.2), we have

$$
\begin{aligned}
&\|S_{n+1}x - S_nx\| = \|U_{n+1,N}x - U_{n,N}x\| \\
\leq\ & \frac{2}{1-k}\|U_{n+1,N-1}x - U_{n,N-1}x\| + |\alpha_1^{n+1,N} - \alpha_1^{n,N}|(\|T_NU_{n,N-1}x\| \\
&+\|U_{n,N-1}x\|) + |\alpha_3^{n+1,N} - \alpha_3^{n,N}|(\|U_{n,N-1}x\| + \|x\|) \\
\leq\ & \frac{2}{1-k}\bigg(\frac{2}{1-k}\|U_{n+1,N-2}x - U_{n,N-2}x\| \\
&+|\alpha_1^{n+1,N-1} - \alpha_1^{n,N-1}|(\|T_{N-1}U_{n,N-2}x\| + \|U_{n,N-2}x\|) \\
&+|\alpha_3^{n+1,N-1} - \alpha_3^{n,N-1}|[\|U_{n,N-2}x\| + \|x\|]\bigg) \\
&+|\alpha_1^{n+1,N} - \alpha_1^{n,N}|(\|T_NU_{n,N-1}x\| + \|U_{n,N-1}x\|) \\
&+|\alpha_3^{n+1,N} - \alpha_3^{n,N}|(\|U_{n,N-1}x\| + \|x\|) \\
=\ & \left(\frac{2}{1-k}\right)^2\|U_{n+1,N-2}x - U_{n,N-2}x\| + \sum_{j=N-1}^{N}\left(\frac{2}{1-k}\right)^{N-j}|\alpha_1^{n+1,j} - \alpha_1^{n,j}|(\|T_jU_{n,j-1}x\| \\
&+\|U_{n,j-1}x\|) + \sum_{j=N-1}^{N}\left(\frac{2}{1-k}\right)^{N-j}|\alpha_3^{n+1,j} - \alpha_3^{n,j}|(\|U_{n,j-1}x\| + \|x\|) \\
&\vdots \\
\leq\ & \left(\frac{2}{1-k}\right)^{N-1}\|U_{n+1,1}x - U_{n,1}x\| + \sum_{j=2}^{N}\left(\frac{2}{1-k}\right)^{N-j}|\alpha_1^{n+1,j} - \alpha_1^{n,j}|(\|T_jU_{n,j-1}x\| \\
&+\|U_{n,j-1}x\|) + \sum_{j=2}^{N}\left(\frac{2}{1-k}\right)^{N-j}|\alpha_3^{n+1,j} - \alpha_3^{n,j}|(\|U_{n,j-1}x\| + \|x\|) \\
=\ & \left(\frac{2}{1-k}\right)^{N-1}|\alpha_1^{n+1,1} - \alpha_1^{n,1}|\|T_1x - x\| + \sum_{j=2}^{N}\left(\frac{2}{1-k}\right)^{N-j} \\
&+|\alpha_1^{n+1,j} - \alpha_1^{n,j}|(\|T_jU_{n,j-1}x\| + \|U_{n,j-1}x\| + \|x\|) + \sum_{j=2}^{N}\left(\frac{2}{1-k}\right)^{N-j} \\
&+|\alpha_3^{n+1,j} - \alpha_3^{n,j}|(\|U_{n,j-1}x\| + \|x\|).
\end{aligned}
$$

This implies by assumption we have that

$$\sum_{n=1}^{\infty}\|S_{n+1}x - S_nx\| < \infty.$$

This complete the proof. □

Lemma 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H, let $\{T_i\}_{i=1}^N$ be a finite family of k_i-strictly pseudo-contraction of C into itself for some $k_i \in [0,1)$ and $k = \max\{k_i : i = 1, 2, \ldots, N\}$ with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let S_n be the S-mapping generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$, where $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}) \in I \times I \times I, I = [0,1], \alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$ and satisfy conditions:*

(1) $k < a \leq \alpha_1^{n,j}, \alpha_3^{n,j} \leq b < 1$ *for all* $k < c \leq \alpha_1^{n,N} \leq 1, k \leq \alpha_3^{n,N} \leq d < 1, k \leq \alpha_2^{n,j} \leq e < 1$ *for all* $j = 1, 2, \ldots, N$

(2) $\sum_{n=1}^{\infty} |\alpha_1^{n,j} - \alpha_1^j| < \infty, \sum_{n=1}^{\infty} |\alpha_2^{n,j} - \alpha_2^j| < \infty, \sum_{n=1}^{\infty} |\alpha_3^{n,j} - \alpha_3^j| < \infty$ *for all* $j = \{1, 2, 3, \ldots, N\}$.

Then for all $x \in H, \lim_{n\to\infty} \|S_n x - Sx\| = 0$.

Proof. Let $x \in C$ and for each $n \in \mathbb{N}$, from the definition of S mapping and Lemma 2.4, we have

$$\begin{aligned}
\|U_{n,1}x - U_1x\| &= \|\alpha_1^{n,1}T_1U_{n,0}x + \alpha_2^{n,1}U_{n,0}x + \alpha_3^{n,1}x - (\alpha_1^1T_1U_0x + \alpha_2^1U_0x + \alpha_3^1x)\| \\
&\leq |\alpha_1^{n,1} - \alpha_1^1|\|T_1x\| + |\alpha_2^{n,1} - \alpha_2^1|\|x\| + |\alpha_3^{n,1} - \alpha_3^1|\|x\|.
\end{aligned}$$

From boundedness and condition (2) we have

$$\lim_{n\to\infty} \|U_{n,1}x - U_1x\| = 0. \tag{3.3}$$

Next, consider

$$\begin{aligned}
\|U_{n,2}x - U_2x\| &= \|\alpha_1^{n,2}T_2U_{n,1}x + \alpha_2^{n,2}U_{n,1}x + \alpha_3^{n,2}x - (\alpha_1^2T_2U_1x + \alpha_2^2U_1x + \alpha_3^2x)\| \\
&\leq \|\alpha_1^{n,2}T_2U_{n,1}x - \alpha_1^{n,2}T_2U_1x + \alpha_1^{n,2}T_2U_1x + \alpha_2^{n,2}U_{n,1}x + \alpha_3^{n,2}x \\
&\quad -(\alpha_1^2T_2U_1x + \alpha_2^2U_1x + \alpha_3^2x)\| \\
&\leq \|\alpha_1^{n,2}(T_2U_{n,1}x - T_2U_1x)\| + \|(\alpha_3^{n,2} - \alpha_3^2)(x)\| + \|(\alpha_1^{n,2} - \alpha_1^2)(T_2U_1x)\| \\
&\quad +\|\alpha_2^{n,2}U_{n,1}x - \alpha_2^2U_1x\| \\
&\leq \alpha_1^{n,2}\|T_2U_{n,1}x - T_2U_1x\| + |\alpha_3^{n,2} - \alpha_3^2|\|x\| + |\alpha_1^{n,2} - \alpha_1^2|\|T_2U_1x\| \\
&\quad +\alpha_2^{n,2}\|U_{n,1}x - U_1x\| + |\alpha_2^{n,2} - \alpha_2^2|\|U_1x\| \\
&\leq \alpha_1^{n,2}\frac{1+k}{1-k}\|U_{n,1}x - U_1x\| + |\alpha_3^{n,2} - \alpha_3^2|\|x\| + |\alpha_1^{n,2} - \alpha_1^2|\|T_2U_1x\| \\
&\quad +\alpha_2^{n,2}\|U_{n,1}x - U_1x\| + |\alpha_2^{n,2} - \alpha_2^2|\|U_1x\|.
\end{aligned}$$

From boundedness, condition (2) and equation (3.3), we have

$$\lim_{n\to\infty} \|U_{n,2}x - U_2x\| = 0. \tag{3.4}$$

Similarly of the proof, we have

$$\lim_{n\to\infty} \|U_{n,N}x - U_Nx\| = 0. \tag{3.5}$$

Since $\|S_nx - Sx\| = \|U_{n,N}x - U_Nx\|$, we have

$$\lim_{n\to\infty} \|S_nx - Sx\| = 0. \tag{3.6}$$

This complete the proof. □

Theorem 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H, let T be a λ-strictly pseudo-contractive mapping and $\{T_i\}_{i=1}^N$ be a finite family of k_i-strictly pseudo-contractive mappings of C into itself for some $k_i \in [0,1)$ and $k = \max\{k_i : i = 1,2,...,N\}$ which $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let S_n be the S-mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1^n, \alpha_2^n, ..., \alpha_N^n$ where $\alpha_j^n = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}) \in I \times I \times I$, $I = [0,1], \alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$ and $k < a \le \alpha_1^{n,j}, \alpha_3^{n,j} \le b < 1$ for all $j = 1,2,...,N-1, k < c \le \alpha_1^{n,N} \le 1, k \le \alpha_3^{n,N} \le d < 1, k \le \alpha_2^{n,j} \le e < 1$ for all $j = 1,2,...N$. Assume that set Ω of solution of general hierarchical problem (1.5) is nonempty. For a mapping $f : C \to C$ is a contraction with $\gamma \in (0,1)$, sequence $\{\alpha_n\}, \{\beta_n\}$ are two real number in $(0,1)$ and assume that the following condition hold:*

(1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\lim_{n\to\infty} \frac{\beta_n}{\alpha_n} = 0$,

(2) $\sum_{n=1}^{\infty} \beta_n = \infty$,

(3) $\lim_{n\to\infty} \frac{1}{\beta_n}|\frac{1}{\alpha_n} - \frac{1}{\alpha_{n-1}}| = 0$, *and* $\lim_{n\to\infty} \frac{1}{\alpha_n}|1 - \frac{\beta_{n-1}}{\beta_n}| = 0$

(4) $\sum_{n=1}^{\infty} |\alpha_1^{n+1,j} - \alpha_1^{n,j}| < \infty, \sum_{n=1}^{\infty} |\alpha_3^{n+1,j} - \alpha_3^{n,j}| < \infty$ *for all* $j = \{1,2,3,\dots,N\}$,

(5) $\sum_{n=1}^{\infty} |\alpha_1^{n,j} - \alpha_1^{j}| < \infty, \sum_{n=1}^{\infty} |\alpha_2^{n,j} - \alpha_2^{j}| < \infty, \sum_{n=1}^{\infty} |\alpha_3^{n,j} - \alpha_3^{j}| < \infty$ *for all* $j = \{1,2,3,\dots,N\}$.

Then the sequence $\{x_n\}$ in (1.6) solve the following variational inequality:

$$\begin{cases} \tilde{x} \in \Omega \\ \langle (I-f)\tilde{x}, x - \tilde{x}\rangle \ge 0, \ \ \forall x \in \Omega. \end{cases} \tag{3.7}$$

Proof. From (1.6), let $y_n = \beta_n f(x_n) + (1-\beta_n)x_n$ and $x^* \in \Omega$ we have

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|\alpha_n S_n x_n + (1-\alpha_n) S_k y_n - x^* \\ &\le \alpha_n \|S_n x_n - x^*\| + (1-\alpha_n)\|S_k y_n - x^*\| \\ &\le \alpha_n \|x_n - x^*\| + (1-\alpha_n)\|y_n - x^*\|. \end{aligned} \tag{3.8}$$

Consider,

$$\begin{aligned} \|y_n - x^*\| &= \|\beta_n f(x_n) + (1-\beta_n)x_n - x^*\| \\ &\le \|\beta_n \gamma \|x_n - x^*\| + \|f(x^*) - x^*\| + (1-\beta_n)\|x_n - x^*\| \\ &= (1 - (1-\gamma)\beta_n)\|x_n - x^*\| + \|f(x^*) - x^*\|. \end{aligned} \tag{3.9}$$

From (3.8) and (3.9), we have

$$\begin{aligned} \|x_{n+1} - x^*\| &\le \alpha_n \|x_n - x^*\| + (1-\alpha_n)\big[(1-(1-\gamma)\beta_n)\|x_n - x^*\| + \|f(x^*) - x^*\|\big] \\ &\le \alpha_n \|x_n - x^*\| + (1-\alpha_n)\|x_n - x^*\| + (1-\alpha_n)\|f(x^*) - x^*\| \\ &= \|x_n - x^*\| + (1-\alpha_n)\|f(x^*) - x^*\| \\ &\le \max\{\|x_0 - x^*\|, \|f(x^*) - x^*\|\}. \end{aligned}$$

Then $\{x_n\}$ and $\{y_n\}$ are bounded and hence $\{f(x_n)\}, \{S_n x_n\}, \{S_\lambda y_n\}$ are also. Now we consider

$$\begin{aligned}\|y_n - y_{n-1}\| &= \|\beta_n f(x_n) - \beta_n f(x_{n-1}) + \beta_n f(x_{n-1}) - \beta_{n-1} f(x_{n-1}) + (1-\beta_n)x_n \\ &- (1-\beta_n)x_{n-1} + (1-\beta_n)x_{n-1} - (1-\beta_{n-1})x_{n-1}\| \\ &\le \beta_n\gamma\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|f(x_{n-1})\| + (1-\beta_n)\|x_n - x_{n-1}\| \\ &+ |\beta_n - \beta_{n-1}|\|x_{n-1}\| \\ &= (1-(1-\gamma)\beta_n)\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|(\|f(x_{n-1})\| + \|x_{n-1}\|).\end{aligned}$$

From definition of $\{x_n\}$ and nonexpansiveness of S_n, we have

$$\begin{aligned}\|x_n - x_{n-1}\| &= \|\alpha_n S_n x_n + (1-\alpha_n)S_\lambda y_n - \alpha_{n-1}S_{n-1}x_{n-1} + (1-\alpha_{n-1})S_\lambda y_{n-1}\| \\ &= \|\alpha_n S_n x_n - \alpha_n S_n x_{n-1} + \alpha_n S_n x_{n-1} - \alpha_{n-1}S_n x_{n-1} + \alpha_{n-1}S_n x_{n-1} \\ &\quad -\alpha_{n-1}S_{n-1}x_{n-1} + (1-\alpha_n)S_\lambda y_{n-1} + (1-\alpha_n)S_\lambda y_{n-1} - (1-\alpha_{n-1})S_\lambda y_{n-1}\| \\ &\le \alpha_n\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|S_n x_{n-1}\| + \alpha_{n-1}\|S_n x_{n-1} - S_{n-1}x_{n-1}\| \\ &\quad +(1-\alpha_n)\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|S_\lambda y_{n-1}\| \\ &\le \alpha_n\|x_n - x_{n-1}\| + (1-\alpha_n)\big[(1-(1-\gamma)\beta_n)\|x_n - x_{n-1}\| \\ &\quad +|\beta_n - \beta_{n-1}|(\|f(x_{n-1})\| + \|x_{n-1}\|)\big] + |\alpha_n - \alpha_{n-1}|(\|S_n x_{n-1}\| + \|S_\lambda y_{n-1}\|) \\ &\quad +\alpha_{n-1}\|S_n x_{n-1} - S_{n-1}x_{n-1}\| \\ &\le \big[\alpha_n + (1-\alpha_n)(1-(1-\gamma)\beta_n)\big]\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|(\|f(x_{n-1})\| + \|x_{n-1}\|) \\ &\quad +|\alpha_n - \alpha_{n-1}|(\|S_n x_{n-1}\| + \|S_\lambda y_{n-1}\|) + \alpha_{n-1}\|S_n x_{n-1} - S_{n-1}x_{n-1}\| \\ &= \big[1-(1-\gamma)\beta_n(1-\alpha_n)\big]\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|(\|f(x_{n-1})\| + \|x_{n-1}\|) \\ &\quad +|\alpha_n - \alpha_{n-1}|(\|S_n x_{n-1}\| + \|S_\lambda y_{n-1}\|) + \alpha_{n-1}\|S_n x_{n-1} - S_{n-1}x_{n-1}\|.\end{aligned}$$

Put $M = \sup\Big\{\|f(x_{n-1})\|, \|S_n x_{n-1}\|, \|S_\lambda y_{n-1}\|\Big\}, \quad n \ge 1$, it follows that

$$\begin{aligned}\|x_{n+1} - x_n\| &\le [1-(1-\gamma)\beta_n((1-\alpha_n)]\|x_n - x_{n-1}\| + (|\beta_n - \beta_{n-1}| + |\alpha_n - \alpha_{n-1}|)M \\ &+ \alpha_{n-1}\|S_n x_{n-1} - S_{n-1}x_{n-1}\|.\end{aligned}$$

Put $\delta_n = \|S_n x_{n-1} - S_{n-1}x_{n-1}\|$, from Lemma 3.1, we have $\Sigma_{n=1}^{\infty}\delta_n < \infty$, it follows that

$$\begin{aligned}\frac{\|x_{n+1} - x_n\|}{\alpha_n} &= [1-(1-\gamma)\beta_n(1-\alpha_n)]\frac{\|x_n - x_{n-1}\|}{\alpha_n} + \frac{|\beta_n - \beta_{n-1}|}{\alpha_n}M + \frac{|\beta_n - \beta_{n-1}|}{\alpha_n} + \alpha_{n-1}\frac{\delta_n}{\alpha_n} \\ &= [1-(1-\gamma)\beta_n(1-\alpha_n)]\frac{\|x_n - x_{n-1}\|}{\alpha_{n-1}} \\ &\quad +[1-(1-\gamma)\beta_n(1-\alpha_n)]\left(\frac{\|x_n - x_{n-1}\|}{\alpha_n} - \frac{\|x_n - x_{n-1}\|}{\alpha_{n-1}}\right) \\ &\quad +\frac{|\beta_n - \beta_{n-1}|}{\alpha_n}M + \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n}M + \alpha_{n-1}\frac{\delta_n}{\alpha_n}\end{aligned}$$

$$\begin{aligned}
&\leq [1-(1-\gamma)\beta_n(1-\alpha_n)]\frac{\|x_n-x_{n-1}\|}{\alpha_{n-1}} \\
&\quad +\Big(\Big|\frac{1}{\alpha_n}-\frac{1}{\alpha_{n-1}}\Big|+\frac{|\alpha_n-\alpha_{n-1}|}{\alpha_n}+\frac{|\beta_n-\beta_{n-1}|}{\alpha_n}+\frac{\delta_n}{\alpha_n}\Big)M \\
&= [1-(1-\gamma)\beta_n(1-\alpha_n)]\frac{\|x_n-x_{n-1}\|}{\alpha_{n-1}} \\
&\quad +(1-\gamma)\beta_n(1-\alpha_n)\Big\{\frac{M}{(1-\gamma)(1-\alpha_n)}\Big(\frac{1}{\beta_n}\Big|\frac{1}{\alpha_n}-\frac{1}{\alpha_{n-1}}\Big|+\frac{1}{\beta_n}\frac{|\alpha_n-\alpha_{n-1}|}{\alpha_n} \\
&\quad +\frac{1}{\beta_n}\frac{|\alpha_n-\alpha_{n-1}|}{\alpha_n}+\frac{1}{\beta_n}\frac{|\beta_n-\beta_{n-1}|}{\alpha_n}+\frac{\delta_n}{\alpha_n}\Big)\Big\}.
\end{aligned}$$

From Lemma 2.5, we obtain that

$$\lim_{n\to\infty}\frac{\|x_{n+1}-x_n\|}{\alpha_n}=0. \tag{3.10}$$

This implies that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.11}$$

From (1.6) and (3.11), we have that

$$\lim_{n\to\infty}\|x_n-S_\lambda y_n\|=0. \tag{3.12}$$

It follows that

$$y_n-x_n=\beta_n(f(x_n)-x_n)\to 0. \tag{3.13}$$

It implies that

$$\|y_n-S_\lambda y_n\|\leq\|y_n-x_n\|+\|x_n-S_\lambda y_n\|\to 0. \tag{3.14}$$

Since the sequence $\{x_n\}$ and $\{y_n\}$ are also bounded. Thus there exists a subsequence of $\{y_n\}$, which is still denoted by $\{y_{n_i}\}$ which converges weakly to a point $\tilde{x}\in H$. Therefore, $\tilde{x}\in Fix(T)$ by (1.6), we observe that

$$x_{n+1}-x_n=\alpha_n(S_nx_n-x_n)+(1-\alpha_n)(S_\lambda y_n-y_n)+(1-\alpha_n)\beta_n(fx_n-x_n),$$

that is,

$$\frac{x_n-x_{n+1}}{\alpha_n}=(I-S_n)x_n+\frac{1-\alpha_n}{\alpha_n}(I-S_\lambda)y_n+\frac{\beta_n(1-\alpha_n)}{\alpha_n}(I-f)x_n.$$

Set $z_n=\frac{(x_n-x_{n+1})}{\alpha_n}$ for each $n\geq 1$, that is

$$z_n=(I-S_n)x_n+\frac{1-\alpha_n}{\alpha_n}(I-S_\lambda)y_n+\frac{\beta_n(1-\alpha_n)}{\alpha_n}(I-f)x_n.$$

Using monotonicity of $I-S_\lambda$ and $I-S_n$, we derive that, for all $u \in Fix(T)$,

$$\begin{aligned}
\langle z_n, x_n - u\rangle &= \langle (I-S_n)x_n, x_n - u\rangle + \frac{1-\alpha_n}{\alpha_n}\langle (I-S_\lambda)y_n - (I-S_\lambda)u, y_n - u\rangle \\
&\quad + \frac{1-\alpha_n}{\alpha_n}\langle (I-S_\lambda)y_n, x_n - y_n\rangle + \frac{\beta_n(1-\alpha_n)}{\alpha_n}\langle (I-f)x_n, x_n-u\rangle \\
&\geq \langle (I-S_n)u, x_n - u\rangle + \frac{\beta_n(1-\alpha_n)}{\alpha_n}\langle (I-f)x_n, x_n-u\rangle \\
&\quad + \frac{(1-\alpha_n)\beta_n}{\alpha_n}\langle (I-S_\lambda y_n, x_n - fx_n)\rangle \\
&= \langle (I-S)u, x_n-u\rangle + \langle (S-S_n)u, x_n - u\rangle + \frac{\beta_n(1-\alpha_n)}{\alpha_n}\langle (I-f)x_n, x_n-u\rangle \\
&\quad + \frac{(1-\alpha_n)\beta_n}{\alpha_n}\langle (I-S_\lambda)y_n, x_n - fx_n\rangle.
\end{aligned}$$

But, since $z_n \to 0$, $\frac{\beta_n}{\alpha_n} \to 0$ and $\lim_{n\to\infty}\|S_nu - Su\| = 0$, it follows from the above inequality that

$$\limsup_{n\to\infty}\langle (I-S)u, x_n - u\rangle \leq 0, \quad \forall u \in Fix(T).$$

It suffices to guarantee that $\omega_w(x_n) \subset \Omega$. As a matter of fact, if we take any $x^* \in \omega_w(x_n)$, then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup x^*$. Therefore, we have

$$\langle (I-S)u, x^* - u\rangle = \lim_{j\to\infty}\langle (I-S)u, x_{n_j} - u\rangle \leq 0, \quad \forall u \in Fix(T).$$

Note that $x^* \in Fix(T)$. Hence x^* solves the following problem:

$$\begin{cases} x^* \in Fix(T) \\ \langle (I-S)u, x^*-u\rangle \geq 0, \ \forall u \in Fix(T). \end{cases}$$

It is obvious that this equivalent to the problem (1.5) by Lemma 3.2, we have $S_n \to S$ uniformly in any bounded set. Thus $x^* \in \Omega$. Let $\tilde{x}$ be the solution of the variational inequality (3.7), by Lemma 2.7 we have $\tilde{x}$ is unique. Now, take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle (I-f)\tilde{x}, x_n - \tilde{x}\rangle = \lim_{i\to\infty}\langle (I-f)\tilde{x}, x_{n_i} - \tilde{x}\rangle.$$

Without loss of generality, we can assume that $x_{n_i} \rightharpoonup x^*$. Then $x^* \in \Omega$. Therefore, we have

$$\limsup_{n\to\infty}\langle (I-f)\tilde{x}, x_n - \tilde{x}\rangle = \langle (I-f)\tilde{x}, x^* - \tilde{x}\rangle \geq 0.$$

This completes the proof. □

Theorem 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H, let T be a λ-strictly pseudo-contractive mapping and $\{T_i\}_{i=1}^N$ be a finite family of k_i-strictly pseudo-contractive mappings of C into itself for some $k_i \in [0,1)$ and $k = \max\{k_i : i = 1,2,...,N\}$ which $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let S_n be the S-mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1^n, \alpha_2^n, ..., \alpha_N^n$ where $\alpha_j^n = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}) \in I \times I \times I$, $I = [0,1], \alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$ and $k < a \leq \alpha_1^{n,j}, \alpha_3^{n,j} \leq b < 1$ for all $j = 1,2,...,N-1, k < c \leq \alpha_1^{n,N} \leq 1, k \leq \alpha_3^{n,N} \leq d < 1, k \leq \alpha_2^{n,j} \leq e < 1$ for all $j = 1,2,...N$. Assume that set Ω of solution of generalized hierarchical problem (1.5) is nonempty. For a mapping $f : C \to C$ is a contraction with $\gamma \in (0,1)$, sequence $\{\alpha_n\}, \{\beta_n\}$ are two real number in $(0,1)$ and assume that the following condition hold:*

(1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\lim_{n\to\infty} \frac{\beta_n}{\alpha_n} = 0$,

(2) $\sum_{n=1}^{\infty} \beta_n = \infty$,

(3) $\lim_{n\to\infty} \frac{1}{\beta_n}|\frac{1}{\alpha_n} - \frac{1}{\alpha_{n-1}}| = 0$ *and* $\lim_{n\to\infty} \frac{1}{\alpha_n}|1 - \frac{\beta_{n-1}}{\beta_n}| = 0$,

(4) $\sum_{n=1}^{\infty} |\alpha_1^{n+1,j} - \alpha_1^{n,j}| < \infty, \sum_{n=1}^{\infty} |\alpha_3^{n+1,j} - \alpha_3^{n,j}| < \infty$ *for all* $j = \{1,2,3,\ldots,N\}$,

(5) $\sum_{n=1}^{\infty} |\alpha_1^{n,j} - \alpha_1^j| < \infty, \sum_{n=1}^{\infty} |\alpha_2^{n,j} - \alpha_2^j| < \infty, \sum_{n=1}^{\infty} |\alpha_3^{n,j} - \alpha_3^j| < \infty$ *for all* $j = \{1,2,3,\ldots,N\}$,

(6) there exists a constant $d > 0$ such that $\|x - S_\lambda x\| \geq \rho Dist(x, F(S_\lambda))$, where

$$Dist(x, F(S_\lambda)) = \inf_{y \in F(S_\lambda)} \|x - y\|.$$

Then the sequence $\{x_n\}$ difined by (1.6) converges strongly to a point $\widetilde{x} \in Fix(T)$, which solve the variational inequality problem (3.7).

Proof. From (1.6), we have

$$x_{n+1} - \tilde{x} = \alpha_n(S_n x_n - S_n\tilde{x}) + \alpha_n(S_n\tilde{x} - \tilde{x}) + (1-\alpha_n)(S_\lambda y_n - \tilde{x}).$$

Thus we have

$$\begin{aligned} & \|x_{n+1} - \tilde{x}\|^2 \\ \leq\ & \|\alpha_n(S_n x_n - S_n\tilde{x}) + (1-\alpha_n)(S_\lambda y_n - \tilde{x})\|^2 + 2\alpha_n\langle S_n\tilde{x} - \tilde{x}, x_{n+1} - \tilde{x}\rangle \\ \leq\ & (1-\alpha_n)\|S_\lambda y_n - \tilde{x}\|^2 + \alpha_n\|S_n x_n - S_n\tilde{x}\|^2 + 2\alpha_n\langle S_n\tilde{x} - \tilde{x}, x_{n+1} - \tilde{x}\rangle \\ \leq\ & (1-\alpha_n)\|y_n - \tilde{x}\|^2 + \alpha_n\|x_n - \tilde{x}\|^2 + 2\alpha_n\langle S_n\tilde{x} - \tilde{x}, x_{n+1} - \tilde{x}\rangle. \end{aligned} \tag{3.15}$$

Now we consider

$$\begin{aligned} \|y_n - \tilde{x}\|^2 &= \|(1-\beta_n)(x_n - \tilde{x}) + \beta_n(fx_n - f\tilde{x}) + \beta_n(f\tilde{x} - \tilde{x})\|^2 \\ &\leq \|(1-\beta_n)(x_n - \tilde{x}) + \beta_n(fx_n - f\tilde{x})\|^2 + 2\beta_n\langle f\tilde{x} - \tilde{x}, y_n - \tilde{x}\rangle \\ &\leq (1-\beta_n)\|x_n - \tilde{x}\|^2 + \beta_n\|(fx_n - f\tilde{x})\|^2 + 2\beta_n\langle f\tilde{x} - \tilde{x}, y_n - \tilde{x}\rangle \\ &\leq (1-\beta_n)\|x_n - \tilde{x}\|^2 + \beta_n\gamma^2\|x_n - \tilde{x}\|^2 + 2\beta_n\langle f\tilde{x} - \tilde{x}, y_n - \tilde{x}\rangle \\ &= \big[1 - \big(1-\gamma^2\big)\beta_n\big]\|x_n - \tilde{x}\|^2 + 2\beta_n\langle f\tilde{x} - \tilde{x}, y_n - \tilde{x}\rangle. \end{aligned} \tag{3.16}$$

Substituting (3.16) into (3.15), we get

$$\begin{aligned}
& \|x_{n+1}-\tilde{x}\|^2 \\
= \ & \alpha_n\|x_n-\tilde{x}\|^2+(1-\alpha_n)\big[1-\big(1-\gamma^2\big)\beta_n\big]\|x_n-\tilde{x}\|^2 \\
& +2\beta_n(1-\alpha_n)\langle f\tilde{x}-\tilde{x},y_n-\tilde{x}\rangle+2\alpha_n\langle S_n\tilde{x}-\tilde{x},x_{n+1}-\tilde{x}\rangle \\
= \ & \big[1-\big(1-\gamma^2\big)\beta_n(1-\alpha_n)\big]\|x_n-\tilde{x}\|^2+2\beta_n(1-\alpha_n)\langle f\tilde{x}-\tilde{x},y_n-\tilde{x}\rangle \\
& +2\alpha_n\langle S_n\tilde{x}-\tilde{x},x_{n+1}-\tilde{x}\rangle \\
= \ & \big[1-\big(1-\gamma^2\big)\beta_n(1-\alpha_n)\big]\|x_n-\tilde{x}\|^2+\big(1-\gamma^2\big)\beta_n(1-\alpha_n) \\
& \times\bigg\{\frac{1}{1-\gamma^2}\langle f\tilde{x}-\tilde{x},y_n-\tilde{x}\rangle+\frac{2}{(1-\gamma^2)(1-\alpha_n)}\times\frac{\alpha_n}{\beta_n}\langle S_n\tilde{x}-\tilde{x},x_{n+1}-\tilde{x}\rangle\bigg\}. \quad (3.17)
\end{aligned}$$

By Theorem 3.3, we note that every weak cluster point of the sequence $\{x_n\}$ is in Ω. Since $y_n - x_n \to 0$, then every weak cluster point of $\{y_n\}$ is also in Ω. Consequently, since $\tilde{x} = proj_{\Omega}(f\tilde{x})$, we easily have

$$\limsup_{n\to\infty}\langle f\tilde{x}-\tilde{x},y_n-\tilde{x}\rangle\leq 0. \quad (3.18)$$

On the other hand, we observe that

$$\langle S_n\tilde{x}-\tilde{x},x_{n+1}-\tilde{x}\rangle=\Big\langle S_n\tilde{x}-\tilde{x},proj_{Fix(S_\lambda)}x_{n+1}-\tilde{x}\Big\rangle+\Big\langle S_n\tilde{x}-\tilde{x},x_{n+1}-proj_{Fix(S_\lambda)}x_{n+1}\Big\rangle$$

Since $\tilde{x}$ is a solution of the problem(1.5) and $proj_{Fix(S_\lambda)}x_{n+1} \in Fix(S_\lambda)$, we have

$$\big\langle S_n\tilde{x}-\tilde{x},proj_{Fix(S_\lambda)}x_{n+1}-\tilde{x}\big\rangle\leq 0.$$

Thus it follows that

$$\begin{aligned}
\langle S_n\tilde{x}-\tilde{x},x_{n+1}-\tilde{x}\rangle &\leq \langle S_n\tilde{x}-\tilde{x},x_{n+1}-proj_{Fix(S_\lambda)}x_{n+1}\rangle \\
&\leq \|S_n\tilde{x}-\tilde{x}\|\big\|x_{n+1}-proj_{Fix(S_\lambda)}x_{n+1}\big\| \\
&= \|S_n\tilde{x}-\tilde{x}\|\times Dist(x_{n+1},Fix(S_\lambda)) \\
&\leq \frac{1}{\rho}\|S_n\tilde{x}-\tilde{x}\|\|x_{n+1}-S_\lambda x_{n+1}\|.
\end{aligned}$$

We note that

$$\begin{aligned}
\|x_{n+1}-S_\lambda x_{n+1}\| &\leq \|x_{n+1}-S_\lambda x_n\|+\|S_\lambda x_n-S_\lambda x_{n+1}\| \\
&\leq \alpha_n\|S_nx_n-S_\lambda x_n\|+(1-\alpha_n)\|S_\lambda y_n-S_\lambda x_n\|+\|x_{n+1}-x_n\| \\
&\leq \alpha_n\|S_nx_n-S_\lambda x_n\|+\|y_n-x_n\|+\|x_{n+1}-x_n\| \\
&\leq \alpha_n\|S_nx_n-S_\lambda x_n\|+\beta_n\|fx_n-x_n\|+\|x_{n+1}-x_n\|.
\end{aligned}$$

Hence we have

$$\begin{aligned}\frac{\alpha_n}{\beta_n}\langle S_n\tilde{x}-\tilde{x}, x_{n+1}-\tilde{x}\rangle \quad \leq \quad & \frac{\alpha_n^2}{\beta_n}\left(\frac{1}{\rho}\|S_n\tilde{x}-\tilde{x}\|\|S_nx_n-S_\lambda x_n\|\right)\\ & +\alpha_n\left(\frac{1}{\rho}\|S_n\tilde{x}-\tilde{x}\|\|fx_n-x_n\|\right)\\ & +\frac{\alpha_n^2}{\beta_n}\frac{\|x_{n+1}-x_n\|}{\alpha_n}\left(\frac{1}{\rho}\|S_n\tilde{x}-\tilde{x}\|\right).\end{aligned}$$

From Theorem 3.3 we have $\lim_{n\to\infty}\frac{\|x_{n+1}-x_n\|}{\alpha_n}=0$. And then, we note that $\{\frac{1}{\rho}\|S_n\tilde{x}-\tilde{x}\|\|S_nx_n-S_\lambda x_n\|\}$, $\{\frac{1}{\rho}\|S_n\tilde{x}-\tilde{x}\|\|fx_n-x_n\|\}$, and $\{\frac{1}{\rho}\|S_n\tilde{x}-\tilde{x}\|\}$ are all bounded. Hence it follows from (1) and the above inequality that

$$\limsup_{n\to\infty}\frac{\alpha_n}{\beta_n}\langle S_n\tilde{x}-\tilde{x}, x_{n+1}-\tilde{x}\rangle \leq 0.$$

Finally, by (3.17) and Lemma 2.5, we conclude that the sequence $\{x_n\}$ converges strongly to a point $\tilde{x}\in Fix(S_\lambda)=Fix(T)$. This completes the proof. □

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References

[1] A. Moudafi, P. E. Mainge, Towards viscosity approximations of hierarchical fixed-point problems, Fixed Point Theory and Applications, vol. 2006, Article ID 95453, 10 pages, 2006.

[2] A. Moudafi, Krasnoselskii-Mann iteration for hierarchical fixed-point problems, Inverse Problems. 23 (4) (2007) 1635–1640.

[3] P. E. Mainge, A. Moudafi, Strong convergence of an iterative method for hierarchical fixed-point problems, Pacific Journal of Optimization. 3 (3) (2007) 529–538.

[4] X. Lu, H. K. Xu, X. Yin, Hybrid methods for a class of monotone variational inequalities, Nonlinear Analysis. 71 (3-4) (2009) 1032–1041.

[5] Y. Yao, Y. J. Cho, P. X. Yang, An Iterative Algorithm for a Hierarchical Problem, Hindawi Publishing Corporation Journal of Applied Mathematics, Volume 2012, Article ID 320421, 13 pages, doi:10.1155/2012/320421.

[6] G. Marino, H. K. Xu, Explicit hierarchical fixed point approach to variational inequalities, Journal of Optimization Theory and Applications. 149 (1) (2011) 61–78.

[7] H. K. Xu, Viscosity method for hierarchical fixed point approach to variational inequalities, Taiwanese Journal of Mathematics. 14 (2) (2009) 463–478.

[8] F. Cianciaruso, V. Colao, L. Muglia, and H. K. Xu, On an implicit hierarchical fixed point approach to variational inequalities, Bulletin of the Australian Mathematical Society. 80 (1) (2009) 117–124.

[9] G. Marino, H.K. Xu, Weak and strong convergence theorems for strictly pseudo-contractions in Hilbert space, J. Math. Anal. Appl. 329 (2007) 336–346.

[10] Y. Yao, Y. C. Liou, An implicit extragradient method for hierarchical variational inequalities, Fixed Point Theory and Applications, vol. 2011, Article ID 697248, 11 pages, 2011.

[11] Y. Yao, Y. C. Liou, C. P. Chen, Hierarchical convergence of a double-net algorithm for equilibrium problems and variational inequality problems, Fixed Point Theory and Applications, vol. 2010, Article ID 642584, 16 pages, 2010.

[12] Y. Yao, Y. J. Cho, Y. C. Liou, Iterative algorithms for hierarchical fixed points problems and variational inequalities, Mathematical and Computer Modelling. 52 (9-10) (2010) 1697–1705.

[13] F. Cianciaruso, G. Marino, L. Muglia, Y. Yao, On a two-step algorithm for hierarchical fixed point problems and variational inequalities, Journal of Inequalities and Applications, vol. 2009, Article ID 208692, 13 pages, 2009.

[14] G. Marino, V. Colao, L. Muglia, Y. Yao, Krasnoselskii-Mann iteration for hierarchical fixed points and equilibrium problem, Bulletin of the Australian Mathematical Society. 79 (2) (2009) 187–200.

[15] Y. Yao, Y. C. Liou, G. Marino, Two-step iterative algorithms for hierarchical fixed point problems and variational inequality problems, Journal of Applied Mathematics and Computing. 31 (1-2) (2009) 433–445.

[16] Y. Yao, Y. J. Cho, Y. C. Liou, Hierarchical convergence of an implicit double-net algorithm for nonexpanseive semigroups and variational inequality problems, Fixed Point Theory and Applications. 2011 (101) (2011).

[17] H. Zhou, Convergence theorems of fixed points for k-strict pseudocontractions in Hilbert space, Nonlinear Anal. 69 (2) (2008) 456–462.

[18] A. Kangtunyakarn, S. Suantai, Strong convergence of a new iterative scheme for a finite family of strict pseudo-contractions, Computers and Mathematics with Applications 60 (2010) 680 - 694.

[19] H. K. Xu, Iterative algorithm for nonlinear operator, Journal of the London Mathematical Society. 66 (1) (2002) 240–256.

[20] Z. Opial, Weak convergence of successive approximations for nonexpansive mappings, Bull. Amer. Math. Soc. 73 (1967) 591–597.

[21] K. Goebel, W. A. Kirk, Topics in Metric Fixed Point Theory, vol. 28 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, UK, 1990.

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