Thai Journal of Mathematics
Volume 12 (2014) Number 3 : 687–698
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

A Generalised Statistical Convergence

Gokulananda Das[†], B. K. Ray[†] and Sakambari Mishra[‡,1]

[†]Institute of Mathematics and Applications
Andharua, Bhubaneswar, Odisha, India
e-mail : gdas100@yahoo.com (G. Das)
raykbraja@yahoo.co.in (B.K. Ray)

[‡]Department of Mathematics
College of Basic Science and Humanities
O.U.A.T., Bhubaneswar, Odisha, India
e-mail : sakambarimishra@gmail.com

Abstract : The object of this present paper is to define and study generalised statistical convergence for the sequences in any locally convex Hausdorff space X whose topology is determined by a set Q of continuous seminorms q and their relation with the nearly convergent sequence space using a bounded modulus function along with regular and almost positive method.

Keywords : modulus function; natural density; regular and almost positive matrices; δ-nearly convergence; statistical convergence.
2010 Mathematics Subject Classification : 40A05; 40A35; 46A45.

---

1 Introduction and notations

The notion of statistical convergence was introduced by Fast [1] and subsequently has been investigated widely,whose basic idea depends on concept of density of a certain subset $E \subseteq N$, the set of natural number (see for example ([2, 3]); also for recent works see ([4–8]). Recall that the natural density of E is

[1]Corresponding author.

denoted by $\delta(E)$ and is defined by Freedman and Sember [9] as

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k) \tag{1.1}$$

where χ_E is the characteristic function of E. A real sequence $x = (x_k)$ is said to be statistical convergent to l, denoted by st-lim $x_k = l$, or $x_k \to l$ (stat), if for every $\epsilon > 0$, the set $E = \{k \le n : \ |x_k - l| \ge \epsilon\}$ has natural density zero.

The concept of statistical convergence has been extended by Maddox [10] for the sequences in any locally convex Hausdorff space X whose topology is determined by a set Q of continuous seminorms q in the following way.

The sequence $x = (x_k) \in X$ is called statistically convergent to $l \in X$ if for all $q \in Q$ and all $\epsilon > 0$,

$$n^{-1} \left|\{k \le n : q(x_k - l) \ge \epsilon\}\right| \to 0 \text{ as } n \to \infty \tag{1.2}$$

where vertical bar denotes the cardinality of the set enclosed, and when this happens, we write this as

$$x_k \to l(S) \tag{1.3}$$

Let $\mathcal{A} = (A^i)$ be the sequence of matrices $A^i = (a_{nk}(i))$ of complex numbers $\mathbb{C}$ and for a sequence $x = (x_k)$ we write

$$A_n^i(x) = \sum_{k=0}^{\infty} a_{nk}(i) x_k$$

if it exists for each n and $i \ge 0$. The sequence x is said to be summable to the value s by the method $\mathcal{A}$ if

$$\lim_{n\to\infty} A_n^i(x) = s \text{ uniformly in } i.$$

The method $\mathcal{A}$ is *conservative* [11] if and only if the following conditions hold:

$$(i)\ \|\mathcal{A}\| = \sup_{i,\ge 0, n\ge 0} \sum_{k=0}^{\infty} |a_{nk}(i)| < \infty$$

$$(ii)\ \exists\ a_k \in \mathbb{C} : \lim_{n\to\infty} a_{nk}(i) = a_k \text{ uniformly in } i$$

$$(iii)\ \exists\ a \in \mathbb{C} : \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk}(i) = a \text{ uniformly in } i$$

The method $\mathcal{A}$ is *regular*, if further $a_k = 0, a = 1$. We write

$$\alpha(\mathcal{A}) = a - \sum_{k=0}^{\infty} a_k. \tag{1.4}$$

The method $\mathcal{A}$ is called *co-null* if $\alpha(\mathcal{A}) = 0$; otherwise *co-regular*.
We write

$$a_{nk}^{+}(i) = \max(a_{nk}(i), 0), \quad a_{nk}^{-}(i) = \max(-a_{nk}(i), 0)$$

so that

$$\sum_{k=0}^{\infty} |a_{nk}(i)| = \sum_{k=0}^{\infty} a_{nk}^{+}(i) + \sum_{k=0}^{\infty} a_{nk}^{-}(i)$$

and

$$\sum_{k=0}^{\infty} a_{nk}(i) = \sum_{k=0}^{\infty} a_{nk}^{+}(i) - \sum_{k=0}^{\infty} a_{nk}^{-}(i)$$

The method $\mathcal{A}$ is called *almost positive* if and only if

$$\lim_{n} \sum_{k=0}^{\infty} a_{nk}^{-}(i) = 0, \text{uniformly in } i. \tag{1.5}$$

This definition is parallel to the definition for almost positive matrices given in [12]. Clearly a regular method $\mathcal{A}$ is *almost positive* if and only if

$$\lim_{n} \sum_{k=0}^{\infty} |a_{nk}(i)| = 1 \text{ uniformly in } i.$$

Before we can begin, it is necessary to introduce some most important definitions and notations.We first propose a density using the concept and axiomatic definition of lower asymptotic density presented by Freedman and Sember [9].

Let $\mathcal{A} = (a_{nk}(i))$ be almost positive and regular. Then for $E \subseteq N$, we write

$$\underline{\delta}_{\mathcal{A}}(E) = \liminf_{n} \inf_{i} \sum_{k} a_{nk}(i)\chi_E(k) \tag{1.6}$$

It is easily verified that $\underline{\delta}_{\mathcal{A}}(E)$ in (1.6), satisfies all the axioms provided by Freedman and Sember [9] to be a *lower asymptotic density* .So the *upper asymptotic density* associated by $\underline{\delta}_{\mathcal{A}}(E)$ is denoted as $\overline{\delta}_{\mathcal{A}}(E)$ and is given by

$$\overline{\delta}_{\mathcal{A}}(E) = 1 - \underline{\delta}_{\mathcal{A}}(E^{'}) = \limsup_{n} \sup_{i} \sum_{k} a_{nk}(i)\chi_E(k) \tag{1.7}$$

where $E^{'}$ is the complement of E. When the above lower and upper asymptotic density coincide,we write it as $\delta_{\mathcal{A}}(E)$, that is, $\underline{\delta}_{\mathcal{A}}(E) = \delta_{\mathcal{A}}(E) = \overline{\delta}_{\mathcal{A}}(E)$, then it is easily seen that,

$$\delta_{\mathcal{A}}(E) = \lim_{n} \sum_{k=0}^{\infty} a_{nk}(i)\chi_E(k) \tag{1.8}$$

exists uniformly in i and $\delta_{\mathcal{A}}(E)$ is said to be a *generalised density* obtained from $\mathcal{A}$ or just $\mathcal{A}$-density.

In this case, $0 \leq \delta_{\mathcal{A}}(E) \leq 1$. Let $E_\epsilon = \{k \in N : q(x_k - l) \geq \epsilon\}; \quad q \in Q$. Thus using the concept of Maddox [10] on statistical convergence of a sequence $x = (x_k) \in X$ is said to be *$\mathcal{A}$-statistical convergent* to $l \in X$ provided that for every $\epsilon > 0, \delta_{\mathcal{A}}(E_\epsilon) = 0$.

In that case we write this as

$$x_k \to l(\mathcal{A} - \text{stat}) \tag{1.9}$$

and we denote $S_{\mathcal{A}}$ as the set of all *$\mathcal{A}$-statistical convergent sequences* in X . Now we write $x_k \underset{E_\epsilon}{\longrightarrow} l$ to mean that for each $\epsilon > 0$, there exists integer $n_0 > 0$ such that $q(x_k - l) < \epsilon$ whenever $k \geq n_0, k \notin E_\epsilon, q \in Q$.

In the case $\mathcal{A} = A = (C, 1)$, (the Cesaro matrix of order one), and q(x)=$|x|$ the $\mathcal{A}$-statistical convergence reduces to usual definition of statistical convergence. Let W be set of all real sequences $x = (x_k) \in X$. For any density $\delta_{\mathcal{A}}$, let

$$W_{\delta_{\mathcal{A}}} = \left\{ x_k \in W : \text{ there exist } l \in X \text{ and } E_\epsilon \subseteq N \text{ with } \overline{\delta}_{\mathcal{A}}(E_\epsilon) = 0 \text{ and } x_k \underset{E_\epsilon}{\longrightarrow} l \right\}.$$

Then $W_{\delta_{\mathcal{A}}}$ is called the space of *$\delta_{\mathcal{A}}$-nearly convergent sequences* and if $x \in W_{\delta_{\mathcal{A}}}$, then we denote

$$x_k \to l(\delta_{\mathcal{A}} - \text{nearly}). \tag{1.10}$$

The above definitions can be considered for real sequence $x = (x_k)$ by using the fact $q(x) = |x|$, in the set E_ϵ

Before we state the theorem of Maddox on *characterisation* of statistical convergence in Hausdorff space we give the following definition.

Recall [10, 13] that a modulus function $f : [0, \infty) \to [0, \infty)$ satisfies

i) $f(x) = 0$ if and only if $x = 0$;

ii) $f(x + y) \leq f(x) + f(y)$ for $x \geq 0, y \geq 0$;

iii) f is increasing;

iv) f is continuous from the right of 0.

A modulus function may be unbounded or bounded; for example: $f(x) = x^p$ where $0 < p < 1$ is unbounded, but $f(x) = \frac{x}{1+x}$ is bounded. Now suppose that we are given a modulus function f .For $(x_k) \in X$ we say that $x = (x_k) \in W_{\mathcal{A}}(f)$ if and only if there exist $l \in X$ such that

$$\lim_n \sum_{k=0}^{\infty} a_{nk}(i) f(q(x_k - l)) = 0 \text{ uniformly in } i, \quad q \in Q. \tag{1.11}$$

When (1.11) holds we write

$$x_k \to l(W_{\mathcal{A}}(f)). \tag{1.12}$$

We also represent (1.11) as $f(q(x_k - l)) \to 0(\mathcal{A})$. The sequences which satisfy (1.12), we call them as *$W_{\mathcal{A}}(f)$-convergent sequences*.

Maddox [10] proved the following theorem.

Theorem 1.1 ([10]). *Let f be a bounded modulus function. A sequence $x \in X$ is statistically convergent to $l \in X$ if and only if*

$$\frac{1}{n}\sum_{k=1}^{n} f(q(x_k - l)) \to 0 \quad (n \to \infty)$$

and for all $q \in Q$.

2 Main Results

Now we shall first prove a theorem to generalize the above result of Maddox, and then we shall correlate it to nearly convergent sequences in any locally convex Hausdorff space X whose topology is determined by a set Q of continuous seminorms q.

Theorem 2.1. *Let f be a bounded modulus function and let $\mathcal{A} = (a_{nk}(i))$ be regular and almost positive. Then for $x = (x_k), l \in X$, $x_k \to l(\mathcal{A}$-stat) if and only if*

$$f(q(x_k - l)) \to 0(\mathcal{A}) \tag{2.1}$$

Proof. Suppose that $x_k \to l(\mathcal{A}$-stat), that is, $\lim_n \sum_{k=0}^{\infty} a_{nk}(i)\chi_{E_\epsilon}(k) = 0$ uniformly in i for $x_k, l \in X$ where $E_\epsilon = \{k \in N : q(x_k - l) \geq \epsilon\}$.

Now since $\mathcal{A}$ is almost positive, (2.1) is equivalent to:

$$\lim_n \sum_{k=0}^{\infty} a^+_{nk}(i) f(q(x_k - l)) = 0, \quad \text{uniformly in } i. \tag{2.2}$$

Now

$$\begin{aligned}
&\sum_{k=0}^{\infty} a^+_{nk}(i) f(q(x_k - l)) \\
&\qquad = \sum_{k=0}^{\infty} a^+_{nk}(i) f(q(x_k - l))\chi_{E_\epsilon}(k) + \sum_{k=0}^{\infty} a^+_{nk}(i) f(q(x_k - l))\chi_{E'_\epsilon}(k) \\
&\qquad \leq \sup f \left\{\sum_{k=0}^{\infty} a^+_{nk}(i)\chi_{E_\epsilon}(k)\right\} + \left\{\sum_{k=0}^{\infty} a^+_{nk}(i)\chi_{E'_\epsilon}(k)\right\} f(\epsilon) \\
&\qquad \to 0 \text{ as } n \to \infty \text{ and } \epsilon \to 0
\end{aligned}$$

uniformly in i.

Conversely let

$$\sum_{k=0}^{\infty} a_{nk}(i) f(q(x_k - l)) \to 0 \text{ as } n \to \infty$$

uniformly in i. Now

$$\sum_{k=0}^{\infty} a_{nk}^{+}(i) f(q(x_k - l)) \geq \sum_{k=0}^{\infty} a_{nk}^{+}(i) f(q(x_k - l)) \chi_{E_\epsilon}(k)$$
$$\geq f(\epsilon) \sum_{k=0}^{\infty} a_{nk}^{+}(i) \chi_{E_\epsilon}(k)$$

As $\sum_{k=0}^{\infty} a_{nk}^{+}(i) f(q(x_k - l)) \to 0$ as $n \to \infty$ uniformly in i, so $\sum_{k=0}^{\infty} a_{nk}^{+}(i) \chi_{E_\epsilon}(k) \to 0$ as $n \to \infty$ uniformly in i, i.e.,

$$\delta_{\mathcal{A}}(E_\epsilon) = 0.$$

Hence $x_k \to l(\mathcal{A}\text{-stat})$. □

Corollary 2.2. *Let $\mathcal{A} = (a_{nk}(i))$ be regular and almost positive. Then $x_k \to 0(\mathcal{A}$-stat) if and only if*

$$\lim_{n} \sum_{k=0}^{\infty} a_{nk}(i) \frac{|x_k|}{1 + |x_k|} = 0 \text{ uniformly in } i.$$

Proof. Taking $f(x) = \dfrac{x}{1+x}$ and $q(x) = |x|$ in Theorem 2.1 with $l = 0$ we obtain the corollary. □

We now establish an important relation for $\delta_{\mathcal{A}}$-*nearly convergent sequences*, with $W_{\mathcal{A}}(f)$- *convergent sequences* over certain class of matrices in our next theorem.

Theorem 2.3. *Let f be a bounded modulus function. Let $\mathcal{A} = (a_{nk}(i))$ be co-regular and $\mathcal{B} = \left(\frac{a_{nk}(i) - a_k}{\alpha(\mathcal{A})}\right) = (b_{nk}(i))$ be almost positive. Then for $x = (x_k), l \in X$, $x_k \to l(\delta_{\mathcal{B}}-$nearly) if and only if $f(q(x_k - l)) \to o(\mathcal{B})$.*

Proof. Necessity: By hypothesis, $\mathcal{B}$ is regular and almost positive. Let $x_k \to l(\delta_{\mathcal{B}}-$nearly), i.e., there exists a set $E_\epsilon \subseteq N$ such that $x_k \underset{E_\epsilon}{\longrightarrow} l$ and $\overline{\delta}_{\mathcal{B}}(E_\epsilon) = 0$. We have, as $\mathcal{B}$ is almost positive,

$$0 = \overline{\delta}_B(E_\epsilon) = \limsup_n \ \sup_i \sum_{k=0}^{\infty} b_{nk}(i) \chi_{E_\epsilon}(k)$$
$$= \limsup_n \ \sup_i \sum_{k=0}^{\infty} b_{nk}^{+}(i) \chi_{E_\epsilon}(k). \tag{2.3}$$

As $x_k \underset{E_\epsilon}{\longrightarrow} l$, then for $\epsilon > 0$, there exists $n_0 \in N$ such that

$$q(x_k - l) < \epsilon, \ k \geq n_0, k \notin E_\epsilon. \tag{2.4}$$

Write

$$1 = \chi_{E_\epsilon}(k) + \chi_{E'_\epsilon}(k) = \chi_{E_\epsilon}(k) + \chi_{E'_\epsilon \cap G}(k) + \chi_{E'_\epsilon \cap G'}(k)$$

where $G = \{k \in N | q(x_k - l) \geq \epsilon\}$. So we have

$$\begin{aligned}
&\sum_{k=0}^{\infty} b^+_{nk}(i) f(q(x_k - l)) \\
&\quad = \sum_{k=0}^{\infty} b^+_{nk}(i) f(q(x_k - l)) \chi_{E_\epsilon}(k) + \sum_{k=0}^{\infty} b^+_{nk}(i) f(q(x_k - l)) \chi_{E'_\epsilon \cap G}(k) \\
&\qquad + \sum_{k=0}^{\infty} b^+_{nk}(i) f(q(x_k - l)) \chi_{E'_\epsilon \cap G'}(k) \\
&\quad = \sum_1 + \sum_2 + \sum_3 . \qquad (2.5)
\end{aligned}$$

Now by (2.3) and since $f(q(x_k - l)) \leq \sup f$ for $x_k \in X$ as f is bounded

$$\begin{aligned}
\limsup_n \ \sup_i \sum_1 &= \limsup_n \ \sup_i \sum_{k=0}^{\infty} b^+_{nk}(i) f(q(x_k - l)) \chi_{E_\epsilon}(k) \\
&\leq \sup f \{ \limsup_n \ \sup_i \sum_{k=0}^{\infty} b^+_{nk}(i) \chi_{E_\epsilon}(k) \} = 0. \qquad (2.6)
\end{aligned}$$

Again using (2.4) for the second sum, we get

$$\begin{aligned}
\limsup_n \ \sup_i \sum_2 &= \limsup_n \ \sup_i \sum_{k=0}^{\infty} b^+_{nk}(i) f(q(x_k - l)) \chi_{E'_\epsilon \cap G}(k) \\
&= \limsup_n \ \sup_i \sum_{k \geq n_0} b^+_{nk}(i) f(q(x_k - l)) \chi_{E'_\epsilon \cap G}(k) \\
&\quad + \limsup_n \ \sup_i \sum_{k < n_0} b^+_{nk}(i) f(q(x_k - l)) \chi_{E'_\epsilon \cap G}(k) \\
&\leq f(\epsilon) \limsup_n \ \sup_i \sum_{k \geq n_0} b^+_{nk}(i) \chi_{E'_\epsilon \cap G}(k) \\
&\quad + \sup f \left\{ \limsup_n \ \sup_i \sum_{k < n_0} b^+_{nk}(i) \chi_{E'_\epsilon \cap G}(k) \right\} \\
&< f(\epsilon) \limsup_n \ \sup_i \sum_{k=0}^{\infty} b^+_{nk}(i) \\
&\quad + \sup f \left\{ \limsup_n \ \sup_i \sum_{k < n_0} b^+_{nk}(i) \right\}. \qquad (2.7)
\end{aligned}$$

The first term in (2.7) refers to just $f(\epsilon)$ and the last term to zero. Therefore

$$\limsup_n \ \sup_i \sum_2 < f(\epsilon). \tag{2.8}$$

Now

$$\begin{aligned}\limsup_n \ \sup_i \sum_3 &= \limsup_n \ \sup_i \sum_{k=0}^{\infty} b_{nk}^{+}(i) f(q(x_k - l)) \chi_{E'_\epsilon \cap G'}(k) \\ &< f(\epsilon) \limsup_n \ \sup_i \sum_{k=0}^{\infty} b_{nk}^{+}(i) \chi_{E'_\epsilon \cap G'}(k) \\ &\le f(\epsilon) \limsup_n \ \sup_i \sum_{k=0}^{\infty} b_{nk}^{+}(i) \chi_{E'_\epsilon}(k) \\ &< f(\epsilon) \limsup_n \ \sup_i \sum_{k=0}^{\infty} b_{nk}^{+}(i) \\ &= f(\epsilon). \end{aligned} \tag{2.9}$$

As ϵ is arbitrary and $f(\epsilon) \to 0$ as $\epsilon \to 0^+$, from (2.5)-(2.9), it follows that

$$\limsup_n \ \sup_i \sum_{k=0}^{\infty} b_{nk}^{+}(i) f(q(x_k - l)) = 0$$

which ensures

$$\limsup_n \ \sup_i \sum_{k=0}^{\infty} b_{nk}(i) f(q(x_k - l)) = 0.$$

Sufficiency: Define

$$E_\alpha = \{k \in N : q(x_k - l) \ge \alpha > 0\}.$$

So

$$\sum_{k=0}^{\infty} b_{nk}^{+}(i) f(q(x_k - l)) \ge \sum_{k=0}^{\infty} b_{nk}^{+}(i) f(q(x_k - l)) \chi_{E_\alpha}(k) \ge f(\alpha) \sum_{k=0}^{\infty} b_{nk}^{+}(i) \chi_{E_\alpha}(k).$$

Now

$$\sum_{k=0}^{\infty} b_{nk}^{+}(i) \chi_{E_\alpha}(k) \le \frac{1}{f(\alpha)} \sum_{k=0}^{\infty} b_{nk}^{+}(i) f(q(x_k - l)) \to 0$$

as $n \to \infty$ uniformly in i (by hypothesis). Hence for $\alpha > 0$,

$$\begin{aligned}\overline{\delta}_{\mathcal{B}}(E_\alpha) &= \limsup_n \ \sup_i \sum_{k=0}^{\infty} b_{nk}^{+}(i) \chi_{E_\alpha}(k) \\ &= 0. \end{aligned} \tag{2.10}$$

Let

$$E_1 = \{k \in N : q(x_k - l) \geq 1\}$$
$$E_2 = \left\{k \in N : q(x_k - l) \geq \frac{1}{2}\right\}$$
$$\vdots$$
$$E_j = \left\{k \in N : q(x_k - l) \geq \frac{1}{j}\right\}.$$

By (2.10)

$$\overline{\delta}_{\mathcal{B}}(E_1) = \overline{\delta}_{\mathcal{B}}(E_2) = \cdots = 0.$$

Hence

$$\underline{\delta}_{\mathcal{B}}(E_1^{'}) = \underline{\delta}_{\mathcal{B}}(E_2^{'}) = \cdots = 1.$$

Note that $E_1^{'} \supset E_2^{'} \supset \cdots \supset E_j^{'} \supset \cdots$. Choose

$$\nu_1 \in E_1^{'} = \{k \in N : q(x_k - l) < 1\}.$$

Since

$$\underline{\delta}_{\mathcal{B}}(E_1^{'}) = \liminf_n \ \inf_i \sum_{k=0}^{\infty} b_{nk}^{+}(i)\chi_{E_1^{'}}(k) = 1$$

choose integer $\nu_2 > \nu_1, \nu_2 \in E_2^{'} = \left\{k \in N : q(x_k - l) < \frac{1}{2}\right\}$ such that for $n > \nu_2$

$$\sum_{k=0}^{\infty} b_{nk}^{+}(i)\chi_{E_2^{'}}(k) > \frac{1}{2}.$$

Similarly, there exists $\nu_3 > \nu_2, \nu_3 \in E_3^{'}$ such that for $n > \nu_3$

$$\sum_{k=0}^{\infty} b_{nk}^{+}(i)\chi_{E_3^{'}}(k) > \frac{2}{3}.$$

Continuing in the same manner, there exist $\nu_j > \nu_{j-1} \in E_j^{'}$ such that for $n > \nu_j$

$$\sum_{k=0}^{\infty} b_{nk}^{+}(i)\chi_{E_j^{'}}(k) > \frac{j-1}{j}.$$

Let

$$G = \{n \in N : \ 1 \leq n \leq \nu_1 \ \vee (\nu_j \leq n \leq \nu_{j+1} \wedge n \in E_j^{'}, j = 1, 2, \ldots)\}$$
$$= \{n \in N : \ 1 \leq n \leq \nu_1\} \cup_{j=1}^{\infty} \{n \in E_j^{'} | \nu_j \leq n \leq \nu_{j+1}\}.$$

Claim 1:

$$\underline{\delta}_{\mathcal{B}}(G) = \liminf_n \ \inf_i \sum_{k=0}^{\infty} b_{nk}^{+}(i)\chi_G(k) = 1. \tag{2.11}$$

Since $G \supset E_j^{'} \; \forall \; j$

$$\sum_{k=0}^{\infty} b_{nk}^{+}(i)\chi_G(k) > \sum_{k=0}^{\infty} b_{nk}^{+}(i)\chi_{E_j^{'}}(k)$$

$$\Rightarrow \liminf_n \; \inf_i \sum_{k=0}^{\infty} b_{nk}^{+}(i)\chi_G(k) > \liminf_n \; \inf_i \sum_{k=0}^{\infty} b_{nk}^{+}(i)\chi_{E_j^{'}}(k) = 1.$$

Also

$$\liminf_n \; \inf_i \sum_{k=0}^{\infty} b_{nk}^{+}(i)\chi_G(k) \leq 1.$$

It follows that the claim (2.11) is established.

Claim 2: If $E = G^{'}$ then

$$x_n \underset{E}{\longrightarrow} l \text{ and } \overline{\delta}_{\mathcal{B}}(E) = 0. \tag{2.12}$$

By (2.11)

$$\overline{\delta}_{\mathcal{B}}(E) = 1 - \underline{\delta}_{\mathcal{B}}(E^{'}) = 1 - \underline{\delta}_{\mathcal{B}}(G) = 0.$$

Let $\epsilon > 0$, we choose a positive integer j (keep it fixed) such that $\frac{1}{j} < \epsilon$. If $\nu_j \leq n \leq \nu_{j+1}$ and $n \notin E$, then $n \in [\nu_j, \nu_{j+1}] \cap G$ which implies that $n \notin E_j$ by the definition of G. Consequently

$$q(x_n - l) < \frac{1}{j} < \epsilon. \tag{2.13}$$

Now we consider the case when $n \in [\nu_{j+k}, \nu_{j+k+1}], k \in N$ and $n \notin E$. Proceeding as above it can be shown that if $\nu_{j+k} \leq n \leq \nu_{j+k+1}$ and $n \notin E$ then

$$q(x_n - l) < \frac{1}{j+k} < \frac{1}{j} < \epsilon. \tag{2.14}$$

As (2.14) holds for every $k \in N$, it follows from (2.13) and (2.14) that

$$q(x_n - l) < \epsilon \text{ for } n \geq n_j \text{ and } n \notin E$$

which ensures (2.12). So sufficiency follows. □

By taking $f(x) = \dfrac{x}{1+x}$ and $q(x) = |x|$ in Theorem 2.3, we obtain the following corollary.

Corollary 2.4. *Let $\mathcal{A} = (a_{nk}(i))$ be coregular and $\mathcal{B} = \left(\frac{a_{nk}(i) - a_k}{\alpha(\mathcal{A})}\right) = (b_{nk}(i))$ be almost positive. Then $x_k \to l$($\delta_{\mathcal{B}}$-nearly) if and only if*

$$\sum_{k=0}^{\infty} b_{nk}(i) \frac{|x_k - l|}{1 + |x_k - l|} \to 0 \text{ as } n \to \infty \text{ uniformly in } i.$$

Theorem 2.5. *Let f be a bounded modulus function with $x = (x_k), l \in X$. Let $\mathcal{A} = (a_{nk}(i))$ be regular and almost positive. Then $S_{\mathcal{A}} = W_{\mathcal{A}}(f) = W_{\delta_{\mathcal{A}}}$.*

Proof. The proof follows immediately from Theorem 2.1 and Theorem 2.3. □

Acknowledgement : The authors are thankful to the referees for their comments and suggestions on the manuscript.

References

[1] H. Fast, Sur la convergence statistique, Colloq. Math. 2 (1951) 241–244.

[2] J.A. Fridy, On Statistical Convergence, Analysis 5 (1985) 301–313.

[3] T. Salat, On statistical convergent sequences of real numbers, Math. Slovaca 30 (2) (1980) 139–150.

[4] S.A. Mohiuddine, Q.M. Danish Lohani, On generalized statistical convergence in intuitionistic fuzzy normed space, Chaos, Solitons and Fractals 42 (2009) 1731–1737.

[5] S.A. Mohiuddine, A. Alotaibi, M. Mursaleen, Statistical convergence of double sequences in locally solid Reisz spaces, Abstact and Applied Analysis, Volume 2012 (2012), Article ID 719729, 9 pages.

[6] S.A. Mohiuddine, M.A. Alghamdi, Statistical summability through a lacunary sequence in locally solid Riesz spaces, Journal of Inequalities and Applications 2012, 2012:225.

[7] S.A. Mohiuddine, M. Aiyub, Lacunary statistical convergence in random 2-normed Spaces, Appl. Math. Inf. Sci. 6 (3) (2012) 581–585.

[8] S.A. Mohiuddine, E. Savas, Lacunary statistically convergent double sequences in probabilistic normed spaces, Ann. Univ. Ferrara 58 (2012) 331–339

[9] A.R. Freedman, J.J. Sember, Densities and Summability, Pacific J. Math. 95 (1981) 293–305.

[10] I.J. Maddox, Statistical convergence in a locally convex space, Math. Proc. Camb. Phil. Soc. 104 (1988) 141–145.

[11] M. Stieglitz, Eine Verallgemeinerung des Begriffs der Fastkonvergenz, Math. Japon. 18 (1973) 53–70.

[12] S. Simons, Banach limits, Infinite matrices and sublinear functionals, J. Math. Anal. Appl. 26 (1969) 640–655.

[13] I.J. Maddox, Sequence space defined by a modulus, Math. Proc. Cambridge Philos. Soc. 100 (1986) 161–166.

(Received 8 November 2012)
(Accepted 26 March 2013)

THAI J. MATH. Online @ http://thaijmath.in.cmu.ac.th