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Coupled Coincidence Point Theorems in Partially Ordered Metric Spaces

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Abstract : In this paper we have shown that under certain assumptions an arbitrary family of mappings will have a coupled coincidence point with another function in a partially ordered metric space. The main results have several corollaries. Several existing results are improved by the main results. Two examples are given.

Keywords : coupled coincidence point; partially ordered set; mixed g-monotone property; compatible mapping.

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1 Introduction

In recent times, fixed point theory has developed rapidly in partially ordered metric spaces. An early result in this direction was established by Turinici in ordered metrizable uniform spaces [1]. Application of fixed point results in partially ordered metric spaces were made subsequently, for example, by Ran and Reurings [2] to solving matrix equations and by Nieto and Rodŕiguez-López [3] to obtain solutions of certain partial differential equations with periodic boundary conditions. Some more recent references in which new fixed point results have been obtained in such spaces are noted in [4–10].

Coupled fixed point problems constitute a special category of problems in fixed point theory. In their paper Bhaskar and Lakshmikantham [11] established a coupled contraction mapping principle in partially ordered metric spaces for mapping having mixed monotone property. An application of their result to differential equations has also been given in the same work. This result was further generalized to coupled coincidence point theorems in [12] and [13] under two separate sets of sufficient conditions. Several other coupled fixed and coincidence point results were proved in works like those noted in references [14–22].

Common fixed point results for commuting mappings in metric spaces were deduced by Jungck [23]. The concept of commuting has been weakened in various directions and in several ways over the years. One such notion which is weaker than commuting is the concept of compatibility introduced by Jungck [24]. In common fixed point problems, this concept and its generalizations have been used extensively. References [25–30] are some examples of such works. Recently, in [13] the concept of compatibility has been introduced in the context of coupled coincidence point problems. Further coupled coincidence point results using compatibility has been obtained in [17].

In this paper we establish three coupled coincidence point theorems for an arbitrary family of mappings $\{F_\alpha : X \times X \longrightarrow X : \alpha \in \Lambda\}$ with a mapping $g : X \to X$ where $(X,\ d)$ is a metric space with a partial ordering. We have used a control function. Khan et al. [31] initiated the use of a control function in metric fixed point theory, which they called an Altering distance function. This function and its generalizations have been used in fixed and coincidence point problems in a large number of works, some of these works are in [17, 30, 32–35].

Our results extend some existing results.

2 Mathematical Preliminaries

Let $(X,\ \preceq)$ be a partially ordered set and $F : X \longrightarrow X$. The mapping F is said to be nondecreasing if for all $x_1,\ x_2 \in X$, $x_1 \preceq x_2$ implies $F(x_1) \preceq F(x_2)$ and nonincreasing if for all $x_1,\ x_2 \in X$, $x_1 \preceq x_2$ implies $F(x_1) \succeq F(x_2)$.

Definition 2.1 ([11])**.** Let $(X, \preceq)$ be a partially ordered set and $F\colon X \times X \longrightarrow X$. The mapping F is said to have the mixed monotone property if F is monotone nondecreasing in its first argument and is monotone nonincreasing in its second argument, that is, if

$$x_1,\ x_2 \in X,\quad x_1 \preceq x_2 \Longrightarrow F(x_1,\ y) \preceq F(x_2,\ y),\quad \text{for all}\ \ y \in X$$

and

$$y_1,\ y_2 \in X,\quad y_1 \preceq y_2 \Longrightarrow F(x,\ y_1) \succeq F(x,\ y_2),\quad \text{for all}\ \ x \in X.$$

Definition 2.2 ([12])**.** Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$. We say that F has the mixed g- monotone property if

$$x_1,\ x_2 \in X,\quad gx_1 \preceq gx_2 \Longrightarrow F(x_1,\ y) \preceq F(x_2,\ y),\quad \text{for all}\ \ y \in X$$

and

$$y_1,\ y_2 \in X,\quad gy_1 \preceq gy_2 \Longrightarrow F(x,\ y_1) \succeq F(x,\ y_2),\quad \text{for all}\ \ x \in X.$$

Definition 2.3 ([11])**.** An element $(x,\ y) \in X \times X$, is called a coupled fixed point of the mapping $F : X \times X \longrightarrow X$ if

$$F(x,\ y) = x \ \text{ and }\ F(y,\ x) = y.$$

Definition 2.4 ([12])**.** An element $(x,\ y) \in X \times X$, is called a coupled coincidence point of the mappings $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ if

$$F(x,\ y) = gx \ \text{ and }\ F(y,\ x) = gy.$$

Definition 2.5 ([13])**.** The mappings g and F, where $g : X \longrightarrow X$ and $F : X \times X \longrightarrow X$, are said to be compatible if

$$\lim_{n\to\infty} d(gF(x_n,\ y_n),\ F(gx_n,\ gy_n)) = 0$$

and

$$\lim_{n\to\infty} d(gF(y_n,\ x_n),\ F(gy_n,\ gx_n)) = 0,$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n\to\infty} F(x_n,\ y_n) = \lim_{n\to\infty} gx_n = x$ and $\lim_{n\to\infty} F(y_n,\ x_n) = \lim_{n\to\infty} gy_n = y$, for some $x,\ y \in X$ are satisfied.

Definition 2.6 ([31])**.** A function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties are satisfied:

(i) ψ is monotone increasing and continuous;

(ii) $\psi(t) = 0$ if and only if $t = 0$.

Definition 2.7 (**P - property**)**.** Let $(X,\ \preceq)$ be a partially ordered set and d be a metric on X. Then X is said to have P- property if $x_n \longrightarrow x$ is a nondecreasing sequence, then $x_n \preceq x$, for all $n \geq 0$; and if $y_n \longrightarrow y$ is a nonincreasing sequence, then $y \preceq y_n$, for all $n \geq 0$.

Theorem 2.8 ([11])**.** Let $(X, \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \longrightarrow X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists $k \in [0, 1)$ such that for all $x \succeq u, y \preceq v$,

$$d(F(x, y), F(u, v)) \leq \frac{k}{2} [d(x, u) + d(y, v)].$$

If there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Theorem 2.9 ([11])**.** Let $(X, \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Assume that X has the following property:

(i) if a nondecreasing sequence $\{x_n\} \longrightarrow x$, then $x_n \preceq x$, for all n;

(ii) if a nonincreasing sequence $\{y_n\} \longrightarrow y$, then $y \preceq y_n$, for all n.

Let $F : X \times X \longrightarrow X$ be a mapping having the mixed monotone property on X. Assume that there exists $k \in [0, 1)$ such that for all $x \succeq u, y \preceq v$,

$$d(F(x, y), F(u, v)) \leq \frac{k}{2} [d(x, u) + d(y, v)].$$

If there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Theorem 2.10 ([20])**.** Let $(X, \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \longrightarrow X$ be a mapping having the mixed monotone property on X such that there exist two elements $x_0, y_0 \in X$ with $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$. Suppose there exist non-negative real numbers α, β and L with $\alpha + \beta < 1$ such that

$$\begin{aligned} d(F(x, y), F(u, v)) \leq \alpha\, d(x, u) + \beta\, d(y, v) & \\ + L \min \{d(F(x, y), u), d(F(u, v), x), & \\ d(F(x, y), x), d(F(u, v), u)\}, & \end{aligned}$$

for all $x, y, u, v \in X$ for which $x \succeq u, y \preceq v$. Suppose either

(a) F is continuous or

(b) X has the following properties:

  (i) if a nondecreasing sequence $\{x_n\} \longrightarrow x$, then $x_n \preceq x$, for all $n \geq 0$;

  (ii) if a nonincreasing sequence $\{y_n\} \longrightarrow y$, then $y \preceq y_n$, for all $n \geq 0$.

Then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$, that is, F has a coupled fixed point in X.

3 Main Results

Theorem 3.1. *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $\phi : [0, \infty) \longrightarrow [0, \infty)$ be a continuous function with $\phi(t) = 0$ if and only if $t = 0$ and ψ be an altering distance function. Let $g : X \to X$ be a continuous mapping and $\{F_\alpha : X \times X \longrightarrow X : \alpha \in \Lambda\}$ be a family of mappings. Suppose there exists $\alpha_0 \in \Lambda$ such that*

(i) F_{α_0} *is continuous,*

(ii) $F_{\alpha_0}(X \times X) \subseteq g(X)$ *and* F_{α_0} *has the mixed g-monotone property on X,*

(iii) *there exists* $x_0, y_0 \in X$ *such that* $gx_0 \preceq F_{\alpha_0}(x_0, y_0)$ *and* $gy_0 \succeq F_{\alpha_0}(y_0, x_0)$*,*

(iv) *the pair* (g, F_{α_0}) *is compatible,*

(v) *there exists a non-negative real number L such that for all $x, y, u, v \in X$ with $gx \succeq gu$, $gy \preceq gv$ and $\alpha \in \Lambda$,*

$$\begin{aligned}\psi(d(F_{\alpha_0}(x, y), F_\alpha(u, v))) \leq\ & \psi(\max\{d(gx, gu), d(gy, gv)\}) \\ & - \phi(\max\{d(gx, gu), d(gy, gv)\}) \\ & + L \min\{d(F_{\alpha_0}(x, y), gu), d(F_\alpha(u, v), gx), \\ & \quad d(F_{\alpha_0}(x, y), gx), d(F_\alpha(u, v), gu)\}.\end{aligned}$$

Then there exist $x, y \in X$ such that $gx = F_\alpha(x, y)$ and $gy = F_\alpha(y, x)$, for all $\alpha \in \Lambda$, that is, g and $\{F_\alpha : \alpha \in \Lambda\}$ have a coupled coincidence point. Moreover, any coupled coincidence point of g and F_{α_0} is a coupled coincidence point of g and $\{F_\alpha : \alpha \in \Lambda\}$.

Proof. First we establish that any coupled coincidence point of g and F_{α_0} is a coupled coincidence point of g and $\{F_\alpha : \alpha \in \Lambda\}$. Suppose that $(w, z) \in X \times X$ be a coupled coincidence point of g and F_{α_0}. Then $gw = F_{\alpha_0}(w, z)$ and $gz = F_{\alpha_0}(z, w)$. From (v), we have

$$\begin{aligned}\psi(d(F_{\alpha_0}(w, z), F_\alpha(w, z))) \leq\ & \psi(\max\{d(gw, gw), d(gz, gz)\}) \\ & - \phi(\max\{d(gw, gw), d(gz, gz)\}) \\ & + L \min\{d(F_{\alpha_0}(w, z), gw), d(F_\alpha(w, z), gw), \\ & \quad d(F_{\alpha_0}(w, z), gw), d(F_\alpha(w, z), gw)\},\end{aligned}$$

that is,

$$\psi(d(gw, F_\alpha(w, z))) = 0,$$

which implies that $d(gw, F_\alpha(w, z)) = 0$, that is, $gw = F_\alpha(w, z)$.

Again, from (v), we have

$$\begin{aligned}\psi(d(F_{\alpha_0}(z, w), F_\alpha(z, w))) \leq\ & \psi(\max\{d(gz, gz), d(gw, gw)\}) \\ & - \phi(\max\{d(gz, gz), d(gw, gw)\}) \\ & + L \min\{d(F_{\alpha_0}(z, w), gz), d(F_\alpha(z, w), gz), \\ & \quad d(F_{\alpha_0}(z, w), gz), d(F_\alpha(z, w), gz)\},\end{aligned}$$

that is,

$$\psi(d(gz,\ F_\alpha(z,\ w))) = 0,$$

which implies that $d(gz,\ F_\alpha(z,\ w)) = 0$, that is, $gz = F_\alpha(z,\ w)$. Therefore, $gw = F_\alpha(w,\ z)$ and $gz = F_\alpha(z,\ w)$, for all $\alpha \in \Lambda$, that is, $(w,\ z) \in X \times X$ is a coupled coincidence point of g and $\{F_\alpha : \alpha \in \Lambda\}$. Hence, any coupled coincidence point of g and F_{α_0} is a coupled coincidence point of g and $\{F_\alpha : \alpha \in \Lambda\}$. The converse part is trivial.

Now it is sufficient to prove that g and F_{α_0} have coupled coincidence point. By the condition (iii) there exist $x_0,\ y_0 \in X$ such that $gx_0 \preceq F_{\alpha_0}(x_0,\ y_0)$ and $gy_0 \succeq F_{\alpha_0}(y_0,\ x_0)$. Since $F_{\alpha_0}(X \times X) \subseteq g(X)$, we can choose $x_1,\ y_1 \in X$ such that $gx_1 = F_{\alpha_0}(x_0,\ y_0)$ and $gy_1 = F_{\alpha_0}(y_0,\ x_0)$. Again we can choose $x_2,\ y_2 \in X$ such that $gx_2 = F_{\alpha_0}(x_1,\ y_1)$ and $gy_2 = F_{\alpha_0}(y_1,\ x_1)$. Continuing this process we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$gx_{n+1} = F_{\alpha_0}(x_n,\ y_n) \ \text{ and } \ gy_{n+1} = F_{\alpha_0}(y_n,\ x_n), \ \text{ for all } \ n \geq 0. \tag{3.1}$$

We shall prove that for all $n \geq 0$,

$$gx_n \preceq gx_{n+1} \tag{3.2}$$

and

$$gy_n \succeq gy_{n+1}. \tag{3.3}$$

Since $gx_0 \preceq F_{\alpha_0}(x_0,\ y_0)$, $gy_0 \succeq F_{\alpha_0}(y_0,\ x_0)$, $gx_1 = F_{\alpha_0}(x_0,\ y_0)$ and $gy_1 = F_{\alpha_0}(y_0,\ x_0)$, we have $gx_0 \preceq gx_1$ and $gy_0 \succeq gy_1$, that is, (3.2) and (3.3) hold for $n = 0$.

We presume that (3.2) and (3.3) hold for some $n > 0$. As F_{α_0} has the mixed g-monotone property and $gx_n \preceq gx_{n+1}$, $gy_n \succeq gy_{n+1}$, from (3.1), we have

$$\begin{aligned} &gx_{n+1} = F_{\alpha_0}(x_n,\ y_n) \preceq F_{\alpha_0}(x_{n+1},\ y_n), \\ &F_{\alpha_0}(y_{n+1},\ x_n) \preceq F_{\alpha_0}(y_n,\ x_n) = gy_{n+1}. \end{aligned} \tag{3.4}$$

Also, for the same reason, we have

$$\begin{aligned} &F_{\alpha_0}(x_{n+1},\ y_n) \preceq F_{\alpha_0}(x_{n+1},\ y_{n+1}) = gx_{n+2}, \\ &F_{\alpha_0}(y_{n+1},\ x_n) \succeq F_{\alpha_0}(y_{n+1},\ x_{n+1}) = gy_{n+2}. \end{aligned} \tag{3.5}$$

From (3.4) and (3.5), we have that $gx_{n+1} \preceq gx_{n+2}$ and $gy_{n+1} \succeq gy_{n+2}$. Then by mathematical induction it follows that (3.2) and (3.3) hold for all $n \geq 0$. Therefore,

$$gx_0 \preceq gx_1 \preceq gx_2 \preceq gx_3 \preceq \cdots \preceq gx_n \preceq gx_{n+1} \preceq \cdots, \tag{3.6}$$

and

$$gy_0 \succeq gy_1 \succeq gy_2 \succeq gy_3 \succeq \cdots \succeq gy_n \succeq gy_{n+1} \cdots. \tag{3.7}$$

Let $R_n = \max\{d(gx_{n+1},\ gx_n),\ d(gy_{n+1},\ gy_n)\}$.

Since $gx_n \succeq gx_{n-1}$ and $gy_n \preceq gy_{n-1}$, applying (v) for $\alpha = \alpha_0$ and using (3.1), we have

$$\begin{aligned}\psi(d(gx_{n+1},\ gx_n)) &= \psi(d(F_{\alpha_0}(x_n,\ y_n),\ F_{\alpha_0}(x_{n-1},\ y_{n-1})))\\ &\leq \psi(\max\{d(gx_n,\ gx_{n-1}),\ d(gy_n,\ gy_{n-1})\})\\ &\quad - \phi(\max\{d(gx_n,\ gx_{n-1}),\ d(gy_n,\ gy_{n-1})\})\\ &\quad + L\min\{d(F_{\alpha_0}(x_n,\ y_n),\ gx_{n-1}),\ d(F_{\alpha_0}(x_{n-1},\ y_{n-1}),\ gx_n),\\ &\qquad d(F_{\alpha_0}(x_n,\ y_n), gx_n),\ d(F_{\alpha_0}(x_{n-1},\ y_{n-1}),\ gx_{n-1})\}\\ &= \psi(\max\{d(gx_n,\ gx_{n-1}),\ d(gy_n,\ gy_{n-1})\})\\ &\quad - \phi(\max\{d(gx_n,\ gx_{n-1}),\ d(gy_n,\ gy_{n-1})\}).\end{aligned} \tag{3.8}$$

Again, since $gy_{n-1} \succeq gy_n$ and $gx_{n-1} \preceq gx_n$, applying (v) for $\alpha = \alpha_0$ and using (3.1), we have

$$\begin{aligned}\psi(d(gy_n,\ gy_{n+1})) &= \psi(d(F_{\alpha_0}(y_{n-1},\ x_{n-1}),\ F_{\alpha_0}(y_n,\ x_n)))\\ &\leq \psi(\max\{d(gy_{n-1},\ gy_n),\ d(gx_{n-1},\ gx_n)\})\\ &\quad - \phi(\max\{d(gy_{n-1},\ gy_n),\ d(gx_{n-1},\ gx_n)\})\\ &\quad + L\min\{d(F_{\alpha_0}(y_{n-1},\ x_{n-1}), gy_n), d(F_{\alpha_0}(y_n,\ x_n), gy_{n-1}),\\ &\qquad d(F_{\alpha_0}(y_{n-1},\ x_{n-1}), gy_{n-1}), d(F_{\alpha_0}(y_n,\ x_n), gy_n)\}\\ &= \psi(\max\{d(gy_{n-1},\ gy_n),\ d(gx_{n-1},\ gx_n)\})\\ &\quad - \phi(\max\{d(gy_{n-1},\ gy_n),\ d(gx_{n-1},\ gx_n)\}).\end{aligned} \tag{3.9}$$

From (3.8) and (3.9) and using the monotone property of ψ, we have

$$\begin{aligned}&\psi(\max\{d(gx_{n+1},\ gx_n), d(gy_{n+1},\ gy_n)\})\\ &\qquad = \max\{\psi(d(gx_{n+1},\ gx_n)),\ \psi(d(gy_n,\ gy_{n+1}))\}\\ &\qquad \leq \psi(\max\{d(gx_n,\ gx_{n-1}),\ d(gy_n,\ gy_{n-1})\})\\ &\qquad\quad - \phi(\max\{d(gx_n,\ gx_{n-1}),\ d(gy_n,\ gy_{n-1})\}),\end{aligned}$$

that is,

$$\psi(R_n) \leq \psi(R_{n-1}) - \phi(R_{n-1}). \tag{3.10}$$

Using a property of ϕ, for all $n \geq 0$ we have

$$\psi(R_n) \leq \psi(R_{n-1}),$$

which, by the monotone property of ψ, implies that

$$R_n \leq R_{n-1}.$$

Therefore, $\{R_n\}$ is a monotone decreasing sequence. Hence there exists an $r \geq 0$ such that

$$R_n \longrightarrow r \ \text{ as } \ n \longrightarrow \infty. \tag{3.11}$$

Taking the limit as $n \to \infty$ in (3.10), using (3.11) and the continuities of ψ and ϕ, we have

$$\psi(r) \leq \psi(r) - \phi(r),$$

which is a contradiction unless $r = 0$. Hence,

$$R_n \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \tag{3.12}$$

Then

$$\lim_{n\to\infty} d(gx_{n+1},\ gx_n) = 0 \tag{3.13}$$

and

$$\lim_{n\to\infty} d(gy_{n+1},\ gy_n) = 0. \tag{3.14}$$

Next we show that both $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences. If possible suppose that at least one of $\{gx_n\}$ and $\{gy_n\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k,

$$n(k) > m(k) > k,$$

$$\max\{d(gx_{m(k)},\ gx_{n(k)}),\ d(gy_{m(k)},\ gy_{n(k)})\} \geq \epsilon$$

and

$$\max\{d(gx_{m(k)},\ gx_{n(k)-1}),\ d(gy_{m(k)},\ gy_{n(k)-1})\} < \epsilon.$$

Now,

$$\begin{aligned}\epsilon \leq\ & \max\{d(gx_{m(k)},\ gx_{n(k)}),\ d(gy_{m(k)},\ gy_{n(k)})\}\\ &\leq \max\{d(gx_{m(k)},\ gx_{n(k)-1}),\ d(gy_{m(k)},\ gy_{n(k)-1})\}\\ &\quad + \max\{d(gx_{n(k)-1},\ gx_{n(k)}),\ d(gy_{n(k)-1},\ gy_{n(k)})\},\end{aligned}$$

that is,

$$\epsilon \leq \max\{d(gx_{m(k)},\ gx_{n(k)}),\ d(gy_{m(k)},\ gy_{n(k)})\} \leq \epsilon + R_{n(k)-1}.$$

Letting $k \longrightarrow \infty$ in the above inequality and using (3.12), we have

$$\lim_{k\to\infty} \max\{d(gx_{m(k)},\ gx_{n(k)}),\ d(gy_{m(k)},\ gy_{n(k)})\} = \epsilon. \tag{3.15}$$

Again,

$$\begin{aligned}&\max\{d(gx_{m(k)+1},\ gx_{n(k)+1}),\ d(gy_{m(k)+1},\ gy_{n(k)+1})\}\\ &\qquad \leq R_{m(k)} + \max\{d(gx_{m(k)},\ gx_{n(k)}),\ d(gy_{m(k)},\ gy_{n(k)})\} + R_{n(k)}\end{aligned}$$

and

$$\begin{aligned}&\max\{d(gx_{m(k)},\ gx_{n(k)}),\ d(gy_{m(k)},\ gy_{n(k)})\}\\ &\quad \leq R_{m(k)} + \max\{d(gx_{m(k)+1},\ gx_{n(k)+1}),\ d(gy_{m(k)+1},\ gy_{n(k)+1})\} + R_{n(k)}.\end{aligned}$$

Letting $k \longrightarrow \infty$ in above inequalities, using (3.12) and (3.15), we have

$$\lim_{k\to\infty} \max\{d(gx_{m(k)+1},\ gx_{n(k)+1}),\ d(gy_{m(k)+1},\ gy_{n(k)+1})\} = \epsilon. \tag{3.16}$$

Since $n(k) > m(k)$, $gx_{n(k)} \succeq gx_{m(k)}$ and $gy_{n(k)} \preceq gy_{m(k)}$, applying (v) for $\alpha = \alpha_0$ and using (3.1), we have

$$\begin{aligned}
&\psi(d(gx_{n(k)+1},\ gx_{m(k)+1}))\\
&\quad = \psi(d(F_{\alpha_0}(x_{n(k)},\ y_{n(k)}),\ F_{\alpha_0}(x_{m(k)},\ y_{m(k)})))\\
&\quad \le \psi(\max\{d(gx_{n(k)},\ gx_{m(k)}),\ d(gy_{n(k)},\ gy_{m(k)})\})\\
&\qquad - \phi(\max\{d(gx_{n(k)},\ gx_{m(k)}),\ d(gy_{n(k)},\ gy_{m(k)})\})\\
&\qquad + L\min\{d(F_{\alpha_0}(x_{n(k)},\ y_{n(k)}),\ gx_{m(k)}),\ d(F_{\alpha_0}(x_{m(k)},\ y_{m(k)}),\ gx_{n(k)}),\\
&\qquad\quad d(F_{\alpha_0}(x_{n(k)},\ y_{n(k)}),\ gx_{n(k)}),\ d(F_{\alpha_0}(x_{m(k)},\ y_{m(k)}),\ gx_{m(k)})\}\\
&\quad = \psi(\max\{d(gx_{n(k)},\ gx_{m(k)}),\ d(gy_{n(k)},\ gy_{m(k)})\})\\
&\qquad - \phi(\max\{d(gx_{n(k)},\ gx_{m(k)}),\ d(gy_{n(k)},\ gy_{m(k)})\})\\
&\qquad + L\min\{d(F_{\alpha_0}(x_{n(k)},\ y_{n(k)}), gx_{m(k)}), d(F_{\alpha_0}(x_{m(k)},\ y_{m(k)}), gx_{n(k)}),\\
&\qquad\quad d(gx_{n(k)+1},\ gx_{n(k)}),\ d(gx_{m(k)+1},\ gx_{m(k)})\}.
\end{aligned} \tag{3.17}$$

Again, since $n(k) > m(k)$, $gy_{m(k)} \succeq gy_{n(k)}$ and $gx_{m(k)} \preceq gx_{n(k)}$, applying (v) for $\alpha = \alpha_0$ and using (3.1), we have

$$\begin{aligned}
&\psi(d(gy_{m(k)+1},\ gy_{n(k)+1}))\\
&\quad = \psi(d(F_{\alpha_0}(y_{m(k)},\ x_{m(k)}),\ F_{\alpha_0}(y_{n(k)},\ x_{n(k)})))\\
&\quad \le \psi(\max\{d(gy_{m(k)},\ gy_{n(k)}),\ d(gx_{m(k)},\ gx_{n(k)})\})\\
&\qquad - \phi(\max\{d(gy_{m(k)},\ gy_{n(k)}),\ d(gx_{m(k)},\ gx_{n(k)})\})\\
&\qquad + L\min\{d(F_{\alpha_0}(y_{m(k)},\ x_{m(k)}),\ gy_{n(k)}),\ d(F_{\alpha_0}(y_{n(k)},\ x_{n(k)}),\ gy_{m(k)}),\\
&\qquad\quad d(F_{\alpha_0}(y_{m(k)},\ x_{m(k)}),\ gy_{m(k)}),\ d(F_{\alpha_0}(y_{n(k)},\ x_{n(k)}),\ gy_{n(k)})\}\\
&\quad = \psi(\max\{d(gx_{n(k)},\ gx_{m(k)}),\ d(gy_{n(k)},\ gy_{m(k)})\})\\
&\qquad - \phi(\max\{d(gx_{n(k)},\ gx_{m(k)}),\ d(gy_{n(k)},\ gy_{m(k)})\})\\
&\qquad + L\min\{d(F_{\alpha_0}(y_{m(k)},\ x_{m(k)}),\ gy_{n(k)}),\ d(F_{\alpha_0}(y_{n(k)},\ x_{n(k)}),\ gy_{m(k)}),\\
&\qquad\quad d(gy_{m(k)+1},\ gy_{m(k)}),\ d(gy_{n(k)+1},\ gy_{n(k)})\}.
\end{aligned} \tag{3.18}$$

From (3.17) and (3.18) and using the monotone property of ψ, we have

$$\begin{aligned}
&\psi(\max\{d(gx_{n(k)+1},\ gx_{m(k)+1}),\ d(gy_{m(k)+1},\ gy_{n(k)+1})\})\\
&\quad = \max\{\psi(d(gx_{n(k)+1},\ gx_{m(k)+1})),\ \psi(d(gy_{m(k)+1},\ gy_{n(k)+1}))\}\\
&\quad \le \psi(\max\{d(gx_{n(k)},\ gx_{m(k)}),\ d(gy_{n(k)},\ gy_{m(k)})\})\\
&\qquad - \phi(\max\{d(gx_{n(k)},\ gx_{m(k)}),\ d(gy_{n(k)},\ gy_{m(k)})\})\\
&\qquad + L\min\{d(F_{\alpha_0}(x_{n(k)},\ y_{n(k)}),\ gx_{m(k)}),\ d(F_{\alpha_0}(x_{m(k)},\ y_{m(k)}),\ gx_{n(k)}),\\
&\qquad\quad d(gx_{n(k)+1},\ gx_{n(k)}), d(gx_{m(k)+1},\ gx_{m(k)})\}\\
&\qquad + L\min\{d(F_{\alpha_0}(y_{m(k)},\ x_{m(k)}),\ gy_{n(k)}),\ d(F_{\alpha_0}(y_{n(k)},\ x_{n(k)}),\ gy_{m(k)}),\\
&\qquad\quad d(gy_{m(k)+1},\ gy_{m(k)}),\ d(gy_{n(k)+1},\ gy_{n(k)})\}.
\end{aligned}$$

Letting $k \to \infty$ in the above inequality, using (3.13), (3.14), (3.15), (3.16) and the continuities of ψ and ϕ, we have

$$\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon),$$

which is a contradiction by virtue of a property of ϕ. Hence both $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in X. From the completeness of X, there exist $x,\ y \in X$ such that

$$\lim_{n\to\infty} F_{\alpha_0}(x_n,\ y_n) = \lim_{n\to\infty} gx_n = x \tag{3.19}$$

and

$$\lim_{n\to\infty} F_{\alpha_0}(y_n,\ x_n) = \lim_{n\to\infty} gy_n = y. \tag{3.20}$$

Since the pair $(g,\ F_{\alpha_0})$ is compatible, from (3.19) and (3.20), we have

$$\lim_{n\to\infty} d(gF_{\alpha_0}(x_n,\ y_n),\ F_{\alpha_0}(gx_n,\ gy_n)) = 0 \tag{3.21}$$

and

$$\lim_{n\to\infty} d(gF_{\alpha_0}(y_n,\ x_n),\ F_{\alpha_0}(gy_n,\ gx_n)) = 0. \tag{3.22}$$

For all $n \geq 0$, we have

$$d(gx,\ F_{\alpha_0}(gx_n,\ gy_n)) \leq d(gx,\ gF_{\alpha_0}(x_n,\ y_n)) + d(gF_{\alpha_0}(x_n,\ y_n),\ F_{\alpha_0}(gx_n,\ gy_n))$$

and

$$d(gy,\ F_{\alpha_0}(gy_n,\ gx_n)) \leq d(gy,\ gF_{\alpha_0}(y_n,\ x_n)) + d(gF_{\alpha_0}(y_n,\ x_n),\ F_{\alpha_0}(gy_n,\ gx_n)).$$

Taking $n \longrightarrow \infty$ in the above inequalities, using (3.19), (3.20), (3.21), (3.22) and the continuities of F_{α_0} and g, we have

$$d(gx,\ F_{\alpha_0}(x,\ y)) = 0 \ \text{ and } \ d(gy,\ F_{\alpha_0}(y,\ x)) = 0,$$

that is,

$$gx = F_{\alpha_0}(x,\ y) \ \text{ and } \ gy = F_{\alpha_0}(y,\ x),$$

that is, $(x,\ y) \in X \times X$ is a coupled coincidence point of the mappings g and F_{α_0}. Then by what we have already proved, $(x,\ y)$ is a coupled coincidence point of g and $\{F_\alpha : \alpha \in \Lambda\}$. □

Theorem 3.2. *Let $(X,\ \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that $(X,\ d)$ is a complete metric space. Assume that X has the P- property. Let $\phi : [0,\ \infty) \longrightarrow [0,\ \infty)$ be a continuous function with $\phi(t) = 0$ if and only if $t = 0$ and ψ be an altering distance function. Let $g : X \to X$ be a monotonic increasing and continuous mapping and $\{F_\alpha : X \times X \longrightarrow X : \alpha \in \Lambda\}$ be a family of mappings. Suppose there exists $\alpha_0 \in \Lambda$ such that*

(i) *$F_{\alpha_0}(X \times X) \subseteq g(X)$ and F_{α_0} has the mixed g-monotone property on X,*

(ii) *there exists* $x_0,\ y_0 \in X$ *such that* $gx_0 \preceq F_{\alpha_0}(x_0,\ y_0)$ *and* $gy_0 \succeq F_{\alpha_0}(y_0,\ x_0)$*,*

(iii) *the pair* $(g,\ F_{\alpha_0})$ *is compatible,*

(iv) *there exists a non-negative real number* L *such that for all* $x,\ y,\ u,\ v \in X$ *with* $gx \succeq gu,\ gy \preceq gv$ *and* $\alpha \in \Lambda$*,*

$$\begin{aligned}\psi(d(F_{\alpha_0}(x,\ y),\ F_\alpha(u,\ v))) \le\ & \psi(max\,\{d(gx,\ gu),\ d(gy,\ gv)\}) \\ & - \phi(max\,\{d(gx,\ gu),\ d(gy,\ gv)\}) \\ & + L\ min\,\{d(F_{\alpha_0}(x,\ y),\ gu),\ d(F_\alpha(u,\ v),\ gx), \\ & \quad d(F_{\alpha_0}(x,\ y),\ gx),\ d(F_\alpha(u,\ v),\ gu)\}.\end{aligned}$$

Then there exist $x,\ y \in X$ *such that* $gx = F_\alpha(x,\ y)$ *and* $gy = F_\alpha(y,\ x)$*, for all* $\alpha \in \Lambda$*, that is,* g *and* $\{F_\alpha : \alpha \in \Lambda\}$ *have a coupled coincidence point. Moreover, any coupled coincidence point of* g *and* F_{α_0} *is a coupled coincidence point of* g *and* $\{F_\alpha : \alpha \in \Lambda\}$*.*

Proof. We take the same sequences $\{x_n\}$ and $\{y_n\}$ as in the proof of Theorem 3.1. Then like in the proof of theorem 3.1, we have (3.1), (3.6), (3.7), (3.12), (3.13), (3.14), (3.19), (3.20), (3.21) and (3.22). Using the P-property of X we have from (3.6), (3.7), (3.19) and (3.20),

$$gx_n \preceq x \ \text{ and }\ gy_n \succeq y,$$

which, by the monotone property of g, implies that

$$ggx_n \preceq gx \ \text{ and }\ ggy_n \succeq gy. \tag{3.23}$$

Since the pair $(g,\ F_{\alpha_0})$ is compatible and g is continuous, by (3.19), (3.20), (3.21) and (3.22), we have

$$\lim_{n\to\infty} ggx_n = gx = \lim_{n\to\infty} gF_{\alpha_0}(x_n,\ y_n) = \lim_{n\to\infty} F_{\alpha_0}(gx_n,\ gy_n) \tag{3.24}$$

and

$$\lim_{n\to\infty} ggy_n = gy = \lim_{n\to\infty} gF_{\alpha_0}(y_n,\ x_n) = \lim_{n\to\infty} F_{\alpha_0}(gy_n,\ gx_n). \tag{3.25}$$

Now,

$$d(F_{\alpha_0}(x,\ y),\ gx) \le d(F_{\alpha_0}(x,\ y),\ ggx_{n+1}) + d(ggx_{n+1},\ gx),$$

that is,

$$d(F_{\alpha_0}(x,\ y),\ gx) \le d(F_{\alpha_0}(x,\ y),\ gF_{\alpha_0}(x_n,\ y_n)) + d(ggx_{n+1},\ gx).$$

Taking $n \longrightarrow \infty$ in the above inequality, using (3.24), we have

$$\begin{aligned}d(F_{\alpha_0}(x,\ y),\ gx) &\le \lim_{n\to\infty} d(F_{\alpha_0}(x,\ y),\ gF_{\alpha_0}(x_n,\ y_n)) + \lim_{n\to\infty} d(ggx_{n+1},\ gx) \\ &\le \lim_{n\to\infty} d(F_{\alpha_0}(x,\ y),\ F_{\alpha_0}(gx_n,\ gy_n)).\end{aligned}$$

Since ψ is continuous and monotone increasing, from the above inequality, we have

$$\begin{aligned}\psi(d(F_{\alpha_0}(x,\ y),\ gx)) &\le \psi(\lim_{n\to\infty} d(F_{\alpha_0}(x,\ y),\ F_{\alpha_0}(gx_n,\ gy_n)))\\ &= \lim_{n\to\infty} \psi(d(F_{\alpha_0}(x,\ y),\ F_{\alpha_0}(gx_n,\ gy_n))).\end{aligned}$$

By virtue of (3.23), applying (iv) for $\alpha = \alpha_0$, we have

$$\begin{aligned}\psi(d(F_{\alpha_0}(x,\ y),\ gx)) \le \lim_{n\to\infty} [&\psi(\max\{d(gx,\ ggx_n),\ d(gy,\ ggy_n)\})\\ &- \phi(\max\{d(gx,\ ggx_n),\ d(gy,\ ggy_n)\})\\ &+ L \min\{d(F_{\alpha_0}(x,\ y), ggx_n), d(F_{\alpha_0}(gx_n,\ gy_n), gx),\\ &\quad d(F_{\alpha_0}(x,\ y), gx), d(F_{\alpha_0}(gx_n,\ gy_n), ggx_n)\}].\end{aligned}$$

Using (3.24), (3.25) and the properties of ψ, ϕ, we have

$$\psi(d(F_{\alpha_0}(x,\ y),\ gx)) = 0,$$

which implies that $d(F_{\alpha_0}(x,\ y),\ gx) = 0$, that is, $gx = F_{\alpha_0}(x,\ y)$.

Again, we have

$$d(gy,\ F_{\alpha_0}(y,\ x)) \le d(gy,\ ggy_{n+1}) + d(ggy_{n+1},\ F_{\alpha_0}(y,\ x)),$$

that is,

$$d(gy,\ F_{\alpha_0}(y,\ x)) \le d(gy,\ ggy_{n+1}) + d(gF_{\alpha_0}(y_n,\ x_n),\ F_{\alpha_0}(y,\ x)).$$

Taking $n \longrightarrow \infty$ in the above inequality, using (3.25), we have

$$\begin{aligned}d(gy,\ F_{\alpha_0}(y,\ x)) &\le \lim_{n\to\infty} d(gy,\ ggy_{n+1}) + \lim_{n\to\infty} d(gF_{\alpha_0}(y_n,\ x_n),\ F_{\alpha_0}(y,\ x))\\ &\le \lim_{n\to\infty} d(F_{\alpha_0}(gy_n,\ gx_n),\ F_{\alpha_0}(y,\ x)).\end{aligned}$$

Since ψ is continuous and monotone increasing, from the above inequality, we have

$$\begin{aligned}\psi(d(gy,\ F_{\alpha_0}(y,\ x))) &\le \psi(\lim_{n\to\infty} d(F_{\alpha_0}(gy_n,\ gx_n),\ F_{\alpha_0}(y,\ x)))\\ &= \lim_{n\to\infty} \psi(d(F_{\alpha_0}(gy_n,\ gx_n),\ F_{\alpha_0}(y,\ x))).\end{aligned}$$

By virtue of (3.23), applying (iv) for $\alpha = \alpha_0$, we have

$$\begin{aligned}\psi(d(gy,\ F_{\alpha_0}(y,\ x))) \le \lim_{n\to\infty} [&\psi(\max\{d(ggy_n,\ gy),\ d(ggx_n,\ gx)\})\\ &- \phi(\max\{d(ggy_n,\ gy),\ d(ggx_n,\ gx)\})\\ &+ L \min\{d(F_{\alpha_0}(gy_n,\ gx_n),\ gy),\ d(F_{\alpha_0}(y,\ x),\ ggy_n),\\ &\quad d(F_{\alpha_0}(gy_n,\ gx_n),\ ggy_n),\ d(F_{\alpha_0}(y,\ x),\ gy)\}].\end{aligned}$$

Using (3.24), (3.25) and the properties of ψ, ϕ, we have

$$\psi(d(gy,\ F_{\alpha_0}(y,\ x))) = 0,$$

which implies that $d(gy,\ F_{\alpha_0}(y,\ x)) = 0$, that is, $gy = F_{\alpha_0}(y,\ x)$. Hence the element $(x,\ y) \in X \times X$, is a coupled coincidence point of the mappings g and F_{α_0}. By what we have already proved in theorem 3.1, $(x,\ y)$ is a coupled coincidence point of g and $\{F_\alpha : \alpha \in \Lambda\}$. □

The compatibility of the pairs $(g,\ F_{\alpha_0})$ and the properties (continuity and monotonicity) of g which are assumed in Theorem 3.2 have been relaxed in the next theorem by taking $g(X)$ to be closed in $(X,\ d)$.

Theorem 3.3. *Let $(X,\ \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that $(X,\ d)$ is a complete metric space. Assume that X has the P- property. Let $\phi : [0,\ \infty) \longrightarrow [0,\ \infty)$ be a continuous function with $\phi(t) = 0$ if and only if $t = 0$ and ψ be an altering distance function. Let $g : X \to X$ be a mapping such that $g(X)$ is closed in X. Let $\{F_\alpha : X \times X \longrightarrow X : \alpha \in \Lambda\}$ be a family of mappings. Suppose there exists $\alpha_0 \in \Lambda$ such that*

(i) $F_{\alpha_0}(X \times X) \subseteq g(X)$ *and* F_{α_0} *has the mixed g-monotone property on X,*

(ii) *there exists* $x_0,\ y_0 \in X$ *such that* $gx_0 \preceq F_{\alpha_0}(x_0,\ y_0)$ *and* $gy_0 \succeq F_{\alpha_0}(y_0,\ x_0)$*,*

(iii) *there exists a non-negative real number L such that for all $x,\ y,\ u,\ v \in X$ with $gx \succeq gu,\ gy \preceq gv$ and $\alpha \in \Lambda$,*

$$\begin{aligned}\psi(d(F_{\alpha_0}(x,\ y),\ F_\alpha(u,\ v))) \leq\ & \psi(max\,\{d(gx,\ gu),\ d(gy,\ gv)\}) \\ & - \phi(max\,\{d(gx,\ gu),\ d(gy,\ gv)\}) \\ & + L\ min\,\{d(F_{\alpha_0}(x,\ y),\ gu),\ d(F_\alpha(u,\ v),\ gx), \\ & \quad d(F_{\alpha_0}(x,\ y),\ gx), d(F_\alpha(u,\ v),\ gu)\}.\end{aligned}$$

Then there exist $x,\ y \in X$ such that $gx = F_\alpha(x,\ y)$ and $gy = F_\alpha(y,\ x)$, for all $\alpha \in \Lambda$, that is, g and $\{F_\alpha : \alpha \in \Lambda\}$ have a coupled coincidence point. Moreover, any coupled coincidence point of g and F_{α_0} is a coupled coincidence point of g and $\{F_\alpha : \alpha \in \Lambda\}$.

Proof. We take the same sequences $\{x_n\}$ and $\{y_n\}$ as in the proof of Theorem 3.1. Then like in the proof of Theorem 3.1, we have (3.1), (3.6), (3.7), (3.12), (3.13), (3.14), (3.19) and (3.20). Since the metric space $(X,\ d)$ is complete and $g(X)$ is closed in X, (3.19) and (3.20) implies that $x,\ y \in g(X)$. Since $x,\ y \in g(X)$, there exist $u,\ v \in X$ such that $x = gu$ and $y = gv$. Then from (3.19) and (3.20), we have

$$\lim_{n\to\infty} F_{\alpha_0}(x_n,\ y_n) = \lim_{n\to\infty}\ gx_n = x = gu \tag{3.26}$$

and

$$\lim_{n\to\infty} F_{\alpha_0}(y_n,\ x_n) = \lim_{n\to\infty}\ gy_n = y = gv. \tag{3.27}$$

By (3.6), (3.7), (3.26), (3.27) and the P- property of X, we have

$$gx_n \preceq gu \ \text{ and } \ gy_n \succeq gv, \ \text{ for all } \ n \geq 0.$$

Then applying (iii) for $\alpha = \alpha_0$, we have

$$\begin{aligned}\psi(d(F_{\alpha_0}(u,\ v),\ F_{\alpha_0}(x_n,\ y_n))) \le\ & \psi(\max\{d(gu,\ gx_n),\ d(gv,\ gy_n)\}) \\ & - \phi(\max\{d(gu,\ gx_n),\ d(gv,\ gy_n)\}) \\ & + L \min\{d(F_{\alpha_0}(u,\ v),\ gx_n),\ d(F_{\alpha_0}(x_n,\ y_n),\ gu), \\ & \quad d(F_{\alpha_0}(u,\ v),\ gu),\ d(F_{\alpha_0}(x_n,\ y_n),\ gx_n)\}.\end{aligned}$$

Taking $n \longrightarrow \infty$ in the above inequality, using (3.26), (3.27) and the properties of ψ and ϕ, we have $d(F_{\alpha_0}(u,\ v),\ gu) = 0$, that is, $gu = F_{\alpha_0}(u,\ v)$. Then applying (iii) for $\alpha = \alpha_0$, we have

$$\begin{aligned}\psi(d(F_{\alpha_0}(y_n,\ x_n),\ F_{\alpha_0}(v,\ u))) \le\ & \psi(\max\{d(gy_n,\ gv),\ d(gx_n,\ gu)\}) \\ & - \phi(\max\{d(gy_n,\ gv),\ d(gx_n,\ gu)\}) \\ & + L \min\{d(F_{\alpha_0}(y_n,\ x_n),\ gv),\ d(F_{\alpha_0}(v,\ u),\ gy_n), \\ & \quad d(F_{\alpha_0}(y_n,\ x_n),\ gy_n),\ d(F_{\alpha_0}(v,\ u),\ gv)\}.\end{aligned}$$

Taking $n \longrightarrow \infty$ in the above inequality, using (3.26), (3.27) and the properties of ψ and ϕ, we have $d(gv,\ F_{\alpha_0}(v,\ u)) = 0$, that is, $gv = F_{\alpha_0}(v,\ u)$. Therefore, $gu = F_{\alpha_0}(u,\ v)$ and $gv = F_{\alpha_0}(v,\ u)$, that is, $(u,\ v) \in X \times X$ is a coupled coincidence point of the mappings $g : X \longrightarrow X$ and $F_{\alpha_0} : X \times X \longrightarrow X$. By what we have already proved, $(u,\ v)$ is a coupled coincidence point of g and $\{F_\alpha : \alpha \in \Lambda\}$. □

Considering $\{F_\alpha : \alpha \in \Lambda\} = \{F\}$ in Theorems 3.1 and 3.2, we have the following corollaries respectively.

Corollary 3.4. *Let $(X,\ \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that $(X,\ d)$ is a complete metric space. Let $\phi : [0,\ \infty) \longrightarrow [0,\ \infty)$ be a continuous function with $\phi(t) = 0$ if and only if $t = 0$ and ψ be an altering distance function. Let $g : X \to X$ and $F : X \times X \longrightarrow X$ be two mappings such that*

(i) g and F are continuous,

(ii) $F(X \times X) \subseteq g(X)$ and F has the mixed g-monotone property on X,

(iii) there exists $x_0,\ y_0 \in X$ such that $gx_0 \preceq F(x_0,\ y_0)$ and $gy_0 \succeq F(y_0,\ x_0)$,

(iv) the pair $(g,\ F)$ is compatible,

(v) there exists a non-negative real number L such that for all $x,\ y,\ u,\ v \in X$ with $gx \succeq gu,\ gy \preceq gv$,

$$\begin{aligned}\psi(d(F(x,\ y),\ F(u,\ v))) \le\ & \psi(\max\{d(gx,\ gu),\ d(gy,\ gv)\}) \\ & - \phi(\max\{d(gx,\ gu),\ d(gy,\ gv)\}) \\ & + L \min\{d(F(x,\ y),\ gu),\ d(F(u,\ v),\ gx), \\ & \quad d(F(x,\ y),\ gx),\ d(F(u,\ v),\ gu)\}.\end{aligned}$$

Then there exist $x,\ y \in X$ such that $gx = F(x,\ y)$ and $gy = F(y,\ x)$, that is, g and F have a coupled coincidence point in X.

Corollary 3.5. *Let $(X,\ \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that $(X,\ d)$ is a complete metric space. Assume that X has the P- property. Let $\phi : [0,\ \infty) \longrightarrow [0,\ \infty)$ be a continuous function with $\phi(t) = 0$ if and only if $t = 0$ and ψ be an altering distance function. Let $g : X \to X$ and $F : X \times X \longrightarrow X$ be two mappings such that*

(i) g is monotonic increasing and continuous,

(ii) $F(X \times X) \subseteq g(X)$ and F has the mixed g-monotone property on X,

(iii) there exists $x_0,\ y_0 \in X$ such that $gx_0 \preceq F(x_0,\ y_0)$ and $gy_0 \succeq F(y_0,\ x_0)$,

(iv) the pair $(g,\ F)$ is compatible,

(v) there exists a non-negative real number L such that for all $x,\ y,\ u,\ v \in X$ with $gx \succeq gu,\ gy \preceq gv$,

$$\begin{aligned}\psi(d(F(x,\ y),\ F(u,\ v))) \le\ & \psi(max\,\{d(gx,\ gu),\ d(gy,\ gv)\}) \\ & - \phi(max\,\{d(gx,\ gu),\ d(gy,\ gv)\}) \\ & + L\ min\,\{d(F(x,\ y),\ gu),\ d(F(u,\ v),\ gx), \\ & \quad d(F(x,\ y),\ gx),\ d(F(u,\ v),\ gu)\}.\end{aligned}$$

Then there exist $x,\ y \in X$ such that $gx = F(x,\ y)$ and $gy = F(y,\ x)$, that is, g and F have a coupled coincidence point in X.

Example 3.6. *Let $X = [0,\ \infty)$. Then $(X,\ \le)$ is a partially ordered set with the natural ordering of real numbers. Let $d(x,\ y) = |x - y|$, for $x,\ y \in X$. Then $(X,\ d)$ is a complete metric space.*

Let $g\colon X \to X$ be given by $gx = x^2$, for all $x \in X$. Also, consider

$$F\colon X \times X \to X, \quad F(x,\ y) = \begin{cases} \frac{1}{3}(x^2 - y^2), & \text{if} \quad x \ge y, \\ 0, & \text{if} \quad x \le y, \end{cases}$$

which obeys the mixed g-monotone property. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that

$$\lim_{n\to\infty} F(x_n,\ y_n) = \lim_{n\to\infty} gx_n = a, \quad \lim_{n\to\infty} F(y_n,\ x_n) = \lim_{n\to\infty} gy_n = b.$$

Then obviously, $a = 0$ and $b = 0$.

Now, for all $n \ge 0$, $gx_n = x_n^2$, $gy_n = y_n^2$, while

$$F(x_n,\ y_n) = \begin{cases} \frac{1}{3}(x_n^2 - y_n^2), & \text{if} \quad x_n \ge y_n, \\ 0, & \text{if} \quad x_n \le y_n, \end{cases}$$

and

$$F(y_n,\ x_n) = \begin{cases} \frac{1}{3}(y_n^2 - x_n^2), & \text{if} \quad y_n \geq x_n, \\ 0, & \text{if} \quad y_n \leq x_n. \end{cases}$$

Then it follows that

$$\lim_{n\to\infty} d(gF(x_n,\ y_n),\ F(gx_n,\ gy_n)) = 0$$

and

$$\lim_{n\to\infty} d(gF(y_n,\ x_n),\ F(gy_n,\ gx_n)) = 0.$$

Hence, the pair $(g,\ F)$ *is compatible in* X.

Let $x_0 = 0$ *and* $y_0 = c(> 0)$ *be two points in* X. *Then*

$$g(x_0) = g(0) = 0 = F(0,\ c) = F(x_0,\ y_0)$$

and

$$g(y_0) = g(c) = c^2 \geq \frac{c^2}{3} = F(c,\ 0) = F(y_0,\ x_0).$$

Let $\psi,\ \phi : [0,\ \infty) \longrightarrow [0,\ \infty)$ *be defined as follows:*

$$\psi(t) = t^2, \quad \phi(t) = \frac{5}{9}t^2.$$

Then ψ *and* ϕ *have the properties mentioned in corollaries 3.4 and 3.5. We now verify the inequality (v) of corollaries 3.4 and 3.5. We take* $x,\ y,\ u,\ v \in X$ *such that* $gx \geq gu$ *and* $gy \leq gv$, *that is,* $x^2 \geq u^2$ *and* $y^2 \leq v^2$.

Let $M = \max\{d(gx,\ gu),\ d(gy,\ gv)\} = \max\{\left|x^2 - u^2\right|, \left|y^2 - v^2\right|\}$. *Then* $M \geq |x^2 - u^2| = x^2 - u^2$ *and* $M \geq |y^2 - v^2| = v^2 - y^2$. *The following are the four possible cases.*

Case 1: $x \geq y$ *and* $u \geq v$. *Then*

$$\begin{aligned} d(F(x,\ y),\ F(u,\ v)) &= d\Big(\frac{x^2 - y^2}{3}, \frac{u^2 - v^2}{3}\Big) = \left|\frac{(x^2 - y^2) - (u^2 - v^2)}{3}\right| \\ &= \left|\frac{(x^2 - u^2) + (v^2 - y^2)}{3}\right| = \frac{(x^2 - u^2) + (v^2 - y^2)}{3} \leq \frac{2}{3}M. \end{aligned}$$

Case 2: $x < y$ *and* $u < v$. *Then*

$$d(F(x,\ y), F(u,\ v)) = d(0,\ 0) = 0 \leq \frac{2}{3}M.$$

Case 3: $x \geq y$ *and* $u \leq v$. *Then*

$$\begin{aligned} d(F(x,\ y), F(u,\ v)) &= d\Big(\frac{x^2 - y^2}{3}, 0\Big) = \frac{x^2 - y^2}{3} = \frac{u^2 + x^2 - y^2 - u^2}{3} \\ &= \frac{(u^2 - y^2) + (x^2 - u^2)}{3} \leq \frac{(v^2 - y^2) + (x^2 - u^2)}{3} \leq \frac{2}{3}M. \end{aligned}$$

Case 4: The case "$x < y$ and $u > v$" is not possible. Under this condition $x^2 < y^2$ and $u^2 > v^2$. Then by the condition $y^2 \leq v^2$, we have $x^2 < y^2 \leq v^2 < u^2$, which contradicts that $x^2 \geq u^2$.

In all above cases, for any $L \geq 0$,

$$\begin{aligned}\psi(d(F(x,\ y),\ F(u,\ v))) &\leq \frac{4}{9}M^2 = M^2 - \frac{5}{9}M^2 \\ &\leq \psi(max\,\{d(gx,\ gu),\ d(gy,\ gv)\}) \\ &\quad - \phi(max\,\{d(gx,\ gu),\ d(gy,\ gv)\}) \\ &\quad + L\ min\,\{d(F(x,\ y),\ gu),\ d(F(u,\ v),\ gx), \\ &\qquad d(F(x,\ y),\ gx),\ d(F(u,\ v),\ gu)\}.\end{aligned}$$

Hence the required conditions of Corollaries 3.4 and 3.5 are satisfied and it is seen that $(0,\ 0)$ is a coupled coincidence point of g and F.

Remark 3.7. *Considering ψ to be the identity mapping and $\phi(t) = (1 - k)\ t$ with $0 < k < 1$ in Corollaries 3.4 and 3.5, we have the generalizations of Theorems 2.1 and 2.2 of Bhaskar and Lakshmikantham [11] respectively and of Theorem 2.1 of Luong and Thuan [20].*

Remark 3.8. *In the above example ψ is not the identity mapping and $\phi(t) \neq (1-k)\ t$ with $0 < k < 1$ and hence the above mentioned generalizations of Theorems 2.1 and 2.2 of Bhaskar and Lakshmikantham [11] and of Theorem 2.1 of Luong and Thuan [20] are not applicable to the above example. Therefore, corollaries 3.4 and 3.5 and hence Theorems 3.1 and 3.2 are actual extensions of Theorems 2.1 and 2.2 of Bhaskar and Lakshmikantham [11] respectively and of Theorem 2.1 of Luong and Thuan [20] which are also noted here as Theorems 2.8, 2.9 and 2.10 respectively.*

Example 3.9. *Let $X = [0,\ \infty)$. Then $(X,\ \leq)$ is a partially ordered set with the natural ordering of real numbers. Let $d(x,\ y) = |x - y|$, for $x,\ y \in X$. Then $(X,\ d)$ is a metric space with the required properties of Theorem 3.3.*

Let $g : X \to X$ be defined as follows:

$$gx = \begin{cases} \dfrac{x}{2}, & if\ 0 \leq x \leq 1, \\ 200, & if\ x > 1. \end{cases}$$

Then g has the properties mentioned in theorem 3.3.

Let $\Lambda = \{1,\ 2,\ 3,\ \ldots\}$. Let the family of mappings $\{F_\alpha : X \times X \longrightarrow X : \alpha \in \Lambda\}$ be defined as follows: for $\alpha \in \Lambda$ with $\alpha \neq 1$,

$$F_\alpha(x,\ y) = \begin{cases} \dfrac{2\alpha}{\alpha+1}, & if\ x > 1,\ and\ y > 1, \\ \dfrac{1}{3}, & if\ x > 1,\ and\ 0 \leq y \leq 1, \\ 0, & otherwise, \end{cases}$$

and

$$F_1(x,\ y) = \begin{cases} \dfrac{1}{3}, & \text{if } x > 1 \text{ and } 0 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then $F_1(X \times X) \subseteq g(X)$ and F_1 has the mixed g-monotone property on X.
Let $\psi,\ \phi : [0,\ \infty) \longrightarrow [0,\ \infty)$ be defined as follows:

$$\psi(t) = t^2, \quad \phi(t) = \frac{5}{9}t^2.$$

Then ψ and ϕ have the properties mentioned in Theorem 3.3.

In following cases, we consider $(x,\ y),\ (u,\ v) \in X \times X$ for which $gx \succeq gu$ and $gy \preceq gv$.

Case 1: $x > 1$ and $0 \le y \le 1$.

(i) $u > 1$ *and* $v > 1$,

(ii) $u > 1$ *and* $0 \le v \le 1$ *with* $y \le v$,

(iii) $0 \le u \le 1$ *and* $v > 1$,

(iv) $0 \le u \le 1$ *and* $0 \le v \le 1$ *with* $y \le v$.

Case 2: $x > 1$ and $y > 1$.

(i) $u > 1$ *and* $v > 1$,

(ii) $0 \le u \le 1$ *and* $v > 1$.

Case 3: $0 \le x \le 1$ and $0 \le y \le 1$.

(i) $0 \le u \le 1$ *with* $u \le x$ *and* $v > 1$,

(ii) $0 \le u \le 1$ *with* $u \le x$ *and* $0 \le v \le 1$ *with* $y \le v$.

Case 4: $0 \le x \le 1$ and $y > 1$.

(i) $0 \le u \le 1$ *with* $u \le x$ *and* $v > 1$.

Let $L = 1$. Then, in all cases, the condition (iii) of Theorem 3.3 is satisfied. Hence all the required conditions of Theorems 3.3 are satisfied. Here, it is seen that $(0,\ 0) \in X \times X$ is a coupled coincidence point of g and $\{F_\alpha : \alpha \in \Lambda\}$.

Remark 3.10. *Theorem 3.3 is a generalization of Theorem 2.2 of Bhaskar and Lakshmikantham [11] and Theorem 2.1 (when the condition (b) holds) of Luong and Thuan [20] which are also noted here as Theorems 2.9 and 2.10 respectively. The above example, in which the family of mappings $\{F_\alpha : \alpha \in \Lambda\}$ contains countably infinite no of functions, is not applicable to above mentioned theorems which are special cases of Theorem 3.3.*

Note 1. In the above example, the function g is not continuous and hence it is not applicable to Theorems 3.1 and 3.2.

Note 2. If $L = 0$, then for $(x,\ y) = (u,\ v)$ the conditions (v) of Theorem 3.1 or the condition (iv) of Theorem 3.2 or the condition (iii) of Theorem 3.3 implies that $F_\alpha(x,\ y) = F_{\alpha_0}(x,\ y)$ for all $\alpha \in \Lambda$, that is, the family of mappings $\{F_\alpha : \alpha \in \Lambda\}$ becomes the single mapping.

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