Positive Solutions for a Class of Fourth-order Singular BVPs on the Positive Half-line[1]

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Abstract : In this work, we are concerned with the existence and multiplicity of positive solutions for the singular fourth-order boundary value problem on the half-line $x^{(4)}(t) - \eta x''(t) + \lambda x(t) = \phi(t) f(t, x(t), x'(t), x''(t), x'''(t)), \quad t \in I = (0, +\infty)$, $x(0) = x''(0) = 0$, $x(+\infty) = x''(+\infty) = 0$ where $f \in C(\mathbb{R}^+ \times I \times \mathbb{R}^3, \mathbb{R}^+)$ and η, λ are real positive constants such that $\eta^2 > 4\lambda$. By using the fixed point index theory on cones in appropriate Banach spaces, we obtained existence results of single and multiple positive solutions.

Keywords : fourth-order; half-line; singular problem; positive solution; fixed point index.

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1   Introduction

This paper is devoted to the study of the existence of multiple positive solutions for the nonlinear fourth-order equation set on the positive half-line:

$$
\begin{cases}
x^{(4)}(t) - \eta x''(t) + \lambda x(t) = \phi(t) f(t, x(t), x'(t), x''(t), x'''(t)), & t \in I, \\
x(0) = x''(0) = 0, \quad \lim_{t \to +\infty} x(t) = \lim_{t \to +\infty} x''(t) = 0,
\end{cases}
\tag{1.1}
$$

where η and λ are positive constants such that $\eta^2 > 4\lambda$. $\mathbb{R}^+$ denotes the set of nonnegative real numbers and $I = (0, +\infty)$. The function $f \in C(\mathbb{R}^+ \times I \times \mathbb{R}^3, \mathbb{R}^+)$ may have a singularity at the second variable and $\phi \in C(I, I)$.

Boundary value problems (bvps for short) on the half line arise in many applications in physics such that in modeling the unsteady flow of a gas through a semi-infinite porous media, in plasma physics, in determining the electrical potential in an isolated neutral atom, or in combustion theory (see [1–4]). In recent years, boundary value problems on the half-line have received a great deal of attention in the literature (see [5–19] and references therein). There are many papers considering the existence of positive solutions for second-order and even higher-order differential equations posed in the finite intervals (see, e.g., [20–29]). However, to the authors' knowledge, there are few results about third and fourth order boundary value problems posed on unbounded intervals. The goal of this paper is to fill the gap in this area. We aim to discuss the question of existence and multiplicity of positive solutions for a class of singular fourth order boundary value problems on the positive half-line with the nonlinearity depending on the derivatives x, x', x'', x'''; we shall define and make use of a fixed point index in a cone of some weighted Banach space. The singularity is treated by means of regularization, approximation, and compactness arguments.

This paper has mainly four sections. In Section 2, we prove some lemmas which are needed to develop subsequent results, we gather together some auxiliary results, and we give a fixed point formulation of the problem. In Section 3, we construct a special cone; then using the fixed point index theory, we prove the existence of at least one positive solution of (1.1). Finally, under a super-linearity condition of the right-hand side, the existence of at least two positive solutions of (1.1) is obtained in Section 4. An example is included to illustrate the applicability of the final existence result.

2   Preliminaries

2.1   Auxiliary Results and a Compactness Criterion

Let $p : I \longrightarrow I$ be a continuous function and let X_p be the set of the functions $x \in C^3([0, \infty), \mathbb{R})$ such that

$$
\sup_{t \in I} \{ [|x(t)| + |x'(t)| + |x''(t)| + |x'''(t)|] p(t) \} < \infty.
$$

Equipped with the norm $\|x\|_p = \sup_{t \in I}\{p(t)\sum_{k=0}^{k=3}|x^{(k)}(t)|\}$, X_p is a Banach space.

Definition 2.1. A set of functions $\Omega \subseteq X_p$ is said to be almost equicontinuous if it is equicontinuous on each interval $[0, T]$.

Lemma 2.2. *Let the functions $\{x \in \Omega \subseteq X_p\}$ and their derivatives x', x'' and x''' be almost equi-continuous on I and uniformly bounded in the sense of the norm $\|x\|_q = \sup_{t \in I}\{[\sum_{k=0}^{k=3}|x^{(k)}(t)|]q(t)\}$, where q is a positive continuous dominant function on I, that is $\lim_{t \to +\infty}\frac{p(t)}{q(t)} = 0$. Then Ω is relatively compact in X_p.*

Proof. If $(x_n)_{n \in \mathbb{N}}$ is a sequence in Ω, uniformly bounded for the norm $\|.\|_q$, then there exists some $M > 0$ such that

$$\forall n \in \mathbb{N}, \; \forall t \in I, \; \sum_{k=0}^{k=3}|x_n^{(k)}(t)| \leq \frac{M}{q(t)}.$$

The functions $(x_n^{(k)})_{n \in \mathbb{N}}$, $k = 0, 1, 2, 3$ are then uniformly bounded on any subinterval of I. In addition, these functions are, by assumption, equicontinuous on subintervals of I. By the Ascoli-Arzela lemma and a diagonal procedure, there exists some subsequence $(x_{n_j}^{(k)})_{j \in \mathbb{N}}$ converging uniformly to some limit function $x^{(k)}$ on every compact subset of I; moreover $\sum_{k=0}^{k=3}|x^{(k)}(t)| \leq \frac{M}{q(t)}$. Let us prove that the sequence $(x_{n_j})_{j \in \mathbb{N}}$, which we shorten to (x_n), converges in X_p for the p-weighted norm. Indeed, for any $T > 0$, we have

$$\|x_n - x\|_p = \sup_{t \in I}\sum_{k=0}^{k=3}|x_n^{(k)}(t) - x^{(k)}(t)|p(t)$$

$$\leq \sup_{t \in [0,T]}\sum_{k=0}^{k=3}|x_n^{(k)}(t) - x^{(k)}(t)|p(t) + \sup_{t > T}\sum_{k=0}^{k=3}|x_n^{(k)}(t) - x^{(k)}(t)|q(t)\frac{p(t)}{q(t)}$$

$$\leq \sum_{k=0}^{k=3}\sup_{t \in [0,T]}|x_n^{(k)}(t) - x^{(k)}(t)|\sup_{t \in [0,T]}p(t) + 2M\sup_{t > T}\frac{p(t)}{q(t)}.$$

Since, $(x_n)_{n \in \mathbb{N}}$ converges uniformly to x on every compact subset of I and $\lim_{t \to +\infty}\frac{p(t)}{q(t)} = 0$, we infer that $\lim_{n \to \infty}\|x_n - x\|_p = 0$, proving our claim. $\square$

Definition 2.3. A nonempty subset $\mathcal{P}$ of Banach space E is called a cone if it is convex, closed, and satisfies the conditions:

(i) $\alpha x \in \mathcal{P}$ for all $x \in \mathcal{P}$ and $\alpha \geq 0$,

(ii) $x, -x \in \mathcal{P}$ implies that $x = 0$.

Definition 2.4. A mapping $A : E \to E$ is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets.

The following two results will be needed in the sequel (for the main properties of the fixed point index, we refer to [30–32]).

Lemma 2.5 (See [30]). *Let Ω be a bounded open set in a real Banach space E, $\mathcal{P}$ a cone of E and $A : \overline{\Omega} \cap \mathcal{P} \to \mathcal{P}$ a completely continuous map. Suppose that $\lambda A x \neq x, \forall\, x \in \partial\Omega \cap \mathcal{P}$ and $\forall\, \lambda \in (0,1]$. Then $i(A, \Omega \cap \mathcal{P}, \mathcal{P}) = 1$.*

Lemma 2.6 (See [30]). *Let Ω be a bounded open set in a real Banach space E, $\mathcal{P}$ a cone of E, and $A : \overline{\Omega} \cap \mathcal{P} \to \mathcal{P}$ a completely continuous map. Suppose $Ax \not\leq x, \forall\, x \in \partial\Omega \cap \mathcal{P}$. Then $i(A, \Omega \cap \mathcal{P}, \mathcal{P}) = 0$.*

2.2 Related Lemmas

With $k_1 = \sqrt{\frac{\eta + \sqrt{\eta^2 - 4\lambda}}{2}}$ and $k_2 = \sqrt{\frac{\eta - \sqrt{\eta^2 - 4\lambda}}{2}}$, bvp (1.1) becomes

$$\begin{cases} x^{(4)}(t) - (k_1^2 + k_2^2)x''(t) + k_1^2 k_2^2 x(t) = \phi(t) f(t, x(t), x'(t), x''(t), x'''(t)), & t \in I, \\ x(0) = x''(0) = 0, \quad \lim\limits_{t \to +\infty} x(t) = \lim\limits_{t \to +\infty} x''(t) = 0. \end{cases}$$

For a given positive constant k, consider the Green's function G of the linear bvp:

$$\begin{cases} x''(t) - k^2 x(t) = 0, & t \in I, \\ x(0) = 0, \quad \lim\limits_{t \to +\infty} x(t) = 0, \end{cases} \tag{2.1}$$

that is

$$G(t,s) = \frac{1}{2k} \begin{cases} e^{-kt}(e^{ks} - e^{-ks}), & 0 \leq s \leq t < +\infty, \\ e^{-ks}(e^{kt} - e^{-kt}), & 0 \leq t \leq s < +\infty, \end{cases}$$

with partial derivative with respect to t

$$G_t(t,s) = \frac{1}{2} \begin{cases} e^{-ks}(e^{kt} + e^{-kt}), & 0 \leq t < s, \\ -e^{-kt}(e^{ks} - e^{-ks}), & 0 \leq s < t. \end{cases}$$

Lemma 2.7. *Let $\gamma(t) := (e^{2kt} - 1)e^{-(\theta + 3k)t}$. Then the function G satisfies the following properties:*

(a) $G(t,s) \geq 0, \quad \forall\, t, s \in \mathbb{R}^+.$

(b) $G(t,s) = G(s,t), \quad \forall t, s \in \mathbb{R}^+.$

(c) $G(t,s) \leq G(s,s) \leq \frac{1}{2k}, \quad \forall t, s \in \mathbb{R}^+.$

(d) $G(t,s)e^{-\theta t} \leq G(s,s)e^{-ks}, \quad \forall t, s \in I, \forall \theta \geq k.$

(e) $G(t,s)e^{-\theta t} \geq \gamma(t)G(\tau,s)e^{-\theta\tau}, \quad \forall t, s, \tau \in \mathbb{R}^+, \forall\, \theta \geq k.$

Proof. It is easy to prove the properties (a), (b), (c), and (d). So we only check part (e):

$$\frac{G(t,s)e^{-\theta t}}{G(\tau,s)e^{-\theta \tau}} = \begin{cases} \frac{e^{-ks}(e^{kt}-e^{-kt})}{e^{-k\tau}(e^{ks}-e^{-ks})}e^{\theta \tau}e^{-\theta t}, & t \leq s \leq \tau, \\[2mm] \frac{e^{-kt}(e^{ks}-e^{-ks})}{e^{-ks}(e^{k\tau}-e^{-k\tau})}e^{\theta \tau}e^{-\theta t}, & \tau \leq s \leq t, \\[2mm] \frac{e^{-kt}}{e^{-k\tau}}e^{\theta \tau}e^{-\theta t}, & s \leq t \leq \tau, \\[2mm] \frac{e^{kt}-e^{-kt}}{e^{k\tau}-e^{-k\tau}}e^{\theta \tau}e^{-\theta t}, & \tau \leq t \leq s, \\[2mm] \frac{e^{-kt}}{e^{-k\tau}}e^{\theta \tau}e^{-\theta t}, & s \leq \tau \leq t, \\[2mm] \frac{e^{kt}-e^{-kt}}{e^{k\tau}-e^{-k\tau}}e^{\theta \tau}e^{-\theta t}, & t \leq \tau \leq s, \end{cases}$$

which implies

$$\frac{G(t,s)e^{-\theta t}}{G(\tau,s)e^{-\theta \tau}} = \begin{cases} \frac{1}{(e^{2ks}-1)}e^{(\theta+k)\tau}(e^{2kt}-1)e^{-(\theta+k)t}, & t \leq s \leq \tau, \\ \frac{(e^{2ks}-1)}{(e^{2k\tau}-1)}e^{(\theta+k)\tau}e^{-(\theta+k)t}, & \tau \leq s \leq t, \\ e^{(\theta+k)\tau}e^{-(\theta+k)t}, & s \leq t \leq \tau, \\ \frac{e^{(\theta+k)\tau}}{(e^{2k\tau}-1)}(e^{2kt}-1)e^{-(\theta+k)t}, & \tau \leq t \leq s, \\ e^{(\theta+k)\tau}e^{-(\theta+k)t} & s \leq \tau \leq t, \\ \frac{e^{(\theta+k)\tau}}{(e^{2k\tau}-1)}(e^{2kt}-1)e^{-(\theta+k)t}, & t \leq \tau \leq s. \end{cases}$$

Since $\theta \geq k$, we get

$$\frac{G(t,s)e^{-\theta t}}{G(\tau,s)e^{-\theta \tau}} \geq \begin{cases} (e^{2kt}-1)e^{-(\theta+k)t}, & t \leq s \leq \tau, \\ e^{-(\theta+k)t}, & \tau \leq s \leq t, \\ e^{-(\theta+k)t}, & s \leq t \leq \tau, \\ (e^{2kt}-1)e^{-(\theta+k)t}, & \tau \leq t \leq s, \\ e^{-(\theta+k)t}, & s \leq \tau \leq t, \\ (e^{2kt}-1)e^{-(\theta+k)t}, & t \leq \tau \leq s. \end{cases}$$

Consequently

$$\frac{G(t,s)e^{-\theta t}}{G(\tau,s)e^{-\theta \tau}} \geq \begin{cases} (e^{2kt}-1)e^{-(\theta+3k)t} = \gamma(t), & t \leq s \leq \tau, \\ (e^{2kt}-1)e^{-(\theta+3k)t} = \gamma(t), & \tau \leq s \leq t, \\ (e^{2kt}-1)e^{-(\theta+3k)t} = \gamma(t), & s \leq t \leq \tau, \\ (e^{2kt}-1)e^{-(\theta+3k)t} = \gamma(t), & \tau \leq t \leq s, \\ (e^{2kt}-1)e^{-(\theta+3k)t} = \gamma(t), & s \leq \tau \leq t, \\ (e^{2kt}-1)e^{-(\theta+3k)t} = \gamma(t), & t \leq \tau \leq s. \end{cases}$$

$\square$

Denote by $G_t(s+0,s)$ the right-hand side derivative of G at (s,s) and $G_t(s-0,s)$ the left-hand side derivative of G at (s,s). Further properties of the Green's function G are presented in the next lemmas.

Lemma 2.8.

(a) $kG(t,s) \leq |G_t(t,s)| \leq k\coth(kt)G(t,s), \quad \forall t,s \in \mathbb{R}^+ \text{ with } t \neq s.$

(b) $G_{tt}(t,s) = k^2 G(t,s) \quad \forall t,s \in \mathbb{R}^+ \text{ with } t \neq s.$

(c) $\frac{\partial^3 G}{\partial^3 t}(t,s) = k^2 G_t(t,s) \quad \forall t,s \in \mathbb{R}^+ \text{ with } t \neq s.$

(d) $G(s,s) + |G_t(s+0,s)| \leq \frac{1+k}{k}, \quad \forall s \in \mathbb{R}^+.$

(e) $G(s,s) + |G_t(s-0,s)| \leq \frac{1+k}{k}, \quad \forall s \in \mathbb{R}^+.$

(f) $[G(t,s) + |G_t(t,s)|]e^{-\theta t} \leq 2[G(s,s) + |G_t(s-0,s)|]e^{-ks}, \quad \forall t < s, \forall \theta > k.$

(g) $[G(t,s) + |G_t(t,s)|]e^{-\theta t} \leq 2[G(s,s)) + |G_t(s+0,s)|]e^{-ks}, \quad \forall s < t, \forall \theta > k.$

(h) $\int_0^{+\infty} G(t,s)\,ds \leq \frac{1}{k^2}, \quad \forall t \geq 0.$

Proof. The properties (d), (e), (f), and (g) are proved in [33]. The properties (b), (c), (h), and the first inequality of (a) are immediate. To prove the second inequality of (a), it is sufficient to notice that

$$\frac{|G_t(t,s)|}{kG(t,s)} = \begin{cases} \frac{(e^{kt}+e^{-kt})}{(e^{kt}-e^{-kt})} = \coth(kt), & 0 \leq t < s < \infty, \\ 1 \leq \coth(kt), & 0 \leq s < t < \infty. \end{cases}$$

$\square$

From [8, Lemma 2.1], we have

Lemma 2.9. *Let* $h \in L^1(0,+\infty)$ *and*

$$x(t) = \int_0^{+\infty} G(t,s)h(s)\,ds. \tag{2.2}$$

Then

$$\begin{cases} -x''(t) + k^2 x(t) = h(t), & t \in I, \\ x(0) = 0, \quad \lim_{t\to+\infty} x(t) = 0. \end{cases}$$

The following two lemmas also hold

Lemma 2.10. *Let* $h \in L^1(0,+\infty)$ *and*

$$x(t) = \int_0^{+\infty} \int_0^{+\infty} G_1(t,s)G_2(s,\tau)h(\tau)\,d\tau\,ds. \tag{2.3}$$

Then $x \in C^3 \cap W^{4,1}(0,+\infty)$ *and* x *is solution of the fourth-order bvp:*

$$\begin{cases} x^{(4)}(t) - (k_1^2 + k_2^2)x''(t) + k_1^2 k_2^2 x(t) = h(t), & t \in I, \\ x(0) = x''(0) = 0, \quad \lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} x''(t) = 0, \end{cases} \tag{2.4}$$

where G_i is the Green's function of (2.1) with $k = k_i$ and $i = 1, 2$. As a consequence

$$x'(t) = \int_0^{+\infty} \int_0^{+\infty} G_{1t}(t, s) G_2(s, \tau) h(\tau) \, d\tau \, ds, \tag{2.5}$$

$$x''(t) = k_1^2 x(t) - \int_0^{+\infty} G_2(t, \tau) h(\tau) \, d\tau, \tag{2.6}$$

$$x'''(t) = k_1^2 x'(t) - \int_0^{+\infty} G_{2t}(t, \tau) h(\tau) \, d\tau. \tag{2.7}$$

Proof. Let x be a solution of (2.3) and let $H(s) = \int_0^{+\infty} G_2(s, \tau) h(\tau) d\tau$. Then $x(t) = \int_0^{+\infty} G_1(t, s) H(s) ds$. G being symmetric, Lemma 2.8(h) and Fubini's Theorem imply

$$\int_0^{+\infty} H(s) ds = \int_0^{+\infty} \int_0^{+\infty} G_2(s, \tau) h(\tau) d\tau \, ds < \frac{1}{k_2^2} \int_0^{+\infty} h(\tau) d\tau < +\infty.$$

Then Lemma 2.9 guarantees that

$$\begin{cases} -x''(t) + k_1^2 x(t) = H(t), & t \in I, \\ x(0) = 0, & \lim_{t \to +\infty} x(t) = 0 \end{cases}$$

and

$$\begin{cases} -H''(t) + k_2^2 H(t) = h(t), & t \in I, \\ H(0) = 0, & \lim_{t \to +\infty} H(t) = 0. \end{cases}$$

This implies that

$$\begin{cases} -(-x''(t) + k_1^2 x(t))''(t) + k_2^2(-x''(t) + k_1^2 x(t)) = h(t), & t \in I, \\ (-x''(0) + k_1^2 x(0)) = 0, & \lim_{t \to +\infty} (-x''(t) + k_1^2 x(t)) = 0. \end{cases}$$

Thus (2.4) holds. Now, it is easy to prove (2.5). So we only check (2.6). If we let $H(s) = \int_0^{+\infty} G_2(s, \tau) h(\tau) d\tau$, then

$$\begin{aligned} x''(t) &= \frac{d}{dt} \int_0^{+\infty} \frac{\partial G_1}{\partial t}(t, s) H(s) ds \\ &= \frac{d}{dt} \left(\int_0^t \frac{\partial G_1}{\partial t}(t, s) H(s) ds \right) + \frac{d}{dt} \left(\int_t^{+\infty} \frac{\partial G_1}{\partial t}(t, s) H(s) ds \right) \\ &= \left[\int_0^t \frac{\partial^2 G_1}{\partial^2 t}(t, s) H(s) ds - G_{1t}(t, t - 0) H(t) \right] \\ &\quad + \left[\int_t^{+\infty} \frac{\partial^2 G_1}{\partial^2 t}(t, s) H(s) ds + G_{1t}(t, t + 0) H(t) \right] \\ &= \left[\int_0^t \frac{\partial^2 G_1}{\partial^2 t}(t, s) H(s) ds - \frac{1}{2} e^{-k_1 t}(e^{k_1 t} - e^{-k_1 t}) H(t) \right] \\ &\quad + \left[\int_t^{+\infty} \frac{\partial^2 G_1}{\partial^2 t}(t, s) H(s) ds - \frac{1}{2} e^{-k_1 t}(e^{k_1 t} + e^{-k_1 t}) H(t) \right] \end{aligned}$$

$$= \int_0^{+\infty} \frac{\partial^2 G_1}{\partial^2 t}(t,s)H(s)ds - H(t)$$

$$= k_1^2 \int_0^{+\infty} G_1(t,s)H(s)ds - H(t)$$

$$= k_1^2 x(t) - \int_0^{+\infty} G_2(t,\tau)h(\tau)d\tau$$

and

$$x'''(t) = \frac{d}{dt}(x''(t))$$

$$= \frac{d}{dt}\left(k_1^2 x(t) - \int_0^{+\infty} G_2(t,\tau)h(\tau)d\tau\right)$$

$$= k_1^2 x'(t) - \int_0^{+\infty} G_{2t}(t,\tau)h(\tau)d\tau.$$

□

Remark 2.11. *We can also prove that x is a solution of (2.4) if and only if x is a solution of*

$$x(t) = \int_0^{+\infty} \int_0^{+\infty} G_2(t,s)G_1(s,\tau)h(\tau)\,d\tau\,ds$$

and for all $t,\tau \in \mathbb{R}^+$

$$\int_0^{+\infty} G_1(t,s)G_2(s,\tau)ds = \int_0^{+\infty} G_2(t,s)G_1(s,\tau)ds.$$

Lemma 2.12. *For all $\theta \geq k_1$ and for all $t,\tau \in \mathbb{R}^+$, we have*

$$\int_0^{+\infty} [G_1(t,s) + |G_{1t}(t,s)|]G_2(s,\tau)e^{-\theta t}\,ds \leq \frac{1+k_1}{k_1^2 k_2}.$$

Proof. Using the properties (d), (e), (f), and (g) of Lemma 2.8, we have

$$\int_0^{+\infty} [G_1(t,s) + |G_{1t}(t,s)|]G_2(s,\tau)e^{-\theta t}\,ds \leq \frac{2(1+k_1)}{k_1} \int_0^{+\infty} G_2(s,\tau)e^{-k_1 s}\,ds$$

$$\leq \frac{2(1+k_1)}{k_1} \int_0^{+\infty} G_2(s,s)e^{-k_1 s}\,ds$$

$$\leq \frac{(1+k_1)}{k_1 k_2} \int_0^{+\infty} e^{-k_1 s}\,ds$$

$$\leq \frac{1+k_1}{k_1^2 k_2}.$$

□

Lemma 2.13. *For all* $t, \tau \in \mathbb{R}^+$, *we have*

$$\int_0^{+\infty} G_1(t,s)G_2(s,\tau)\, ds \geq k' G_1(t,t)G_2(t,\tau),$$

where $k' = \min\left\{ \frac{1-e^{-k_1}}{k_1}, \frac{e^{-(k_1+k_2)}}{k_1+k_2} \right\}$.

Proof. (1) If $t \geq \tau$, we have

$$\int_0^{+\infty} G_1(t,s)G_2(s,\tau)\, ds \geq \int_t^{+\infty} G_1(t,s)G_2(s,\tau)\, ds$$

$$= \frac{1}{4k_1k_2}(e^{k_1t} - e^{-k_1t})(e^{k_2\tau} - e^{-k_2\tau})\int_t^{+\infty} e^{-(k_1+k_2)s}\, ds$$

$$= \frac{1}{4k_1k_2(k_1+k_2)}(e^{k_1t} - e^{-k_1t})(e^{k_2\tau} - e^{-k_2\tau})e^{-(k_1+k_2)t}$$

$$\geq \frac{1}{(k_1+k_2)}G_1(t,t)G_2(t,\tau)$$

$$\geq k' G_1(t,t)G_2(t,\tau).$$

(2) If $t \leq \tau$, then we distinguish between two cases:

(a) If $t < t+1 \leq \tau$, we have the estimates:

$$\int_0^{+\infty} G_1(t,s)G_2(s,\tau)\, ds \geq \int_t^{t+1} G_1(t,s)G_2(s,\tau)\, ds$$

$$= \frac{1}{4k_1k_2}e^{-k_2\tau}(e^{k_1t} - e^{-k_1t})\int_t^{t+1} e^{-k_1s}(e^{k_2s} - e^{-k_2s})\, ds$$

$$= \frac{1}{4k_1k_2}e^{-k_2\tau}(e^{k_1t} - e^{-k_1t})(e^{k_2t} - e^{-k_2t})\int_t^{t+1} e^{-k_1s}\, ds$$

$$= \frac{1}{4k_1^2k_2}e^{-k_2\tau}(e^{k_1t} - e^{-k_1t})(e^{k_2t} - e^{-k_2t})(e^{-k_1t} - e^{-k_1(t+1)})$$

$$= \frac{1}{k_1}G_1(t,t)G_2(t,\tau)(1 - e^{-k_1})$$

$$\geq k' G_1(t,t)G_2(t,\tau).$$

(b) If $t \leq \tau \leq t+1$, then we have

$$\int_0^{+\infty} G_1(t,s)G_2(s,\tau)\, ds \geq \int_{t+1}^{+\infty} G_1(t,s)G_2(s,\tau)\, ds$$

$$= \frac{1}{4k_1k_2}(e^{k_1t} - e^{-k_1t})(e^{k_2\tau} - e^{-k_2\tau})\int_{t+1}^{+\infty} e^{-(k_1+k_2)s}\, ds$$

$$= \frac{1}{4k_1k_2(k_1+k_2)}(e^{k_1t} - e^{-k_1t})(e^{k_2\tau} - e^{-k_2\tau})e^{-(k_1+k_2)(t+1)}$$

$$= \frac{e^{-(k_1+k_2)}}{4k_1k_2(k_1+k_2)}(e^{k_1t} - e^{-k_1t})(e^{k_2\tau} - e^{-k_2\tau})e^{-(k_1+k_2)t}$$

$$= \frac{e^{-(k_1+k_2)}}{2k_2(k_1+k_2)} G_1(t,t)(e^{k_2\tau} - e^{-k_2\tau})e^{-k_2t}$$

$$= \frac{e^{-(k_1+k_2)}}{2k_2(k_1+k_2)} G_1(t,t)e^{-k_2\tau}(e^{2k_2\tau}e^{-k_2t} - e^{-k_2t})$$

$$\geq \frac{e^{-(k_1+k_2)}}{2k_2(k_1+k_2)} G_1(t,t)e^{-k_2\tau}(e^{k_2t} - e^{-k_2t})$$

$$= \frac{e^{-(k_1+k_2)}}{(k_1+k_2)} G_1(t,t)G_2(t,\tau)$$

$$\geq k' G_1(t,t)G_2(t,\tau).$$

$\square$

2.3   General Setting and Assumptions

Given a real parameter $\theta > k_1$, consider the weighted Banach space

$$E = \left\{ x \in C^3(\mathbb{R}^+, \mathbb{R}) : \sup_{t \geq 0} \sum_{k=0}^{k=3} |x^{(k)}(t)|e^{-\theta t} < \infty \right\}$$

endowed with the weighted Bielecki's sup-norm

$$\|x\|_\theta = \sup_{t \geq 0} \sum_{k=0}^{k=3} |x^{(k)}(t)|e^{-\theta t}.$$

We have

Lemma 2.14. *If x is solution of (2.4), then*

$$x(t) \geq \widetilde{k}\gamma_1(t)\|x\|_\theta,$$

where $\widetilde{k} = \frac{1}{4} \min \left\{ 1, \frac{1}{k_1}, \frac{k'}{k_1(k_1k'+2)}, \frac{k'}{k_1(k_1^2k'+2k_2)} \right\}$ *and* $\gamma_1(t) := (e^{2k_1t} - 1)e^{-(\theta+3k_1)t}$.

Proof. Let x be a solution of (2.4). Lemma 2.7(e) guarantees that

$$x(t) = \int_0^{+\infty} \int_0^{+\infty} G_1(t,s)G_2(s,\tau)h(\tau)d\tau ds$$

$$= e^{\theta t} \int_0^{+\infty} \int_0^{+\infty} e^{-\theta t}G_1(t,s)G_2(s,\tau)h(\tau)d\tau ds$$

$$\geq e^{\theta t}\gamma_1(t) \int_0^{+\infty} \int_0^{+\infty} e^{-\theta r}G_1(r,s)G_2(s,\tau)h(\tau)d\tau ds$$

$$\geq \gamma_1(t)e^{-\theta r}x(r), \ \forall r \in \mathbb{R}^+.$$

Passing to the supremum over r, we get the lower bound

$$x(t) \geq \gamma_1(t) \sup_{t \in \mathbb{R}^+} |x(t)| e^{-\theta t}. \qquad (2.8)$$

In the other hand, by Lemma 2.8(a), we have

$$\begin{aligned}
|x'(t)| &\leq \int_0^{+\infty} \int_0^{+\infty} |G_{1t}(t,s)| G_2(s,\tau) h(\tau) d\tau ds \\
&\leq k_1 \coth(k_1 t) \int_0^{+\infty} \int_0^{+\infty} G_1(t,s) G_2(s,\tau) h(\tau) d\tau ds \\
&\leq k_1 \coth(k_1 t) x(t),
\end{aligned}$$

which implies that $\tanh(k_1 t)|x'(t)| \leq k_1 x(t)$. Since $\sup_{t \in \mathbb{R}^+} \tanh(k_1 t) = 1 < \infty$ and $\sup_{t \in \mathbb{R}^+} |x'(t)| e^{-\theta t} < \infty$, then

$$\begin{aligned}
\sup_{t \in \mathbb{R}^+} |x'(t)| e^{-\theta t} &= \sup_{t \in \mathbb{R}^+} \tanh(kt) \sup_{t \in \mathbb{R}^+} |x'(t)| e^{-\theta t} \\
&= \sup_{t \in \mathbb{R}^+} \tanh(k_1 t)|x'(t)| e^{-\theta t} \\
&\leq k_1 \sup_{t \in \mathbb{R}^+} |x(t)| e^{-\theta t}.
\end{aligned}$$

Moreover, using (2.8), we obtain

$$x(t) \geq \frac{1}{k_1} \gamma_1(t) \sup_{t \in \mathbb{R}^+} |x'(t)| e^{-\theta t}. \qquad (2.9)$$

Furthermore, Lemma 2.10 guarantees that

$$x''(t) = k_1^2 x(t) - \int_0^{+\infty} G_2(t,\tau) h(\tau) \, d\tau$$

and from Lemma 2.13, we get the estimates

$$\begin{aligned}
G_1(t,t)|x''(t)| &\leq k_1^2 G_1(t,t) x(t) + \int_0^{+\infty} G_1(t,t) G_2(t,\tau) h(\tau) \, d\tau \\
&\leq \frac{k_1}{2} x(t) + \frac{1}{k'} \int_0^{+\infty} \int_0^{+\infty} G_1(t,s) G_2(s,\tau) h(\tau) \, d\tau \, ds \\
&\leq \frac{k_1}{2} x(t) + \frac{1}{k'} x(t) \\
&\leq \frac{k_1 k' + 2}{2k'} x(t).
\end{aligned}$$

Since $\sup_{t\in\mathbb{R}^+} G_1(t,t) = \frac{1}{2k_1}$, then

$$\sup_{t\in\mathbb{R}^+} |x(t)|e^{-\theta t} \geq \frac{2k'}{k_1 k' + 2} \sup_{t\in\mathbb{R}^+} G_1(t,t)|x''(t)|e^{-\theta t}$$

$$\geq \frac{2k'}{k_1 k' + 2} \sup_{t\in\mathbb{R}^+} G_1(t,t) \sup_{t\in\mathbb{R}^+} |x''(t)|e^{-\theta t}$$

$$\geq \frac{k'}{k_1(k_1 k' + 2)} \sup_{t\in\mathbb{R}^+} |x''(t)|e^{-\theta t}.$$

(2.8) yields

$$x(t) \geq \frac{k'}{k_1(k_1 k' + 2)}\gamma_1(t) \sup_{t\in\mathbb{R}^+} |x''(t)|e^{-\theta t}. \qquad (2.10)$$

Moreover, from Lemma 2.10, we have

$$x'''(t) = k_1^2 x'(t) - \int_0^{+\infty} G_{2t}(t,\tau)h(\tau)\, d\tau$$

and from Lemma 2.13 and Lemma 2.8(a), we have

$$G_1(t,t)|x'''(t)| \leq k_1^2 G_1(t,t)|x'(t)| + \int_0^{+\infty} G_1(t,t)|G_{2t}(t,\tau)|h(\tau)\, d\tau$$

$$\leq k_1^2 G_1(t,t)|x'(t)| + k_2 \coth(k_2 t) \int_0^{+\infty} G_1(t,t)G_2(t,\tau)h(\tau)\, d\tau$$

$$\leq \frac{k_1}{2}|x'(t)| + \frac{k_2}{k'} \coth(k_2 t) \int_0^{+\infty}\int_0^{+\infty} G_1(t,s)G_2(s,\tau)h(\tau)\, d\tau\, ds$$

$$\leq \frac{k_1}{2}|x'(t)| + \frac{k_2}{k'} \coth(k_2 t)x(t).$$

Hence

$$\tanh(k_2 t)G_1(t,t)|x'''(t)| \leq \frac{k_1}{2}|x'(t)| + \frac{k_2}{k'}x(t).$$

Then

$$\sup_{t\in\mathbb{R}^+} \tanh(k_2 t)G_1(t,t)|x'''(t)|e^{-\theta t} \leq \frac{k_1}{2} \sup_{t\in\mathbb{R}^+} |x'(t)|e^{-\theta t} + \frac{k_2}{k'} \sup_{t\in\mathbb{R}^+} |x(t)|e^{-\theta t}$$

which implies that

$$\sup_{t\in\mathbb{R}^+} |x'''(t)|e^{-\theta t} \leq k_1^2 \sup_{t\in\mathbb{R}^+} |x'(t)|e^{-\theta t} + \frac{2k_1 k_2}{k'} \sup_{t\in\mathbb{R}^+} |x(t)|e^{-\theta t}.$$

By (2.8) and (2.9), we deduce that

$$\gamma_1(t) \sup_{t\in\mathbb{R}^+} |x'''(t)|e^{-\theta t} \leq k_1^2 \gamma_1(t) \sup_{t\in\mathbb{R}^+} |x'(t)|e^{-\theta t} + \frac{2k_1 k_2}{k'}\gamma_1(t) \sup_{t\in\mathbb{R}^+} |x(t)|e^{-\theta t}$$

$$\leq k_1^3 x(t) + \frac{2k_1 k_2}{k'}x(t)$$

$$\leq \frac{k_1(k_1^2 k' + 2k_2)}{k'}x(t).$$

Hence

$$x(t) \geq \frac{k'}{k_1(k_1^2 k' + 2k_2)} \gamma_1(t) \sup_{t \in \mathbb{R}^+} |x'''(t)| e^{-\theta t}. \tag{2.11}$$

Finally, using (2.8), (2.9), (2.10) and (2.11), we obtain that

$$x(t) \geq \widetilde{k}\gamma_1(t) \sup_{t \in \mathbb{R}^+} \left[|x(t)| + |x'(t)| + |x'(t)| + |x'''(t)| \right] e^{-\theta t} = \widetilde{k}\gamma_1(t)\|x\|_\theta.$$

$\square$

Throughout this paper, we denote $F(t, x, y, u, v) = f(t, e^{\theta t} x, e^{\theta t} y, e^{\theta t} u, e^{\theta t} v)$ and $\widetilde{\gamma_1}(t) = \gamma_1(t)e^{-\theta t}$ and we list the following hypotheses:

$(\mathcal{H}_1)$ There exist $p \in C(I, \mathbb{R}^+)$ and $g \in C(\mathbb{R}^+ \times \mathbb{R}^3, \mathbb{R}^+)$ such that g is nondecreasing in the second, third, and fourth variable with

$$F(t, x, y, u, v) \leq p(x)g(t, y, u, v), \quad \forall (t, x, y, u, v) \in \mathbb{R}^+ \times I \times \mathbb{R}^3 \tag{2.12}$$

and there exists a decreasing function $h \in C(I, I)$ such that $\frac{p}{h}$ is nondecreasing function with

$$\int_0^{+\infty} \phi(\tau)h(c\widetilde{\gamma_1}(\tau))g(\tau, c', c', c')d\tau < +\infty, \text{ for each } c, c' > 0. \tag{2.13}$$

$(\mathcal{H}_2)$ For each $c, c' > 0$, there exists a function $\psi_{c,c'} \in C(\mathbb{R}^+, I)$ such that

$$F(t, x, y, u, v) \geq \psi_{c,c'}(t), \quad \forall t \in \mathbb{R}^+, \ \forall (x, y, u, v) \in (0, c] \times [-c', c']^3$$

with

$$\int_0^{+\infty} \phi(\tau)\psi_{c,c'}(\tau)d\tau < +\infty. \tag{2.14}$$

$(\mathcal{H}_3)$

$$\sup_{c>0} \frac{ch(c)}{p(c) \int_0^{+\infty} \phi(\tau)h(\widetilde{k}\widetilde{\gamma_1}(\tau)c)g(\tau, c, c, c)d\tau} > \frac{(1 + k_1)(1 + k_1^2 + 2k_1 k_2)}{k_1^2 k_2}.$$

3   Existence Result

Given $f \in C(\mathbb{R}^+ \times I \times \mathbb{R}^3, \mathbb{R}^+)$, define a sequence of functions $\{f_n\}_{n \geq 1}$ by

$$f_n(t, x, y, u, v) = f(t, \max\{e^{\theta t}/n, x\}, y, u, v), \quad n \in \{1, 2, \ldots\}. \tag{3.1}$$

Let $\mathcal{P}$ be the positive cone defined in E by

$$\mathcal{P} = \{x \in E : \ x(t) \geq \widetilde{k}\gamma_1(t)\|x\|_\theta, \ \forall t \geq 0\},$$

where $\widetilde{k}$ is the constant defined in Lemma 2.14. For $x \in \mathcal{P}$, define a sequence of operators by

$$A_n x(t) = \int_0^{+\infty} \int_0^{+\infty} G_1(t,s) G_2(s,\tau) \phi(\tau) f_n(\tau, x(\tau), x'(\tau), x''(\tau), x'''(\tau)) d\tau ds,$$
$$(3.2)$$

for $n \in \{1, 2, \ldots\}$.

Lemma 3.1. *Suppose that $(\mathcal{H}_1)$ holds. Then, for each $n \geq 1$, the operator A_n sends $\mathcal{P}$ into $\mathcal{P}$ and is completely continuous.*

Proof. Step 1. First, we show that $A_n \mathcal{P} \subseteq \mathcal{P}$. For each $x \in \mathcal{P}$, we have by Lemma 2.10

$$(A_n x)''(t) = k_1^2 (A_n x)(t) - \int_0^{+\infty} G_2(t,\tau) \phi(\tau) f_n(\tau, x(\tau), x'(\tau), x''(\tau), x'''(\tau)) d\tau$$

and

$$(A_n x)'''(t) = k_1^2 (A_n x)'(t) - \int_0^{+\infty} G_{2t}(t,\tau) \phi(\tau) f_n(\tau, x(\tau), x'(\tau), x''(\tau), x'''(\tau)) d\tau.$$

Using Lemmas 2.8 and 2.12 and the condition $(\mathcal{H}_1)$, we obtain the estimates

$$\sum_{k=0}^{k=3} |(A_n x)^{(k)}(t)| e^{-\theta t}$$

$$\leq (1 + k_1^2)[|A_n x(t)| + |(A_n x)'(t)|] e^{-\theta t}$$

$$+ \int_0^{+\infty} [G_2(t,\tau) + |G_{2t}(t,\tau)|] e^{-\theta t} \phi(\tau) f_n(\tau, x(\tau), x'(\tau), x''(\tau), x'''(\tau)) d\tau$$

$$\leq (1 + k_1^2) \int_0^{+\infty} \int_0^{+\infty} [G_1(t,s) + |G_{1t}(t,s)|] G_2(s,\tau) e^{-\theta t} \phi(\tau)$$

$$\times f_n(\tau, x(\tau), x'(\tau), x''(\tau), x'''(\tau)) d\tau ds$$

$$+ \int_0^{+\infty} (G_2(t,\tau) + |G_{2t}(t,\tau)|) e^{-\theta t} \phi(\tau) f_n(\tau, x(\tau), x'(\tau), x''(\tau), x'''(\tau)) d\tau$$

$$\leq (1 + k_1^2) \int_0^{+\infty} \left(\int_0^{+\infty} [G_1(t,s) + |G_{1t}(t,s)|] G_2(s,\tau) e^{-\theta t} ds \right)$$

$$\times \phi(\tau) f_n(\tau, x(\tau), x'(\tau), x''(\tau), x'''(\tau)) d\tau$$

$$+ \int_0^{+\infty} [G_2(t,\tau) + |G_{2t}(t,\tau)|] e^{-\theta t} \phi(\tau) f_n(\tau, x(\tau), x'(\tau), x''(\tau), x'''(\tau)) d\tau$$

$$\leq \frac{(1+k_1)(1+k_1^2)}{k_1^2 k_2} \int_0^{+\infty} \phi(\tau) f_n(\tau, x(\tau), x'(\tau), x''(\tau), x'''(\tau)) \, d\tau$$

$$+ \frac{2(1+k_1)}{k_1} \int_0^{+\infty} \phi(\tau) f_n(\tau, x(\tau), x'(\tau), x''(\tau), x'''(\tau)) d\tau$$

$$\leq \frac{(1 + k_1)(1 + k_1^2 + 2k_1 k_2)}{k_1^2 k_2}$$

$$\times \int_0^{+\infty} \phi(\tau) F(\tau, \max\{1/n, x(\tau)e^{-\theta\tau}\}, x'(\tau)e^{-\theta\tau}, x''(\tau)e^{-\theta\tau}, x'''(\tau)e^{-\theta\tau}) \, d\tau$$

$$\leq \frac{(1 + k_1)(1 + k_1^2 + 2k_1 k_2)}{k_1^2 k_2} \frac{p(\max\{1/n, \|x\|_\theta\})}{h(\max\{1/n, \|x\|_\theta\})}$$

$$\times \int_0^{+\infty} \phi(\tau) h(\widetilde{k}\widetilde{\gamma_1}(\tau)\|x\|_\theta) g(\tau, \|x\|_\theta, \|x\|_\theta, \|x\|_\theta) \, d\tau < +\infty.$$

Then $A_n x \in E$. As in Lemma 2.14, we can prove that for each $n \geq 1$, $A_n x \in \mathcal{P}$.

Step 2. We will show that, for each n, the operator $A_n : \mathcal{P} \longrightarrow \mathcal{P}$ is completely continuous.

(a) Let $\{x_j\}_j \subseteq \mathcal{P}$, $x_0 \in \mathcal{P}$ with $\lim_{j \to +\infty} x_j = x_0$. Then there exists $M > 1$ such that $\|x_j\|_\theta \leq M$, $\forall j \in \{0, 1, 2, \ldots\}$. Then, for any $j \in \{0, 1, 2, \ldots\}$, we have

$$f_n(\tau, x_j(\tau), x_j'(\tau), x_j''(\tau), x_j'''(\tau))$$

$$= F(\tau, \max\{1/n, x_j(\tau)e^{-\theta\tau}\}, x_j'(\tau)e^{-\theta\tau}, x_j''(\tau)e^{-\theta\tau}, x_j'''(\tau)e^{-\theta\tau})$$

$$\leq p(\max\{1/n, x_j(\tau)e^{-\theta\tau}\}) g(\tau, x_j'(\tau)e^{-\theta\tau}, x_j''(\tau)e^{-\theta\tau}, x_j'''(\tau)e^{-\theta\tau})$$

$$\leq \frac{p(\max\{1/n, x_j(\tau)e^{-\theta\tau}\})}{h(\max\{1/n, x_j(\tau)e^{-\theta\tau}\})} h(\max\{1/n, x_j(\tau)e^{-\theta\tau}\})$$

$$\times g(\tau, x_j'(\tau)e^{-\theta\tau}, x_j''(\tau)e^{-\theta\tau}, x_j'''(\tau)e^{-\theta\tau})$$

$$\leq \frac{p(\max\{1/n, \|x_j\|_\theta\})}{h(\max\{1/n, \|x_j\|_\theta\})} h(1/n) g(\tau, \|x\|_\theta, \|x\|_\theta, \|x\|_\theta)$$

$$\leq \frac{p(M)}{h(M)} h(1/n\widetilde{\gamma_1}(\tau)) g(\tau, M, M, M).$$

By the continuity of f, for $\tau \in \mathbb{R}^+$

$$|f_n(\tau, x_j(\tau), x_j'(\tau), x_j''(\tau), x_j'''(\tau)) - f_n(\tau, x_0(\tau), x_0'(\tau), x_0''(\tau), x_0'''(\tau))| \longrightarrow 0,$$

$$\text{as } j \to +\infty.$$

Moreover,

$$\|A_n x_j - A_n x_0\|_\theta$$

$$= \sum_{k=0}^{k=3} \sup_{t \in \mathbb{R}^+} |(A_n x_j)^{(k)}(t) - (A_n x_0)^{(k)}(t)| e^{-\theta t}$$

$$\leq \frac{(1 + k_1)(1 + k_1^2 + 2k_1 k_2)}{k_1^2 k_2} \int_0^{+\infty} \phi(\tau) |f_n(\tau, x_j(\tau), x_j'(\tau), x_j''(\tau), x_j'''(\tau))$$

$$- f_n(\tau, x_0(\tau), x_0'(\tau), x_0''(\tau), x_0'''(\tau))| \, d\tau.$$

Since

$$|f_n(\tau, x_j(\tau), x'_j(\tau), x''_j(\tau), x'''_j(\tau)) - f_n(\tau, x_0(\tau), x'_0(\tau), x''_0(\tau), x'''_0(\tau))|$$
$$\leq 2\frac{p(M)}{h(M)}h(1/n\widetilde{\gamma_1}(\tau))g(\tau, M, M, M),$$

then the condition $(\mathcal{H}_1)$ and the Lebesgue dominated convergence theorem imply that the right-hand side term tends to zero, as $j \to +\infty$.

(b) Let $D \subset \mathcal{P}$ be a bounded set; then there exists an $M > 1$ such that

$$\|x\|_\theta \leq M, \ \forall\, x \in D.$$

(i) $A_n(D)$ is uniformly bounded. To prove this, let $\mu \in (k, \theta)$ and take $q(t) = e^{-\mu t}$ in Lemma 2.2. Then, for each $x \in D$, we have

$$\|A_n x\|_\mu \leq (1 + k_1^2)[|A_n x(t)| + |(A_n x)'(t)|]e^{-\mu t}$$
$$+ \int_0^{+\infty} [G_2(t, \tau) + |G_{2t}(t, \tau)|]e^{-\mu t}\phi(\tau)f_n(\tau, x_j(\tau), x'_j(\tau), x''_j(\tau), x'''_j(\tau)\, d\tau$$
$$\leq \frac{(1 + k_1)(1 + k_1^2 + 2k_1 k_2)}{k_1^2 k_2}\frac{p(M)}{h(M)}\int_0^{+\infty} \phi(\tau)h(1/n\widetilde{\gamma_1}(\tau))g(\tau, M, M, M)\, d\tau < +\infty.$$

(ii) The functions of the sets $A_n(D)$, $(A_n(D))'$, $(A_n(D))''$, and $(A_n(D))'''$ are almost equicontinuous. For a given $T > 0$, $x \in D$, and $t, t' \in [0, T]$, we have the estimates

$$|A_n x(t) - A_n x(t')|$$
$$\leq \int_0^{+\infty}\int_0^{+\infty} |G_1(t, s) - G_1(t', s)|G_2(s, \tau)\phi(\tau)f_n(\tau, x(\tau), x'(\tau), x''(\tau), x'''(\tau))\, ds\, d\tau$$
$$\leq \int_0^{+\infty}\int_0^{+\infty} |G_1(t, s) - G_1(t', s)|G_2(\tau, \tau)\phi(\tau)f_n(\tau, x(\tau), x'(\tau), x''(\tau), x'''(\tau))\, ds\, d\tau$$
$$= \int_0^{+\infty} |G_1(t, s) - G_1(t', s)|\, ds \int_0^{+\infty} G_2(\tau, \tau)\phi(\tau)f_n(\tau, x(\tau), x'(\tau), x''(\tau), x'''(\tau))\, d\tau$$
$$\leq \left(\int_0^T |G_1(t, s) - G_1(t', s)|\, ds + \int_T^{+\infty} |G_1(t, s) - G_1(t', s)|\, ds\right)$$
$$\times \left(\int_0^{+\infty} G_2(\tau, \tau)\phi(\tau)f_n(\tau, x(\tau), x'(\tau), x''(\tau), x'''(\tau))\, d\tau\right)$$
$$\leq \frac{1}{2k_2}\frac{p(M)}{h(M)}\left(\int_0^T |G_1(t, s) - G_1(t', s)|\, ds + [(e^{k_1 t} - e^{-k_1 t})\right.$$
$$\left. -(e^{k_1 t'} - e^{-k_1 t'})]\int_T^{+\infty} e^{-k_1 s}\, ds\right)\left(\int_0^{+\infty} \phi(\tau)h(1/n\widetilde{\gamma_1}(\tau))g(\tau, M, M, M)\, d\tau\right).$$

In the same way, we have

$$|(A_nx)'(t) - (A_nx)'(t')|$$

$$\leq \int_0^{+\infty} \int_0^{+\infty} |G_{1t}(t,s) - G_{1t}(t',s)|G_2(s,\tau)\phi(\tau)f_n(\tau,x(\tau),x'(\tau),x''(\tau),x'''(\tau))\,ds\,d\tau$$

$$\leq \frac{1}{2k_2} \int_0^{+\infty} |G_{1t}(t,s) - G_{1t}(t',s)|ds \int_0^{+\infty} \phi(\tau)f_n(\tau,x(\tau),x'(\tau),x''(\tau),x'''(\tau))\,d\tau$$

$$\leq \frac{1}{2k_2} \frac{p(M)}{h(M)} \int_0^{+\infty} |G_{1t}(t,s) - G_{1t}(t',s)|ds \int_0^{+\infty} \phi(\tau)h(1/n\widetilde{\gamma_1}(\tau))g(\tau,M,M,M)\,d\tau,$$

$$|(A_nx)''(t) - (A_nx)''(t')|$$
$$\leq k_1^2|(A_nx)(t) - (A_nx)(t')|$$
$$+ \int_0^{+\infty} |G_2(t,\tau) - G_2(t',\tau)|\phi(\tau)f_n(\tau,x(\tau),x'(\tau),x''(\tau),x'''(\tau))\,d\tau,$$

and

$$|(A_nx)'''(t) - (A_nx)'''(t')|$$
$$\leq k_1^2|(A_nx)'(t) - (A_nx)'(t')|$$
$$+ \int_0^{+\infty} |G_{2t}(t,\tau) - G_{2t}(t',\tau)|\phi(\tau)f_n(\tau,x(\tau),x'(\tau),x''(\tau),x'''(\tau))\,d\tau.$$

Now since $\int_0^{+\infty} \phi(\tau)h(1/n\widetilde{\gamma_1}(\tau))g(\tau,M,M,M)d\tau < +\infty$ and $\int_T^{+\infty} e^{-k_1s}\,ds < +\infty$, then (A_nD), $(A_nD)'$, $(A_nD)''$, and $(A_nD)'''$ are almost equicontinuous. Hence Lemma 2.2 with $q(t) = e^{-\mu t}$ guarantees that $A_n(D)$ is relatively compact in E; hence $A_n : \mathcal{P} \longrightarrow \mathcal{P}$ is completely continuous. $\square$

Theorem 3.2. *Assume that Assumptions $(\mathcal{H}_1) - (\mathcal{H}_3)$ hold. Then Problem (1.1) has at least one positive solution $x \in \mathcal{P}$.*

Proof. From Lemma 3.1, we know that $A_n : \mathcal{P} \longrightarrow \mathcal{P}$ is completely continuous. Using Lemma 2.5, we show that it has a fixed point.

Step 1. From $(\mathcal{H}_3)$, there exists $R > 0$ such that

$$\frac{Rh(R)}{p(R) \int_0^{+\infty} \phi(\tau)h(\widetilde{k}\widetilde{\gamma_1}(\tau)R)g(\tau,R,R,R)d\tau} > \frac{(1+k_1)(1+k_1^2+2k_1k_2)}{k_1^2k_2}. \quad (3.3)$$

Let

$$\Omega_1 = \{x \in E : \|x\|_\theta < R\}.$$

We claim that $x \neq \lambda A_nx$ for any $x \in \partial\Omega_1 \cap \mathcal{P}, \lambda \in (0,1]$ and $n \geq n_0 > 1/R$. On the contrary, suppose that there exist $n \geq n_0$, $x_0 \in \partial\Omega_1 \cap \mathcal{P}$ and $\lambda_0 \in (0,1]$ such that $x_0 = \lambda_0 A_nx_0$. Since $x_0 \in \partial\Omega_1 \cap \mathcal{P}$, we have

$$x_0(t) \geq \widetilde{k}\gamma_1(t)\|x_0\|_\theta = \widetilde{k}\gamma_1(t)R, \ \forall\,t \geq 0.$$

As a consequence, we have

$$
\begin{aligned}
R &= \|x_0\|_\theta \\
&\leq \|A_n x_0\|_\theta \\
&\leq (1 + k_1^2) \sup_{t \in \mathbb{R}^+} [|A_n x_0(t)| + |A_n x_0'(t)|] e^{-\theta t} \\
&\quad + \sup_{t \in \mathbb{R}^+} \int_0^{+\infty} [G_2(t,\tau)) + |G_{2t}(t,\tau)|] e^{-\theta t} \phi(\tau) f_n(\tau, x_0(\tau), x_0'(\tau), x_0''(\tau), x_0'''(\tau)) \, d\tau \\
&\leq \frac{(1+k_1)(1 + k_1^2 + 2k_1 k_2)}{k_1^2 k_2} \int_0^{+\infty} \phi(\tau) f_n(\tau, x_0(\tau), x_0'(\tau), x_0''(\tau), x_0'''(\tau)) \, d\tau \\
&\leq \frac{(1+k_1)(1 + k_1^2 + 2k_1 k_2)}{k_1^2 k_2} \frac{p(R)}{h(R)} \int_0^{+\infty} \phi(\tau) h(\widetilde{k}\widetilde{\gamma_1}(\tau)R) g(\tau, R, R, R) d\tau.
\end{aligned}
$$

This yields

$$
\frac{R h(R)}{p(R) \int_0^{+\infty} \phi(\tau) h(\widetilde{k}\widetilde{\gamma_1}(\tau)R) g(\tau, R, R, R) d\tau} \leq \frac{(1+k_1)(1 + k_1^2 + 2k_1 k_2)}{k_1^2 k_2},
$$

contradicting (3.3). From Lemma 2.5, we can compute the index fixed point of the mapping A_n,

$$
i(A_n, \Omega_1 \cap \mathcal{P}, \mathcal{P}) = 1, \quad \text{for all} \quad n \in \{n_0, n_0 + 1, \ldots\}. \tag{3.4}
$$

By the nonvanishing property (existence property) of the index, we obtain the existence of $x_n \in \Omega_1 \cap \mathcal{P}$ such that $A_n x_n = x_n$, $\forall n \geq n_0$. Since $\|x_n\|_\theta \leq R$, then from $(\mathcal{H}_3)$, we have that $\|x_n\|_\theta < R$. From $(\mathcal{H}_2)$, there exists a function $\psi_{R,R} \in C(\mathbb{R}^+, (0, +\infty))$ such that

$$
f_n(t, x_n(t), x_n'(t), x_n''(t), x_n'''(t)) \geq \psi_{R,R}(t), \quad \forall t \geq 0
$$

with

$$
\int_0^{+\infty} \phi(\tau) \psi_{R,R}(\tau) d\tau < +\infty.
$$

Then

$$
\begin{aligned}
x_n(t) &= A_n x_n(t) \\
&= \int_0^{+\infty} \int_0^{+\infty} G_1(t,s) G_2(s,\tau) \phi(\tau) f_n(\tau, x_n(\tau), x_n'(\tau), x_n''(\tau), x_n'''(\tau)) \, ds \, d\tau \\
&\geq \int_0^{+\infty} \int_0^{+\infty} G_1(t,s) G_2(s,\tau) \phi(\tau) \psi_{R,R}(\tau) ds d\tau.
\end{aligned}
$$

Lemma 2.7 guarantees that

$$
\begin{aligned}
x_n(t) e^{-\theta t} &\geq e^{-\theta t} \int_0^{+\infty} \int_0^{+\infty} G_1(t,s) G_2(s,\tau) \phi(\tau) \psi_{R,R}(\tau) ds d\tau \\
&\geq \gamma_1(t) \int_0^{+\infty} \int_0^{+\infty} e^{-\theta r} G_1(r,s) G_2(s,\tau) \phi(\tau) \psi_{R,R}(\tau) ds d\tau, \quad \forall r \geq 0.
\end{aligned}
$$

Passing to the supremum over r, we get

$$x_n(t)e^{-\theta t} \geq \widetilde{\gamma_1}(t)q^*,$$

where

$$q^* = \sup_{r \in \mathbb{R}^+} \int_0^{+\infty} \int_0^{+\infty} e^{-\theta r} G_1(r,s)G_2(s,\tau)\phi(\tau)\psi_{R,R}(\tau)dsd\tau.$$

Step 2. The sequence $\{x_n\}_{n \geq n_0}$ is relatively compact.

(a) Let $\mu \in (k, \theta)$. By the condition $(\mathcal{H}_1)$, we have

$$\|x_n\|_\mu \leq \frac{(1+k_1)(1+k_1^2+2k_1k_2)}{k_1^2 k_2} \int_0^{+\infty} \phi(\tau)f_n(\tau, x_n(\tau), x_n'(\tau), x_n''(\tau), x_n'''(\tau))\, d\tau$$

$$\leq \frac{(1+k_1)(1+k_1^2+2k_1k_2)}{k_1^2 k_2} \frac{p(R)}{h(R)} \int_0^{+\infty} h(q^* \widetilde{\gamma_1}(\tau)R)g(\tau, R, R, R)\, d\tau < +\infty.$$

Then the sequence $\{x_n\}_n$ is uniformly bounded in the sense of the norm $\|.\|_\mu$.

(b) For any $T > 0$ and $t, t' \in [0, T]$, we have

$$|x_n(t) - x_n(t')|$$

$$= \int_0^{+\infty} \int_0^{+\infty} |G_1(t,s) - G_1(t',s)|G_2(s,\tau)\phi(\tau)f_n(\tau, x_n(\tau), x_n'(\tau), x_n''(\tau), x_n'''(\tau))\, ds\, d\tau$$

$$\leq \left(\int_0^T |G_1(t,s) - G_1(t',s)|\, ds + \int_T^{+\infty} |G_1(t,s) - G_1(t',s)|\, ds \right)$$

$$\times \left(\int_0^{+\infty} G_2(\tau,\tau)\phi(\tau)f_n(\tau, x_n(\tau), x_n'(\tau), x_n''(\tau), x_n'''(\tau))\, d\tau \right)$$

$$\leq \frac{1}{2k_2} \frac{p(R)}{h(R)} \left(\int_0^T |G_1(t,s) - G_1(t',s)|\, ds + [(e^{k_1 t} - e^{-k_1 t}) - (e^{k_1 t'} - e^{-k_1 t'})] \int_T^{+\infty} e^{-k_1 s}\, ds \right)$$

$$\times \left(\int_0^{+\infty} \phi(\tau)h(q^* \widetilde{\gamma_1}(\tau))g(\tau, R, R, R)\, d\tau \right).$$

In the same way, we have

$$|x_n'(t) - x_n'(t')|$$

$$= \int_0^{+\infty} \int_0^{+\infty} |G_{1t}(t,s) - G_{1t}(t',s)|G_2(s,\tau)\phi(\tau)f_n(\tau, x_n(\tau), x_n'(\tau), x_n''(\tau), x_n'''(\tau))\, ds\, d\tau$$

$$\leq \frac{1}{2k_2} \frac{p(R)}{h(R)} \int_0^{+\infty} |G_{1t}(t,s) - G_{1t}(t',s)|\, ds \int_0^{+\infty} \phi(\tau)h(q^* \widetilde{\gamma_1}(\tau)R)g(\tau, R, R, R)\, d\tau.$$

Furthermore

$$|x_n''(t) - x_n''(t')|$$

$$\leq k_1^2 |x_n(t) - x_n(t')|$$

$$+ \int_0^{+\infty} |G_2(t,\tau) - G_2(t',\tau)|\phi(\tau)f_n(\tau, x_n(\tau), x_n'(\tau), x_n''(\tau), x_n'''(\tau))\, d\tau$$

and

$$|x_n'''(t) - x_n'''(t')|$$
$$\leq k_1^2 |x_n'(t) - x_n'(t')|$$
$$+ \int_0^{+\infty} |G_{2t}(t,\tau) - G_{2t}(t',\tau)| \phi(\tau) f_n(\tau, x_n(\tau), x_n'(\tau), x_n''(\tau), x_n'''(\tau)) \, d\tau.$$

Therefore, as in the proof of Lemma 3.1, the functions $\{x_n : n \geq n_0\}$, $\{x_n' : n \geq n_0\}$ $\{x_n'' : n \geq n_0\}$, and $\{x_n''' : n \geq n_0\}$ are equicontinuous and hence the sequence $\{x_n\}_{n \geq n_0}$ is relatively compact in E. As a consequence, there exists a subsequence $\{x_{n_k}\}_{k \geq 1}$ such that $\lim_{k \to +\infty} x_{n_k} = x_0$. Since $x_{n_k}(t) \geq \widetilde{\gamma_1}(t) q^*, \forall k \geq 1$ it follows that $x_0(t) \geq \widetilde{\gamma_1}(t) q^* \ \forall t \geq 0$. Therefore, by Lemma 2.8(h), we have

$$\int_0^{+\infty} \int_0^{+\infty} G_1(t,s) G_2(s,\tau) \phi(\tau) f(\tau, x_0(\tau), x_0'(\tau), x_0''(\tau), x_0'''(\tau)) \, ds d\tau$$
$$\leq \int_0^{+\infty} \left[\int_0^{+\infty} G_1(t,s) \, ds \right] G_2(\tau,\tau) \phi(\tau)) f(\tau, x_0(\tau), x_0'(\tau), x_0''(\tau), x_0'''(\tau)) d\tau$$
$$\leq \frac{1}{2k_1^2 k_2} \frac{p(R)}{h(R)} \int_0^{+\infty} \phi(\tau) h(q^* \widetilde{\gamma_1}(\tau) R) g(\tau, R, R, R) d\tau < +\infty.$$

Now, the continuity of f guarantees that, for all $\tau \geq 0$,

$$\lim_{k \to +\infty} f_{n_k}(\tau, x_{n_k}(\tau), x_{n_k}'(\tau), x_{n_k}''(\tau), x_{n_k}'''(\tau))$$
$$= \lim_{k \to +\infty} f(\tau, \max\{e^{\theta\tau}/n_k, x_{n_k}(\tau)\}, x_{n_k}'(\tau), x_{n_k}''(\tau), x_{n_k}'''(\tau))$$
$$= f(\tau, \max\{0, x_0(\tau)\}, x_0'(\tau), x_0''(\tau), x_0'''(\tau))$$
$$= f(\tau, x_0(\tau), x_0'(\tau), x_0''(\tau), x_0'''(\tau)).$$

Applying the Lebesgue dominated convergence theorem twice, we find

$$x_0(t) = \lim_{k \to +\infty} x_{n_k}(t)$$
$$= \lim_{k \to +\infty} \int_0^{+\infty} \int_0^{+\infty} G_1(t,s) G_2(s,\tau) \phi(\tau) f_{n_k}(\tau, x_{n_k}(\tau), x_{n_k}'(\tau), x_{n_k}''(\tau), x_{n_k}'''(\tau)) \, ds \, d\tau$$
$$= \int_0^{+\infty} \int_0^{+\infty} G_1(t,s) G_2(s,\tau) \phi(\tau) f(\tau, x_0(\tau), x_0'(\tau), x_0''(\tau), x_0'''(\tau)) \, ds \, d\tau.$$

Finally, it is clear that $\|x_0\|_\theta \leq R$ and from (3.3), $\|x_0\|_\theta < R$ and x_0 is a positive solution of Problem (1.1). □

4   A Multiplicity Result

Theorem 4.1. *Assume that Assumptions* $(\mathcal{H}_1) - (\mathcal{H}_3)$ *hold together with*

$(\mathcal{H}_4)$ *there exist* $0 < a < b < +\infty$ *such that*

$$\lim_{x \to +\infty} \frac{f(t, x, y, u, v)}{x} = +\infty, \quad \textit{uniformly in } (t, y, u, v) \in [a, b] \times \mathbb{R}^3.$$

Then Problem (1.1) has at least two positive solutions in $\mathcal{P}$.

Proof. Let

$$N^* = 1 + \frac{1}{r \int_a^b \phi(\tau)\, d\tau \min\limits_{t,\tau \in [a,b]} \int_0^{+\infty} G_1(t,s) G_2(s,\tau) e^{-\theta t}\, ds},$$

where $r = \widetilde{k} \min_{t \in [a,b]} \gamma_1(t)$. By $(\mathcal{H}_4)$, there exists some $R' > R$ such that

$$f(t, x, y, u, v) > N^* R', \quad \forall\, x \geq R',\ \forall\, (t, y, u, v) \in [a, b] \times \mathbb{R}^3.$$

Let

$$\Omega_2 = \{x \in E : \ \|x\|_\theta < R'/r\}.$$

Without loss of generality, assume that $R' > \max\{1, Rr\}$. We show that $A_n x \nleq x$ for all $x \in \partial\Omega_2 \cap \mathcal{P}$. Suppose on the contrary that there exists an $x_0 \in \partial\Omega_2 \cap \mathcal{P}$ with $A_n x_0 \leq x_0$. Since $x_0 \in \mathcal{P}$, we have

$$x_0(t) \geq \widetilde{k}\gamma_1(t)\|x_0\|_\theta \geq \widetilde{k} \min_{s \in [a,b]} \gamma_1(s) \frac{R'}{r} = R', \ \forall\, t \in [a, b].$$

Then for any $t \in [a, b]$, we have

$$
\begin{aligned}
x_0(t) e^{-\theta t} &\geq A_n x_0(t) e^{-\theta t} \\
&= \int_0^{+\infty} \int_0^{+\infty} G_1(t,s) G_2(s,\tau) e^{-\theta t} \phi(\tau) f_n(\tau, x_0(\tau), x_0'(\tau), x_0''(\tau), x_0'''(\tau))\, ds\, d\tau \\
&\geq \int_a^b \left(\int_0^{+\infty} G_1(t,s) G_2(s,\tau) e^{-\theta t}\, ds \right) \phi(\tau) f_n(\tau, x_0(\tau), x_0'(\tau), x_0''(\tau), x_0'''(\tau))\, d\tau \\
&\geq \int_a^b \left(\int_0^{+\infty} G_1(t,s) G_2(s,\tau) e^{-\theta t}\, ds \right) \phi(\tau) N^* R'\, d\tau \\
&\geq N^* R' \min_{t,\tau \in [a,b]} \int_0^{+\infty} G_1(t,s) G_2(s,\tau) e^{-\theta t}\, ds \int_a^b \phi(\tau)\, d\tau \\
&> \frac{R'}{r},
\end{aligned}
$$

and a contradiction with $\|x_0\|_\theta = \frac{R'}{r}$ is reached. Finally Lemma 2.6 yields

$$i(A_n, \Omega_2 \cap \mathcal{P}, \mathcal{P}) = 0, \tag{4.1}$$

while (3.4) and (4.1) imply that

$$i(A_n, (\Omega_2 \setminus \overline{\Omega}_1) \cap \mathcal{P}, \mathcal{P}) = -1, \text{ for all } n \in \{n_0, n_0 + 1, \ldots\}. \qquad (4.2)$$

Therefore A_n has a second fixed point $y_n \in (\Omega_2 \setminus \overline{\Omega}_1) \cap \mathcal{P}$. Since $y_n(t) \geq \widetilde{k}\gamma_1(t)R$, $\forall t \in \mathbb{R}^+$ and $\|y_n\|_\theta \leq \frac{R'}{r}$, then arguing as in Theorem 3.2, we can show that $\{y_n\}_{n \geq n_0}$ has a convergent subsequence $\{y_{n_j}\}_{j \geq 1}$ with $\lim_{j \to +\infty} y_{n_j} = y_0$. From (3.3), we have $\|y_0\|_\theta > R$ and thus y_0 is a solution of Problem (1.1). In addition $\|x_0\|_\theta < R < \|y_0\|_\theta < \frac{R'}{r}$. Hence x_0 and y_0 are two distinct positive solutions of Problem (1.1). □

Remark 4.2. *Another way to prove the results of Theorems 3.2 and 4.1 is to consider the sequence of operators, for $n \in \{1, 2, \ldots\}$,*

$$A_n x(t) = \int_0^{+\infty} \int_0^{+\infty} G_2(t, s) G_1(s, \tau) \phi(\tau) f_n(\tau, x(\tau), x'(\tau), x''(\tau), x'''(\tau)) d\tau ds,$$

and the following Banach space

$$E = \left\{ x \in C^3(\mathbb{R}^+, \mathbb{R}) : \sup_{t \in \mathbb{R}^+} \sum_{k=0}^{k=3} |x^{(k)}(t)| e^{-\theta t} < \infty \right\}, \quad \theta > k_2.$$

In this case, the cone of solutions is defined by

$$\mathcal{P} = \{ x \in E : x(t) \geq \widetilde{k}\gamma_2(t)\|x\|_\theta, \ \forall t \geq 0 \},$$

where $\gamma_2(t) := (e^{2k_2 t} - 1)e^{-(\theta+3k_2)t}$ and

$$\widetilde{k} = \frac{1}{4} \min \left\{ 1, \frac{1}{k_2}, \frac{k'}{k_2(k_2 k' + 2)}, \frac{k'}{k_2(k_2^2 k' + 2k_1)} \right\}$$

with $k' = \min \left\{ \frac{1-e^{-k_2}}{k_2}, \frac{e^{-(k_1+k_2)}}{k_1+k_2} \right\}$.

Example 4.3. *Consider the singular fourth-order boundary value problem*

$$\begin{cases} x^4(t) - \frac{5}{2}x''(t) + \frac{9}{16}x(t) = \phi(t)f(t, x(t), x'(t), x''(t), x'''(t)), \\ x(0) = x''(0) = 0, \lim_{t \to +\infty} x(t) = \lim_{t \to +\infty} x''(t) = 0, \end{cases} \qquad (4.3)$$

where

$$f(t, x, y, u, v) = \frac{Ae^{-(2b+1)t}(x^a e^{2at} + 1)(\psi(ye^{2t}) + \psi(ue^{2t}) + \psi(ve^{2t}))}{x^b}$$

and $\phi(t) = (e^{3t} - 1)^b e^{-\frac{17}{2}bt}$. A, a, b are positive constants with $a > b + 1$, and ψ is defined by

$$\psi(t) = \begin{cases} e^y, & y < 0, \\ 1, & y \geq 0. \end{cases}$$

Here $k_1 = \frac{3}{2}$ and $k_2 = \frac{1}{2}$ and for $\theta = 2$, we have $\gamma_1(t) = (e^{3t} - 1)e^{-\frac{13}{2}t}$, $\widetilde{\gamma_1}(t) = (e^{3t} - 1)e^{-\frac{17}{2}t}$ and

$$F(t, x, y, u, v) = Ae^{-t} \frac{(x^a + 1)(\psi(y) + \psi(u) + \psi(v))}{x^b}.$$

Let $p(x) = \frac{x^a+1}{x^b}$, $h(x) = \frac{1}{x^b}$, and $g(t, y, u, v) = Ae^{-t}(\psi(y) + \psi(u) + \psi(v))$. Then h is decreasing, $\frac{p}{h}$ is nondecreasing and $g(t, \cdot, \cdot, \cdot)$ is nondecreasing in y, u, v with

$$\int_0^{+\infty} \phi(\tau)h(c\widetilde{\gamma_1}(\tau))g(\tau, c', c', c')d\tau = \frac{3A}{c^b} < +\infty, \text{ for each } c, c', c'' > 0.$$

Then the condition $(\mathcal{H}_1)$ hold. In addition, for each $c, c' > 0$ and for $(t, x, y, u, v) \in \mathbb{R}^+ \times (0, c] \times [-c', c']^3$, we have

$$F(t, x, y, u, v) \geq \frac{3Ae^{-c'}}{c^b} e^{-t} = \psi_{c,c'}(t)$$

and $\int_0^{+\infty} \phi(\tau)\psi_{c,c'}(\tau)d\tau < +\infty$. Then the condition $(\mathcal{H}_2)$ hold. Also, we have

$$\sup_{c>0} \frac{k_1^2 k_2 ch(c)}{(1 + k_1)(1 + k_1^2 + 2k_1k_2)p(c) \int_0^{+\infty} \phi(\tau)h(\widetilde{k}\widetilde{\gamma_1}(\tau)c)g(\tau, c, c, c)d\tau}$$

$$= \frac{k_1^2 k_2 \widetilde{k}^b}{3A(1 + k_1)(1 + k_1^2 + 2k_1k_2)} \sup_{c>0} \frac{c^{b+1}}{c^a + 1}.$$

If we choose A small enough, say $0 < A < \frac{k_1^2 k_2 \widetilde{k}^b}{3(1+k_1)(1+k_1^2+2k_1k_2)} \sup_{c>0} \frac{c^{b+1}}{c^a+1}$, then the condition $(\mathcal{H}_3)$ holds. Finally it is clear that the condition $(\mathcal{H}_4)$ also holds. Therefore all conditions of Theorem 4.1 are satisfied and then Problem (4.3) has at least two positive solutions.

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