Thai Journal of Mathematics
Volume 12 (2014) Number 3 : 631–637
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Lipschitz Type Analogue of Strict Contractive Conditions and Common Fixed Points

Ravindra K. Bisht

Department of Mathematics-Applied Sciences and Humanities
Bipin Triphati Kumaon Institute of Technology
Dwarahat, 262553, Almora, Uttarakhand, India
e-mail : ravindra.bisht@yahoo.com

Abstract : In this paper, we obtain a common fixed point theorem for a pair of self mappings satisfying a Lipschitz type analogue of strict contractive condition by using a relatively new notion of conditional reciprocal continuity wherein we never require conditions on the completeness of the space, noncompatibility or property (E.A.), continuity of any mapping and completeness (or closedness) of the range of any one of the involved mappings. Our results substantially improve the results of Pant [Discontinuity and fixed points, J. Math. Anal. Appl. 240 (1999) 284–289], Pant and Pant [Common fixed points under strict contractive conditions, J. Math. Anal. Appl. 248 (2000) 327–332], Imdad et al. [Coincidence fixed points in symmetric spaces under strict contractions, J. Math. Anal. Appl. 320 (2006) 352–360] and Jin-Xuan and Yang [Common fixed point theorems under strict contractive conditions in Menger spaces, Nonlinear Anal. 70 (2009) 184–193].

Keywords : common fixed point; compatible mappings; variants of compatible maps.

2010 Mathematics Subject Classification : 47H09; 47H10.

1 Introduction

In the area of fixed point theory, strict contractive conditions constitute a very important class of mappings and include contraction mappings as their subclass. It may be observed that strict contractive conditions do not ensure the existence of common fixed points unless some strong condition is assumed either on the space or on the mappings. In such cases either the space is taken to be compact or some sequence of iterates is assumed to be Cauchy sequence. The study of common fixed points of strict contractive conditions using noncompatibility was initated by Pant [1]. Motivated by Pant [1] researchers of this domain obtained common fixed point results for strict contractive conditions under generalized metric spaces [2–8]. The significance of this paper lies in the fact that we can obtain fixed point theorems for conditionally reciprocally continuous mappings under generalized strict contractive conditions without assuming any strong conditions on the space or on the mappings.

2 Preliminaries

In 1986, Jungck [9] generalized the notion of weakly commuting maps [10] by introducing the concept of compatible maps.

Definition 2.1. Two selfmaps f and g of a metric space (X, d) are called compatible iff $\lim_n d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X.

The definition of compatibility implies that the mappings f and g will be noncompatible if there exists a sequence $\{x_n\}$ in X such that for some t in X but $\lim_n d(fgx_n, gfx_n)$ is either non zero or nonexistent.

In a recent work, Aamri and Moutawakil [2] introduced the idea of (E.A.) property, which is more general than noncompatible mappings.

Definition 2.2 ([2])**.** A pair (f, g) of selfmappings of a metric space (X, d) is said to satisfy the property (E.A.) if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t, \qquad \text{for some } t \in X.$$

In 1993, Jungck et al. [11] further generalized the notion of weakly commuting maps [10] by introducing the new concept of compatible of type (A) .

Definition 2.3 ([11])**.** Two selfmaps f and g of a metric space (X, d) are called compatible of type (A) iff $\lim_n d(ffx_n, gfx_n) = 0$ and $\lim_n d(fgx_n, ggx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X.

In 1997, Pathak and Khan [12] further introduced some interesting generalized noncommuting conditions analogous to the notion of compatibility by defining the notions of $f-$compatibility and $g-$compaibility.

Definition 2.4 ([12]). Two selfmaps f and g of a metric space (X, d) are called $f-$compatible iff $\lim_n d(fgx_n, ggx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X.

Definition 2.5 ([12]). Two selfmaps f and g of a metric space (X, d) are called $g-$compatible iff $\lim_n d(ffx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X.

It may be observed that compatibility of type (A) implies $f-$compatibility or $g-$compaibility, but the converse is not true in general [12].

Pathak et al. [13] also obtained some fixed point theorems in metric spaces and probabilistic metric spaces by using the notion of compatible of type (P).

Definition 2.6 ([13]). Two selfmaps f and g of a metric space (X, d) are called compatible of type (P) iff $\lim_n d(ffx_n, ggx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X.

In 1998, Pant [14] introduced the notion of reciprocal continuity and as an application of this concept obtained the first result that established a situation in which a collection of mappings has a fixed point which is a point of discontinuity for all the mappings.

Definition 2.7 ([14]). Two selfmappings f and g of a metric space (X, d) are called reciprocally continuous iff $\lim_n fgx_n = ft$ and $\lim_n gfx_n = gt$, whenever $\{x_n\}$ is a sequence such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X.

In the setting of common fixed point theorems for compatible maps satisfying contractive conditions, continuity of one of the mappings f and g implies their reciprocal continuity but not conversely [14].

Remark 2.8. *If f and g fail to be reciprocally continuous then there exists a sequence $\{x_n\}$ in X such that $fx_n \to t$ and $gx_n \to t$ for some t in X but either* $\lim_n fgx_n \neq ft$ *or* $\lim_n gfx_n \neq gt$ *or one of fgx_n, gfx_n is not convergent.*

More recently, Pant and Bisht [6] further generalized reciprocal continuity by introducing the new concept of conditional reciprocal continuity, which turns out to be the necessary condition for the existence of common fixed points. This notion is applicable to compatible as well as noncompatible mappings.

Definition 2.9 ([6]). Two selfmappings f and g of a metric space (X, d) are called conditionally reciprocally continuous (CRC) iff whenever the set of sequences $\{x_n\}$ satisfying $\lim_n fx_n = \lim_n gx_n$ is nonempty, there exists a sequence $\{y_n\}$ satisfying $\lim_n fy_n = \lim_n gy_n = t(say)$ for some t in X such that $\lim_n fgy_n = ft$ and $\lim_n gfy_n = gt$.

If f and g are reciprocally continuous then they are obviously conditionally reciprocally continuous but, as shown in Example 3.2 below, the converse is not true.

The question whether there exists a contractive definition which is strong enough to generate a fixed point but which does not force the map to be continuous at the fixed point was reiterated by Rhoades in [15] as an existing open problem. Pant [1, 14], Pant and Pant [7], Pant and Bisht [6], Imdad et al. [3] and Singh et al. [8] have provided some solutions to this problem. It is of worth to note that in all the results proved by us, none of the mappings under consideration has been assumed continuous. In fact, the mappings become discontinuous at the fixed point. We, thus, also provide one more answer to the open problem of Rhoades [15].

3 Main Results

Theorem 3.1. *Let f and g be conditionally reciprocally continuous selfmappings of a metric space (X, d) satisfying*

1. $fX \subseteq gX$
2. $d(fx, fy)$
$$< \max\{d(gx, gy), k[d(fx, gx) + d(fy, gy)]/2, [d(fx, gy) + d(fy, gx)]/2\},$$
$1 \leq k < 2$,

whenever the right hand side is nonzero. Suppose f and g are not reciprocally continuous. If f and g are either compatible or compatible of type (A) or $g-$compatible or $f-$compatible then f and g have a unique common fixed point.

Proof. Since f and g are not reciprocally continuous, there exists a sequence $\{x_n\}$ in X such that $fx_n \to t$ and $gx_n \to t$ for some t in X but either $\lim_n fgx_n \neq ft$ or $\lim_n gfx_n \neq gt$ or one of fgx_n, gfx_n is not convergent. Since f and g are conditionally reciprocally continuous and $fx_n = gx_n \to t$ there exists a sequence $\{y_n\}$ satisfying $\lim_n fy_n = \lim_n gy_n = u$ such that $\lim_n fgy_n = fu$ and $\lim_n gfy_n = gu$. Since $fX \subseteq gX$, for each y_n there exists z_n in X such that $fy_n = gz_n$. Thus $fy_n \to u$, $gy_n \to u$ and $gz_n \to u$ as $n \to \infty$. By virtue of this and using (ii) we obtain $fz_n \to u$. Therefore, we have

$$fy_n = gz_n \to u, gy_n \to u, fz_n \to u. \tag{3.1}$$

Suppose that f and g are compatible. Then $\lim_n d(fgy_n, gfy_n) = 0$, i.e., $fu = gu$. Since compatibility implies commutativity at coincidence points, i.e., $fgu = gfu$ and, hence $ffu = fgu = gfu = ggu$. If $fu \neq ffu$ then by using (ii), we get $d(ffu, fu) < \max\{d(gfu, gu), k[d(ffu, gfu) + d(fu, gu)]/2, [d(ffu, gu) + d(fu, gfu)]/2 = d(ffu, fu)\}$, a contradiction. Hence $fu = ffu = gfu$ and fu is a common fixed point of f and g.

Next, suppose that f and g are compatible of type (A). Then $\lim_n d(ffy_n, gfy_n) = 0$ and $\lim_n d(fgy_n, ggy_n) = 0$, i.e., $ffy_n \to gu$ and $ggy_n \to fu$. We assert that $fu = gu$. If not, using (ii) we get $d(ffy_n, fu) < \max\{d(gfy_n, gu), k[d(ffy_n, gfy_n) + d(fu, gu)]/2, [d(ffy_n, gu) + d(fu, gfy_n)/2]\}$. On letting $n \to \infty$ this yields $d(gu, fu)$

$\leq \frac{k}{2}d(fu, gu)$, a contradiction unless $fu = gu$. Since compatibility of type (A) implies commutativity at coincidence points, i.e., $fgu = gfu$ and, hence $ffu = fgu = gfu = ggu$. If $fu \neq ffu$ then by using (ii), we get $d(ffu, fu) < \max\{d(gfu, gu), k[d(ffu, gfu) + d(fu, gu)]/2, [d(ffu, gu) + d(fu, gfu)]/2 = d(ffu, fu)\}$, a contradiction. Hence $fu = ffu = gfu$ and fu is a common fixed point of f and g.

Now, suppose that f and g are $g-$compatible. Then $\lim_n d(ffy_n, gfy_n) = 0$, i.e., $ffy_n \to gu$. We assert that $fu = gu$. If not, using (ii) we get $d(ffy_n, fu) < \max\{d(gfy_n, gu), k[d(ffy_n, gfy_n) + d(fu, gu)]/2, [d(ffy_n, gu) + d(fu, gfy_n)/2]\}$. On letting $n \to \infty$ this yields $d(gu, fu) \leq \frac{k}{2}d(fu, gu)$, a contradiction unless $fu = gu$. Since $g-$compatibility implies commutativity at coincidence points, i.e., $fgu = gfu$ and, hence $ffu = fgu = gfu = ggu$. If $fu \neq ffu$ then by using (ii), we get $d(ffu, fu) < \max\{d(gfu, gu), k[d(ffu, gfu) + d(fu, gu)]/2, [d(ffu, gu) + d(fu, gfu)]/2 = d(ffu, fu)\}$, a contradiction. Hence $fu = ffu = gfu$ and fu is a common fixed point of f and g.

Finally suppose that f and g are $f-$compatible. Then $\lim_n d(fgz_n, ggz_n) = 0$. Using $ggz_n = gfy_n \to gu$, this yields $fgz_n \to gu$. If $fu \neq gu$, the inequality $d(fgz_n, fu) < \max\{d(ggz_n, gu), k[d(fgz_n, ggz_n) + d(fu, gu)]/2, [d(fgz_n, gu) + d(fu, ggz_n)]/2\}$, on letting $n \to \infty$ we get $d(gu, fu) \leq \frac{k}{2}d(fu, gu)$, a contradiction. This implies $fu = gu$. Again, $f-$compatibility of f and g implies that $fgu = gfu$ and, hence, $ffu = fgu = gfu = ggu$. If $fu \neq ffu$ then by using (ii), we get $d(ffu, fu) < \max\{d(gfu, gu), k[d(ffu, gfu) + d(fu, gu)]/2, [d(ffu, gu) + d(fu, gfu)]/2 = d(ffu, fu)\}$, a contradiction. Hence $fu = ffu = gfu$ and fu is a common fixed point of f and g. This completes the proof of the theorem. □

The next example illustrates the above theorem.

Example 3.2. *Let $X = [2, 20]$ and d be the usual metric on X. Define $f, g : X \to X$ as follows*

$$fx = 2 \text{ if } x = 2 \text{ or } x > 5,\ fx = 6 \text{ if } 2 < x \leq 5,$$
$$g2 = 2,\ gx = 12, \text{ if } 2 < x \leq 5,\ gx = \frac{(x+1)}{3} \text{ if } x > 5.$$

Then f and g satisfy all the conditions of Theorem 3.1 and have a common fixed point at $x = 2$. It can be verified in this example that f and g satisfy the condition (ii). Furthermore, f and g are $g-$compatible. It can also be noted that f and g are conditionally reciprocally continuous. To see this, let $\{x_n\}$ be the constant sequence given by $x_n = 2$. Then $fx_n \to 2$, $gx_n \to 2$. Also $fgx_n \to 2 = f2$ and $gfx_n \to 2 = g2$. Hence f and g are conditionally reciprocally continuous. It is also obvious that f and g are not reciprocally continuous. To see this, let $\{y_n\}$ be a sequence in X given by $y_n = 5 + \epsilon_n$ where $\epsilon_n \to 0$ as $n \to \infty$. Then $fy_n \to 2$, $gy_n = (2 + \frac{\epsilon_n}{3}) \to 2$, $\lim_n fgy_n = f(2 + \frac{\epsilon_n}{3}) = 6 \neq f2$ and $\lim_n gfy_n = g2 = 2$. Thus $\lim_n gfy_n = g2$ but $\lim_n fgy_n \neq f2$. Hence f and g are not reciprocally continuous mappings.

As a direct consequence of the above theorem we get the following corollary.

Corollary 3.3. *Let f and g be conditionally reciprocally continuous selfmappings of a metric space (X, d) satisfying*

1. $fX \subseteq gX$
2. $d(fx, fy)$
$$< \max\{d(gx, gy), [d(fx, gx) + d(fy, gy)]/2, [d(fx, gy) + d(fy, gx)]/2\},$$

whenever the right hand side is nonzero. Suppose f and g are not reciprocally continuous. If f and g are either compatible or $g-$compatible or $f-$compatible then f and g have a unique common fixed point.

Proof. The corollary follows from Theorem 3.1 by putting $k = 1$. □

Remark 3.4. *In the above result we have not asummed strong conditions, e.g., completeness of the space, noncompatibility or property $(E.A.)$, closedness of the range of any one of the involved mappings and continuity of any mapping. Prior to this, there is perhaps no common fixed point theorem obtained without assuming noncompatibility and property $(E.A.)$ in the setting of Lipschitz type analogue of a strict contractive condition. Thus, our results substantially improve the results of Pant [1], Pant and Pant [7], Imdad et al. [3], Jin-Xuan and Yang [16], Kubiaczyk and Sharma [5] and many others.*

Remark 3.5. *In this paper we have proved a result using generalized strict contractive condition. It may be observed that strict contractive conditions do not ensure the existence of common fixed points unless the space is assumed compact or the strict contractive condition is replaced by some strong conditions, e.g., a Banach type contractive condition or a $\phi-$ contractive condition or a Meir-Keeler type contractive condition.*

Remark 3.6. *In the result established in this paper, we have not assumed any mapping to be continuous. Thus we provide more answers to the problem posed by Rhoades [15] regarding existence a contractive definition which is strong enough to generate a fixed point, but which does not force the map to be continuous at the fixed point.*

References

[1] R.P. Pant, Discontinuity and fixed points, J. Math. Anal. Appl. 240 (1999) 284–289.

[2] A. Aamri, D. El Moutawakil, Some new common fixed point theorems under strict contractive conditions, J. Math. Anal. Appl. 270 (2002) 181–188.

[3] M. Imdad, J. Ali, L. Khan, Coincidence fixed points in symmetric spaces under strict contractions, J. Math. Anal. Appl. 320 (2006) 352–360.

[4] T. Kamran, Coincidence and fixed points for hybrid strict contractions, J. Math. Anal. Appl. 299 (2004) 235–283.

[5] I. Kubiaczyk, S. Sharma, Some common fixed point theorems in Menger space under strict contractive conditions, Southeast Asian Bull. Math. 32 (2008) 117–124.

[6] R.P. Pant, R.K. Bisht, Common fixed point theorems under a new continuity condition, Ann. Univ. Ferrara 58 (1) (2012) 127–141.

[7] R.P. Pant, V. Pant, Common fixed points under strict contractive conditions, J. Math. Anal. Appl. 248 (2000) 327–332.

[8] S.L. Singh, Y.J. Cho, A. Kumar, Fixed points of Meir-Keeler type hybrid contractions, Panamerican Mathematical Journal 16 (4) (2006) 35–54.

[9] G. Jungck, Compatible mappings and common fixed points, Internat. J. Math. Math. Sci. 9 (1986) 771–779.

[10] S. Sessa, On a weak commutativity condition in fixed point considerations, Publ. Inst. Math. (Beograd) (NS) 34 (46) (1982) 149–153.

[11] G. Jungck, P.P. Murthy, Y.J. Cho, Compatible mappings of type (A) and common fixed points, Math. Japonica 36 (1993) 381–390.

[12] H.K. Pathak, M.S. Khan, A comparison of various types of compatible maps and common fixed points, Indian J. Pure Appl. Math. 28 (4) (1997) 477–485.

[13] H.K. Pathak, Y.J. Cho, S. Chang, S.M. Kang, Compatible mappings of type (P) and fixed point theorem in metric spaces and probabilistic metric spaces, Novi Sad J. Math. 26 (2) (1996) 87–109.

[14] R.P. Pant, Common fixed points of four mappings, Bull. Cal. Math. Soc. 90 (1998) 281–286.

[15] B.E. Rhoades, Contractive definitions and continuity, Contemporary Math. (Amer. Math. Soc.) 72 (1988) 233–245.

[16] F. Jin-Xuan, Gao Yang, Common fixed point theorems under strict contractive conditions in Menger spaces, Nonlinear Anal. 70 (2009) 184–193.

(Received 16 July 2012)
(Accepted 17 April 2013)