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Graceful Labeling of some classes of Spider Graphs with Three legs greater than one

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Abstract : A spider graph is a tree with at most one vertex of degree greater than 2. A graceful labeling of a tree T with n edges is a bijection $f : V(T) \to \{0, 1, 2, \ldots, n\}$ such that $\{|f(u) - f(v)| : \{u, v\}$ is an edge of $T\} = \{1, 2, \ldots, n\}$. We show that some classes of spider graphs admit graceful labeling.

Keywords : Spider graphs; Graceful labeling; Inverse transformatin
2010 Mathematics Subject Classification : 05C70; 05B30 (2000 MSC)

1 Introduction

A *graceful labeling* of a tree T with n edges is a bijection $f : V(T) \to \{0, 1, 2, \ldots, n\}$ such that $\{|f(u) - f(v)| : \{u, v\}$ is an edge of $T\} = \{1, 2, \ldots, n\}$. We denote $|f(u) - f(v)|$ by $\bar{f}(uv)$. A tree which has a graceful labeling is called a *graceful tree*. A *spider graph* or *spider* is a tree with at most one vertex (called the branch point) of degree greater than 2. A leg of a spider graph is a path from the branch point to a leaf of the tree. Let $S_n(k, l, m)$ denote a spider of n legs in which all of its legs have lengths one except three legs of length k, l and m. Sometimes we allow k, l or m to be 1.

In 1964, Ringel and Kotzig [1], [2] gave the famous and unsolved "graceful tree

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conjecture" which states that all trees are graceful. In [3], the special case of this conjecture about spider graphs is still open and very few classes of spider graphs are known to be graceful. Huang et al. [4] proved that all spiders with three or four legs are graceful. Bahl et al. [5] proved that every spider in which lengths of any two of its legs differ by at most one is graceful. In this paper we prove that some classes of spider graphs which are different from above are graceful.

2 Preliminaries

To prove our results we need some terminologies and known results which are described below.

Lemma 2.1. [6] *Let T be a tree with n edges and a graceful labeling f. Then, the function $f' : V(T) \to \{0, 1, 2, \dots, n\}$ defined by $f'(v) = n - f(v)$ is also a graceful labeling of T.*

Proof. Note that f' is also a bijection and for each $uv \in E(T)$ we have $\bar{f}'(uv) = |f'(u) - f'(v)| = |n - f(u) - n + f(v)| = |f(u) - f(v)| = \bar{f}(uv)$. Thus f' is also a graceful labeling of T. □

The next lemma is elementary but quite useful in the later parts.

Lemma 2.2. *If $S_n(k, l, m)$ has a graceful labeling f such that $f(u) = 0$ where u is the leaf of the leg of length k, then $S_n(p, l, m)$ is graceful for each $p \geq k$.*

Proof. Let f be a graceful labeling of $S_n(k, l, m)$ such that $f(u) = 0$ where u is the leaf of the leg of length k. Let T_1 be the spider graph obtained from $S_n(k, l, m)$ by adding the vertex z_1 and the edge uz_1. Extending f to T_1 by letting $f(z_1) = n + k + l + m - 2$, we get a graceful labeling f of T_1. We have the desired result if $p = k + 1$.

Suppose that $p > k + 1$. Let $f_1(v) = n + k + l + m - 2 - f(v)$ for all $v \in V(T_1)$. By Lemma 2.1, the function f_1 is a graceful labeling of T_1 such that $f_1(z_1) = 0$. Let T_2 be the spider graph obtained from T_1 by adding the vertex z_2 and the edge z_1z_2. Extending f_1 to T_2 by letting $f_1(z_2) = n + k + l + m - 1$, we get a graceful labeling f_1 of T_2. We have the desired result if $p = k + 2$. Continuing this process for $p > k + 2$, we get the desired result. □

Next consider the special labeling of a graph as the following. Let T be a spider graph of n legs as Fig. 1. It can be seen that $|V(T)| = n + m + k + l - 2$. Now we introduce the labeling f of T in which we use to generate graceful labelings in the subsequent part. Note that the construction of f depends on the parity of m.

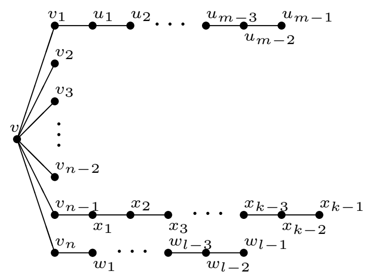

Figure 1:

Case $m-1$ is odd. Let $m-1 = 2p+1$, we have $|V(T)| = n+2p+k+l$. Let

$$
\begin{aligned}
f(u_{m-1}) &= 0, & f(v) &= p+1,\\
f(u_{m-2}) &= n+2p+k+l-1, & f(v_2) &= n+p+k+l-2,\\
f(u_{m-3}) &= 1, & f(v_3) &= n+p+k+l-3,\\
&\vdots & &\vdots\\
f(u_2) &= n+p+k+l, & f(v_{n-2}) &= p+k+l+2,\\
f(u_1) &= p, & f(v_{n-1}) &= p+k+l+1,\\
f(v_1) &= n+p+k+l-1, & f(v_n) &= p+k+l.
\end{aligned}
$$

Next we label the vertices $w_1, w_2, \ldots, w_{l-2}, w_{l-1}, x_1, x_2, \ldots, x_{k-2}, x_{k-1}$ alternating between the highest and the lowest remaining unused labels by starting with $f(w_1) = p+2$ and $f(w_2) = p+k+l-1$. We see that f is injective with $f(V(T)) = \{0, 1, 2, \ldots, n+2p+k+l-1\}$.

Let T' be obtained from T by deleting the edge $v_{n-1}x_1$ and adding the edge $w_{l-1}x_1$. To prove that $\bar{f}(E(T')) = \{1, 2, 3, \ldots, n+2p+k+l-1\}$, it suffices to show the edge labeling $\bar{f}$ is injective. This can be seen by noting that edge labels in each branch are all different and the minimum edge label in a branch containing v_i is greater than all edge labels in a branch containing v_{i+1}. Thus f is a graceful labeling of T'.

Case $m-1$ is even. Let $m-1 = 2p'$, we have $|V(T)| = n+2p'+k+l-1$.

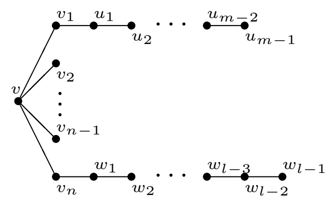

Figure 2:

Let

$$\begin{aligned} f(u_{m-1}) &= 0, & f(v) &= n+p'+k+l-2,\\ f(u_{m-2}) &= n+2p'+k+l-2, & f(v_2) &= p'+1,\\ f(u_{m-3}) &= 1, & f(v_3) &= p'+2,\\ &\vdots & &\vdots\\ f(u_2) &= p'-1, & f(v_{n-2}) &= n+p'-3,\\ f(u_1) &= n+p'+k+l-1, & f(v_{n-1}) &= n+p'-2,\\ f(v_1) &= p', & f(v_n) &= n+p'-1. \end{aligned}$$

Next we label the vertices $w_1, w_2, \ldots, w_{l-2}, w_{l-1}, x_1, x_2, \ldots, x_{k-2}, x_{k-1}$ alternating between the highest and the lowest remaining unused labels by starting with $f(w_1) = n + p' + k + l - 3$ and $f(w_2) = n + p'$. We see that f is injective with $f(V(T)) = \{0, 1, 2, \ldots, n + 2p' + k + l - 2\}$ and if T' is obtained from T as in the previous case, then we also get $\bar{f}(E(T')) = \{1, 2, 3, \ldots, n + 2p' + k + l - 2\}$.

We will use the labeling in two cases above to generate graceful labelings of some spider graphs in the later part. For convenience, we call the labelings constructed in the first case and the second case as Type I and Type II, respectively.

First we use a special labeling f on T to acheive the following theorem.

Theorem 2.3. *Let T be a spider graph with at most two of its legs have lengths more than one. Then T is graceful.*

Proof. Using the previously described labeling f on T as Fig. 2 with $k = 1$ and $l \geq 1$, one can see that f is a graceful labeling by an argument similar to the above. □

3 Graceful labeling of some spider graphs

Next we will consider the labeling of some spider graphs with only three legs of lengths greater than 1.

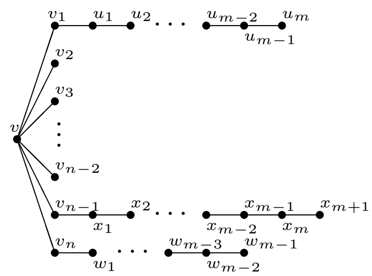

Figure 3:

Lemma 3.1. *Let $T = S_n(m, m+1, m+2)$ with $m > 1$. Then there is a graceful labeling f of T such that $f(u) = 0$ where u is the leaf of the leg of length $m+1$.*

Proof. We see that $|V(T)| = n + 3m + 1$. To construct a graceful labeling of T, we consider two cases.

Case m is odd. Suppose that $m = 2p+1$, so $|V(T)| = n + 6p + 4$. Let T be a spider graph as Fig. 3 and let f be a labeling of T of Type I, we get $f(v_{n-1}) = 5p+5$, $f(x_1) = 2p+2$, $f(w_{m-1}) = 4p+4$ and $f(x_{m+1}) = 3p+3$. Hence $|f(v_{n-1}) - f(x_{m+1})| = 2p+2$ and $|f(w_{m-1}) - f(x_1)| = 2p+2$. Next we change the values of f at $x_1, x_2, \ldots, x_{m+1}$ by reversing the values of them. Comparing vertex labelings of T and T' (the later obtained by transforming T as in Preliminaries), we see that vertices and their labels in branches containing v_i where $i \le n-2$ in both graphs are the same which implies that sets of edge labels in these branches of T and T' are the same. Moreover, one can see that sets of edge labels of T and T' in remaining branches are also the same. Thus the labeling f of T is also graceful.

Case m is even. Let $m = 2p'$, we have $|V(T)| = n + 6p' + 1$. Let T be a spider graph as Fig. 3 and let f be a labeling of T of Type II, we get $f(v_{n-1}) = n + p' - 2$, $f(x_1) = n + 2p' - 1$, $f(w_{m-1}) = n + 4p'$ and $f(x_{m+1}) = n + 3p' - 1$. Hence $|f(v_{n-1}) - f(x_{m+1})| = 2p'+1$ and $|f(x_1) - f(w_{m-1})| = 2p'+1$. Using an argument similar to the above case, one can conclude f is a desired graceful labeling. □

Theorem 3.2. *Let $T = S_n(m, m+2, l)$ with $m > 1$, $l \ge m+1$. Then, T is a graceful graph.*

Proof. The theorem follows directly from lemmas 2.2 and 3.1. □

Lemma 3.3. *Let $T = S_n(m, m+1, m+2)$ with $m \ge 1$ and n is large enough. Then, there is a graceful labeling f of T such that $f(u) = 0$ where u is the leaf of the leg of length m.*

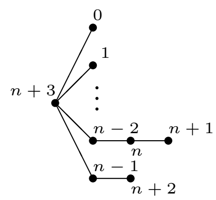

Figure 4:

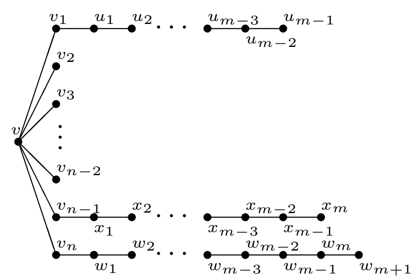

Figure 5:

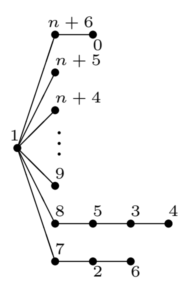

Figure 6:

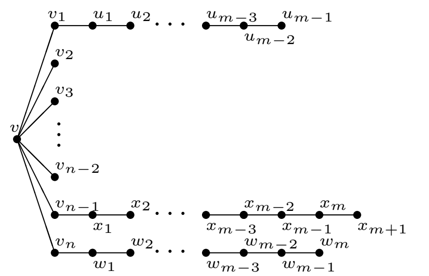

Figure 7:

Proof. If $m = 1$, then we get the spider graph T and a graceful labeling as Fig. 4.

Suppose that $m > 1$, we see that $|V(T)| = n + 3m + 1$. To construct the graceful labeling of T, we consider two cases.

Case $m-1$ is odd. Let $m-1 = 2p+1$, we have $|V(T)| = n+6p+7$. Let T be a spider graph as Fig. 5 and let f be a labeling of T of Type I, we get $f(v_{n-1}) = 5p+8$, $f(w_{m+1}) = 2p + 3$ and $f(x_1) = 4p + 5$. Hence $|f(x_1) - f(w_{m+1})| = 2p + 2$ and $|f(x_1) - f(v_{n-1})| = p + 3$. We see that $2p + 2 \geq p + 4$ if $p \geq 1$ (i.e., $m \geq 4$). Since n is large enough, there is $i \leq n - 1$ such that $f(v_i) = 6p + 7$. If we switch the values of f at v_i and v_{n-1} to be $f(v_{n-1}) = 6p + 7$ and $f(v_i) = 5p + 8$, then $|f(v_{n-1} - f(x_1)| = 2p + 2$. Note that the sets of edge labels in T and T' are the same where T' and its resulting edge labels are defined as in Preliminaries. But the set of edge labels of T' is $\{1, 2, 3, \ldots, n + 3m\}$. Hence f is a graceful labeling of T. For the case $p = 0$, we get the graph T and a graceful labeling as Fig. 6.

Case $m-1$ is even. Let $m-1 = 2p'$, we have $|V(T)| = n+6p'+4$. Let T be the graph as Fig. 7 and let f be a labeling of T of Type II, we get $f(v_{n-1}) = n+p'-2$, $f(w_m) = n + 4p' + 2$ and $f(x_1) = n + 2p'$. Hence $|f(x_1) - f(w_m)| = 2p' + 2$ and $|f(x_1) - f(v_{n-1})| = p' + 2$. We see that $2p' + 2 \geq p' + 2$ for all p' (i.e., for all m). Since n is large enough, there is $i \leq n - 1$ such that $f(v_i) = n - 2$. If we change the value of f at v_i and v_{n-1} to be $f(v_{n-1}) = n - 2$ and $f(v_i) = n + p' - 2$, then $|f(v_{n-1}) - f(x_1)| = 2p' + 2$. Note that the sets of edge labels in T and T' are the same where T' and its resulting edge labels are defined as in Preliminaries. But the set of edge labels of T' is $\{1, 2, 3, \ldots, n + 3m\}$. Hence f is a graceful labeling of T. □

Lemma 3.4. *Let $T = S_n(m, m + 1, l)$ with $m \geq l > 1$ and n is large enough. Then there is a graceful labeling f of T such that $f(u) = 0$ where u is the leaf of the leg with length m.*

Proof. If $l = m$, then consider the subgraph $T_1 = S_n(m - 1, l, m + 1)$ of T. By Lemma 3.3, there is a graceful labeling f of T_1 such that $f(u) = 0$ where u is the leaf of the leg of length $m - 1$. So T can be obtained from T_1 by adding vertex

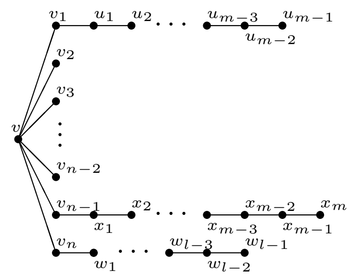

Figure 8:

v_1 and edge uv_1. Extending f to T by letting $f(v_1) = n + 3m - 2$, we get f is a graceful labeling of T. Let $f_1(v) = n + 3m - 2 - f(v)$ for all $v \in V(T)$. By Lemma 2.1, we get f_1 is a graceful labeling of T with the leaf of a leg of length m has a label zero.

Suppose that $m > l$, let T be a spider graph as Fig. 8, we see that $|V(T)| = n + 2m + l - 1$. To construct a graceful labeling of T, we consider two cases.

Case $m - 1$ is odd. Let f be a labeling of T of Type I. Since $m > l$, we get $|f(v_{n-1}) - f(x_m)| \le |f(w_{l-1}) - f(x_1)|$. Suppose that $|f(w_{l-1}) - f(x_1)| - |f(v_{n-1}) - f(x_m)| = d$. Since n is large enough, there is $i \le n - 1$ such that $f(v_i) = f(v_{n-1}) + d$. If we switch the values of f at v_i and v_{n-1}, then $|f(v_{n-1}) - f(x_m)| = |f(w_{l-1}) - f(x_1)|$. After this we change the values of f at $x_1, x_2, \ldots, x_m$ by reversing the order of their labels, we get $\bar{f}(E(T)) = \{1, 2, 3, \ldots, n+2m+l-2\}$. Comparing with $f(T')$ in Preliminaries, one can conclude that a labeling f of T is a graceful labeling with $f(u_{m-1}) = 0$.

Case $m - 1$ is even. Let f be a labeling of T of Type II. Since $m > l$, we get $|f(v_{n-1}) - f(x_m)| \le |f(w_{l-1}) - f(x_1)|$. Suppose that $|f(w_{l-1}) - f(x_1)| - |f(v_{n-1}) - f(x_m)| = d$. Since n is large enough, there is $j \le n - 1$ such that $f(v_j) = f(v_{n-1}) - d$. If we switch the values of f at v_j and v_{n-1}, then $|f(v_{n-1}) - f(x_m)| = |f(w_{l-1}) - f(x_1)|$. After this we change the values of f at $x_1, x_2, \ldots, x_m$ by reversing the order of their labels, we get $\bar{f}(E(T)) = \{1, 2, 3, \ldots, n+2m+l-2\}$. Comparing with $f(T')$ in Preliminaries, one can conclude that a labeling f of T is a graceful labeling with $f(u_{m-1}) = 0$. □

Lemma 3.5. *Let $T = S_n(m, m+1, m+1)$ with $m > 1$ and n is large enough. Then there is a graceful labeling f of T such that $f(u) = 0$ where u is the leaf of the leg of length m.*

Proof. Let T be a spider graph as Fig. 9, we see that $|V(T)| = n + 3m$. To construct a graceful labeling of T, we consider two cases.

Case $m-1$ is odd. Let $m-1 = 2p+1$, we get $|V(T)| = n+6p+6$. Let f be a labeling of T of Type I, we have $f(v_{n-1}) = 5p+7$, $f(v_n) = 5p+6$, $f(w_1) = p+2$,

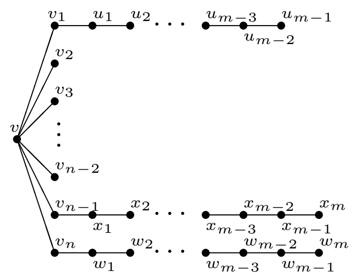

Figure 9:

$f(w_m) = 4p+5$, $f(x_1) = 2p+3$ and $f(x_m) = 3p+4$. Hence $|f(w_m)-f(x_1)| = 2p+2$ and $|f(v_n) - f(w_1)| = 4p + 4$. Since n is large enough, there are $i < n-1$ and $j < n-1$ such that $f(v_i) = 6p+7$ and $f(v_j) = 7p+8$. We change the values of f at v_i, v_j, v_n and v_{n-1} to be $f(v_n) = 6p+7$, $f(v_{n-1}) = 7p+8$, $f(v_i) = 5p+6$ and $f(v_j) = 5p+7$, change the values of f at $x_1, x_2, \ldots, x_m$ by reversing the order of their labels, and change the values of f at $w_1, w_2, \ldots, w_m$ by reversing the order of their labels. Comparing with $f(T')$ in Preliminaries, one can conclude that a labeling f of T is a graceful labeling with $f(u_{m-1}) = 0$.

Case $m-1$ is even. Let $m-1 = 2p'$, we get $|V(T)| = n + 6p' + 3$. Let f be a labeling of T of Type II, we have $f(v_{n-1}) = n + p' - 2$, $f(v_n) = n + p' - 1$, $f(w_1) = n + 5p' + 1$, $f(w_m) = n + 4p' + 1$, $f(x_1) = n + 2p'$ and $f(x_m) = n + 3p'$. Hence $|f(w_m) - f(x_1)| = 2p' + 1$ and $|f(v_n) - f(w_1)| = 4p' + 2$. Since n is large enough, there is $i < n-1$ such that $f(v_i) = n-1$. We change the values of f at v_i, v_n and v_{n-1} to be $f(v_n) = n-1$, $f(v_{n-1}) = n + p' - 1$ and $f(v_i) = n + p' - 2$, change the values of f at $x_1, x_2, \ldots, x_m$ by reversing the order of their labels and change the values of f at $w_1, w_2, \ldots, w_m$ by reversing the order of their labels. Comparing with $f(T')$ in Preliminaries, one can conclude that a labeling f of T is a graceful labeling with $f(u_{m-1}) = 0$. □

Theorem 3.6. *Let $T = S_n(k, l, m)$ with $1 < m \le k \le l$ and n is large enough. Then T is graceful.*

Proof. To prove the theorem we consider three cases.

Case $m = k \le l$ and $m = 2$. We get the spider subgraph of T and a graceful labeling as Fig. 10. Lemma 2.2 implies T is graceful.

Case $m = k \le l$ and $m > 2$. Lemma 3.5 implies $S_n(m, m, m-1)$ has a graceful labeling f such that $f(u) = 0$ where u is the leaf of the leg of length $m-1$. Lemma 2.2 implies T is graceful.

Case $m < k \le l$. By Lemma 3.4, there is a graceful labeling f of $S_n(m, k, k-1)$ such that $f(u) = 0$ where u is the leaf of the leg of length $k-1$. Lemma 2.2 implies T is graceful. □

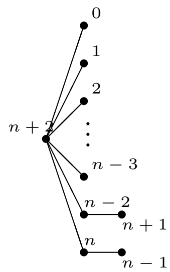

Figure 10:

4 Remarks

The main tool which we use to investigate the spider graph is a graceful labeling f with $f(u) = 0$ where u is the leaf of a leg. We believe these lemmas are still true without condition n is large enough.

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