Thai Journal of Mathematics
Volume 12 (2014) Number 3 : 613–620
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Fixed Point Theorems for Mappings Satisfying a Contractive Condition of Rational Expression on a Ordered Partial Metric Space

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Abstract : The purpose of this manuscript is to present a fixed point theorem using a contractive condition of rational expression in the context of ordered partial metric spaces.

Keywords : partial metric spaces; fixed point; ordered set.
2010 Mathematics Subject Classification : 47H10; 47H04; 54H25.

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1 Introduction and Preliminaries

Partial metric is one of the generalizations of metric was introduced by Matthews [1] in 1992 to study denotational semantics of data flow networks. In fact, partial metric spaces constitute a suitable framework to model several distinguished examples of the theory of computation and also to model metric spaces via domain theory [2–7]. Recently, many researchers have obtained fixed, common fixed and coupled fixed point results on partial metric spaces and ordered partial metric spaces [4, 8–11].

Motivated the interesting paper of Jaggi [12], in [13] Harjani et al. proved the following fixed point theorem in partially ordered metric spaces.

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Theorem 1.1 ([13]). *Let $(X, \leq)$ be a ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T : X \to X$ be a non-decreasing mapping such that*

$$d(Tx, Ty) \leq \alpha \frac{d(x, Tx)d(y, Ty)}{d(x, y)} + \beta d(x, y) \quad \text{for } x, y \in X, x \geq y, x \neq y,$$

and for some $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$.Also, assume either T is continuous or X has the property that

$\{x_n\}$ *is a nondecreasing sequence in X such that $x_n \to x$,then $x = \sup\{x_n\}$.*

If there exists $x_0 \in X$ such that $x_0 \leq Tx_0$, then T has a fixed point.

In this paper we extend the result of Harjani et al. [13] to the case of partial metric spaces. An example is considered to illustrate our obtained results.

First, we recall some definitions of partial metric space and some of their properties [1, 3, 8, 9, 11].

Definition 1.2. A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}_+$ such that for all $x, y, z \in X$:

(PM1) $p(x, y) = p(y, x)$ (symmetry);

(PM2) if $0 \leq p(x, x) = p(x, y) = p(y, y)$ then $x = y$ (equality);

(PM3) $p(x, x) \leq p(x, y)$ (small self-distances);

(PM4) $p(x, z) + p(y, y) \leq p(x, y) + p(y, z)$ (triangularity); for all $x, y, z \in X$.

For a partial metric p on X, the function $d_p : X \times X \to \mathbb{R}_+$ given by

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

is a (usual) metric on X. Each partial metric p on X generates a T_0 topology τ_p on X with a base of the family of open p-balls $\{B_p(x, \epsilon) : x \in X, \epsilon > 0\}$, where $B_p(x, \epsilon) = \{y \in X : p(x, y) < p(x, x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$.

Definition 1.3. Let (X, p) be a partial metric space.

(i) A sequence $\{x_n\}$ in a PMS (X, p) is converges to $x \in X$ iff $p(x, x) = \lim_{n\to\infty} p(x, x_n)$.

(ii) A sequence $\{x_n\}$ in a PMS (X, p) is called Cauchy iff $\lim_{n,m\to\infty} p(x_n, x_m)$ exists (and is finite).

(iii) A PMS (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

(iv) A mapping $T : X \to X$ is said to be continuous at $x_0 \in X$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that $T(B_p(x_0, \delta)) \subset B_p(T(x_0), \epsilon)$.

Lemma 1.4. *Let (X,p) be a partial metric space. Then*

(i) A sequence $\{x_n\}$ is a Cauchy sequence in the PMS (X,p) if and only if $\{x_n\}$ is Cauchy in a metric space (X,d_p).

(ii) A PMS (X,p) is complete if and only if a metric space (X,d_p) is complete. Moreover,

$$\lim_{n\to\infty} d_p(x,x_n)=0 \Leftrightarrow p(x,x)=\lim_{n\to\infty} p(x,x_n)=\lim_{n,m\to\infty} p(x_n,x_m).$$

2 Main Results

Theorem 2.1. *Let $(X,\leq)$ be a partially ordered set and suppose that there exists a partial metric p in X such that (X,p) is a complete partial metric space. Let $T:X\to X$ be a continuous and nondecreasing mapping such that*

$$p(Tx,Ty)\leq \frac{\alpha p(x,Tx)p(y,Ty)}{p(x,y)}+\beta p(x,y),\quad \text{for } x,y\in X, x\geq y, x\neq y, \tag{2.1}$$

with $\alpha\geq 0,\beta\geq 0,\alpha+\beta<1$. If there exists $x_0\in X$ with $x_0\leq Tx_0$, then T has fixed point $z\in X$ and $p(z,z)=0$.

Proof. If $Tx_0=x_0$, then the proof is done. Suppose that $x_0\leq Tx_0$. Since T is a nondecreasing mapping, we obtain by induction that

$$x_0\leq Tx_0\leq T^2x_0\leq\cdots\leq T^nx_0\leq T^{n+1}x_0\leq\cdots.$$

Put $x_{n+1}=Tx_n$. If there exists $n\geq 1$ such that $x_{n+1}=x_n$, then from $x_{n+1}=Tx_n=x_n$, x_n is a fixed point. Suppose that $x_{n+1}\neq x_n$ for $n\geq 1$. That is x_n and x_{n-1} are comparable, we get, for $n\geq 1$,

$$\begin{aligned} p(x_{n+1},x_n)&=p(Tx_n,Tx_{n-1})\\ &\leq \frac{\alpha p(x_n,Tx_n)p(x_{n-1},Tx_{n-1})}{p(x_n,x_{n-1})}+\beta p(x_n,x_{n-1})\\ &\leq \alpha p(x_n,x_{n+1})+\beta p(x_n,x_{n-1}). \end{aligned}$$

The last inequality gives us

$$\begin{aligned} p(x_{n+1},x_n)&\leq kp(x_n,x_{n-1}),\ \ k=\frac{\beta}{1-\alpha}<1\\ &\ \ \vdots\\ &\leq k^np(x_1,x_0). \end{aligned} \tag{2.2}$$

Moreover, by the triangular inequality, we have, for $m \geq n$,

$$\begin{aligned} p(x_m, x_n) &\leq p(x_m, x_{m-1}) + \cdots + p(x_{n+1}, x_n) - \sum_{i=1}^{m-n-1} p(x_{m-k}, x_{m-k}) \\ &\leq [k^{m-1} + \cdots + k^n] p(x_1, x_0) \\ &= k^n \frac{1 - k^{m-n}}{1-k} p(x_1, x_0) \\ &\leq \frac{k^n}{1-k} p(x_1, x_0). \end{aligned}$$

Hence, $\lim_{n,m\to\infty} p(x_n, x_m) = 0$, that is, $\{x_n\}$ is a Cauchy sequence in (X, p). By Lemma 1.4, $\{x_n\}$ is also Cauchy in (X, d_p). In addition, since (X, p) is complete, (X, d_p) is also complete. Thus there exists $z \in X$ such that $x_n \to z$ in (X, d_p); moreover, by Lemma 1.4,

$$p(z, z) = \lim_{n\to\infty} p(z, x_n) = \lim_{n,m\to\infty} p(x_n, x_m) = 0.$$

Given that T is continuous in (X, p). Therefore, $Tx_n \to Tz$ in (X, p).

$$\text{i.e.,} \;\; p(Tz, Tz) = \lim_{n\to\infty} p(Tz, Tx_n) = \lim_{n,m\to\infty} p(Tx_n, Tx_m).$$

But, $p(Tz, Tz) = \lim_{n,m\to\infty} p(Tx_n, Tx_m) = \lim_{n,m\to\infty} p(x_{n+1}, x_{m+1}) = 0$.

We will show next that z is the fixed point of T.

$$\begin{aligned} p(Tz, z) &\leq p(Tz, Tx_n) + p(Tx_n, z) - p(Tx_n, Tx_n) \\ &\leq p(Tz, Tx_n) + p(x_{n+1}, z). \end{aligned}$$

As $n \to \infty$, we obtain $p(Tz, z) \leq 0$. Thus, $p(Tz, z) = 0$. Hence $p(z, z) = p(Tz, Tz) = p(Tz, z) = 0$. Therefore, by (PM2) we get $Tz = z$ and $p(z, z) = 0$ which completes the proof. □

In what follows we prove that Theorem 2.1 is still valid for T not necessarily continuous, assuming X has the property that

$$\{x_n\} \text{ is a nondecreasing sequence in } X \text{ such that } x_n \to x, \text{then } x = \sup\{x_n\}. \tag{2.3}$$

Theorem 2.2. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a partial metric p in X such that (X, p) is a complete partial metric space. Assume that X satisfies (2.3) in (X, p). Let $T : X \to X$ be a nondecreasing mapping such that*

$$p(Tx, Ty) \leq \frac{\alpha p(x, Tx) p(y, Ty)}{p(x, y)} + \beta p(x, y), \quad \textit{for } x, y \in X, x \geq y, x \neq y, \tag{2.4}$$

with $\alpha \geq 0, \beta \geq 0, \alpha + \beta < 1$. If there exists $x_0 \in X$ with $x_0 \leq Tx_0$, then T has fixed point $z \in X$ and $p(z, z) = 0$.

Proof. Following the proof of Theorem 2.1, we only have to check that $Tz = z$. As $\{x_n\}$ is a nondecreasing sequence in X and $x_n \to z$, then, by (2.3), $z = \sup\{x_n\}$. In particularly, $x_n \le z$ for all $n \in \mathbb{N}$. Since T is a nondecreasing mapping, then $Tx_n \le Tz$, for all $n \in \mathbb{N}$ or, equivalently, $x_{n+1} \le Tz$ for all $n \in \mathbb{N}$. Moreover, as $x_0 < x_1 \le Tz$ and $z = \sup\{x_n\}$, we get $z \le Tz$.

Suppose that $z < Tz$. Using a similar argument that in the proof of Theorem 2.1 for $x_0 \le Tx_0$, we obtain that $\{T^n z\}$ is a nondecreasing sequence such that

$$p(y,y) = \lim_{n\to\infty} p(T^n z, y) = \lim_{m,n\to\infty} p(T^n z, T^m z) = 0 \text{ for some } y \in X. \tag{2.5}$$

By the assumption of (2.3), we have $y = \sup\{T^n z\}$.

Moreover, from $x_0 \le z$, we get $x_n = T^n x_0 \le T^n z$ for $n \ge 1$ and $x_n < T^n z$ for $n \ge 1$ because $x_n \le z < Tz \le T^n z$ for $n \ge 1$.

As x_n and $T^n z$ are comparable and distinct for $n \ge 1$, applying the contractive condition we get

$$\begin{aligned} p(T^{n+1}z, x_{n+1}) &= p(T(T^n z), Tx_n) \\ &\le \frac{\alpha p(T^n z, T^{n+1} z) p(x_n, Tx_n)}{p(T^n z, x_n)} + \beta p(T^n z, x_n), \end{aligned}$$

$$p(T^{n+1}z, x_{n+1}) \le \frac{\alpha p(T^n z, T^{n+1} z) p(x_n, x_{n+1})}{p(T^n z, x_n)} + \beta p(T^n z, x_n). \tag{2.6}$$

From $\lim_{n\to\infty} p(x_n, z) = \lim_{n\to\infty} p(T^n z, y) = 0$, we have

$$\lim_{n\to\infty} p(T^n z, x_n) = p(y,z). \tag{2.7}$$

As, $n \to \infty$ in (2.6) and using that (2.2) and (2.7), we obtain

$$p(y,z) \le \beta p(y,z).$$

As $\beta < 1$, $p(y,z) = 0$. Hence $p(z,z) = p(y,y) = p(y,z) = 0$. Therefore, by (PM2) $y = z$. Particularly, $y = z = \sup\{T^n z\}$ and, consequently, $Tz \le z$ and this is a contradiction. Hence, we conclude that $z = Tz$ and $p(z,z) = 0$. □

Theorem 2.3. *In addition to the hypotheses of Theorem 2.1 (or Theorem 2.2), suppose that*

$$\textit{for every } x, y \in X, \textit{ there exists } z \in X \textit{ that is comparable to } x \textit{ and } y, \tag{2.8}$$

then T has a unique fixed point.

Proof. From Theorem 2.1 or Theorem 2.2, the set of fixed points of T is nonempty. Suppose that there exists $z, y \in X$ which are fixed point. By Theorem 2.1 or Theorem 2.2, we get $p(z,z) = 0$ and $p(y,y) = 0$. We distinguish two cases.

Case 1: If y and z are comparable and $y \neq z$, then using the contractive condition we have

$$\begin{aligned} p(y,z) &= p(Ty,Tz) \\ &\leq \frac{\alpha p(y,Ty)p(z,Tz)}{p(y,z)} + \beta p(y,z). \end{aligned}$$

Since y is fixed point and $p(y,y) = 0$. We obtain, $p(y,z) \leq \beta p(y,z)$ which is a contradiction to $\beta < 1$. Thus $y = z$.

Case 2: If y is not comparable to z, then by (2.8) there exists $x \in X$ comparable to y and z. Monotonicity implies that $T^n x$ is comparable to $T^n y = y$ and $T^n z = z$ for $n = 0, 1, 2, \dots$.

Suppose there exists $n_0 \geq 1$ such that $T^{n_0} x = y$, then $T^n x = y = Ty$ for all $n \geq n_0$. Therefore, $\lim_{n\to\infty} p(T^n x, y) = p(y,y) = 0$.

On the other hand, if $T^n x \neq y$ for $n \geq 1$, using the contractive condition, we obtain, for $n \geq 1$,

$$\begin{aligned} p(T^n x, y) &= p(T^n x, T^n y) \\ &\leq \frac{\alpha p(T^{n-1}x, T^n x)p(T^{n-1}y, T^n y)}{p(T^{n-1}x, T^{n-1}y)} + \beta p(T^{n-1}x, T^{n-1}y) \\ &\leq \frac{\alpha p(T^{n-1}x, T^n x)p(y,y)}{p(T^{n-1}x, y)} + \beta p(T^{n-1}x, y). \end{aligned}$$

Since y is fixed point and $p(y,y) = 0$. We obtain,

$$p(T^n x, y) \leq \beta p(T^{n-1}x, y).$$

Therefore, $p(T^n x, y) \leq \beta^n p(x,y)$, for $n \geq 2$. As $n \to \infty$, we get

$$\lim_{n\to\infty} p(T^n x, y) = 0.$$

Using a similar argument, we can prove that $\lim_{n\to\infty} p(T^n x, z) = 0$.

$$\begin{aligned} 0 \leq p(y,z) &\leq p(y, T^n x) + p(T^n x, z) - p(T^n x, T^n x) \\ &\leq p(y, T^n x) + p(T^n x, z). \end{aligned}$$

As $n \to \infty$, we get $p(y,z) = 0$. By (PM2), we obtain $y = z$. Hence the proof is completed. □

Example 2.4. *Let $X = [0,\infty)$ endowed with the usual partial metric p defined by $p : X \times X \to \mathbb{R}_+$ with $p(x,y) = \max\{x,y\}$, for all $x, y \in X$. We consider the ordered relation in X as follows*

$$x \preccurlyeq y \Leftrightarrow p(x,x) = p(x,y) \Leftrightarrow x = \max\{x,y\} \Leftarrow y \leq x$$

where $\leq$ be the usual ordering.

It is clear that $(X, \preccurlyeq)$ *is totally ordered. The partial metric space* (X, p) *is complete because* (X, d_p) *is complete. Indeed, for any* $x, y \in X$,

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y) = 2\max\{x,y\} - (x+y) = |x-y|$$

Thus, $(X, d_p) = ([0,\infty), |.|)$ *is the usual metric space, which is complete.*

Let $T : X \to X$ *be given by* $T(x) = \dfrac{x}{2}$, $x \geq 0$. *The function* T *is continuous on* (X,p). *Indeed, let* $\{x_n\}$ *be a sequence converging to* x *in* (X,p), *then* $\lim_{n\to\infty} \max\{x_n, x\} = \lim_{n\to\infty} p(x_n, x) = p(x,x) = x$ *and*

$$\lim_{n\to\infty} p(Tx_n, Tx) = \lim_{n\to\infty} \max\{Tx_n, Tx\} = \lim_{n\to\infty} \frac{\max\{x_n, x\}}{2} = \frac{x}{2} = p(Tx, Tx) \tag{2.9}$$

that is $\{T(x_n)\}$ *converges to* $T(x)$ *in* (X,p). *Since* $x_n \to x$ *and by the definition* T *we have,* $\lim_{n\to\infty} d_p(x_n, x) = 0$ *and*

$$\lim_{n\to\infty} d_p(Tx_n, Tx) = 0. \tag{2.10}$$

From (2.9) and (2.10) we get T *is continuous on* (X,p). *Any* $x, y \in X$ *are comparable, so for example we take* $x \preccurlyeq y$, *and then* $p(x,x) = p(x,y)$, *so* $y \leq x$. *Since* $T(y) \leq T(x)$, *so* $T(x) \preccurlyeq T(y)$, *giving that* T *is non-decreasing with respect to* $\preccurlyeq$. *In particular, for any* $x \preccurlyeq y$, *we have*

$$p(x,y) = x, p(Tx,Ty) = Tx = \frac{x}{2}, p(x,Tx) = x, p(y,Ty) = y.$$

Now we have to show that T *satisfies the inequality (2.1). For any* $x, y \in X$ *with* $x \preccurlyeq y$ *and* $x \neq y$, *we have*

$$p(Tx,Ty) = \frac{x}{2} \text{ and } \frac{\alpha p(x,Tx)p(y,Ty)}{p(x,y)} + \beta p(x,y) = \frac{\alpha xy}{x} + \beta x.$$

Therefore, choose $\beta \geq \frac{1}{2}$ *and* $\alpha + \beta < 1$, *then (2.1) holds. All the hypotheses of Theorem 2.3 are satisfied, so* T *has a unique fixed point in* X *which is* 0 *and* $p(0,0) = 0$.

Acknowledgement : We would like to thank the referees for there comments and suggestions to improve the manuscript.

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(Received 29 March 2012)
(Accepted 26 March 2013)