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An Efficient Method for Solving Fractional Partial Differential Equations

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Abstract : In this paper, a novel algorithm based on new modified Homotopy Perturbation Method, called NHPM, for fractional partial differential equations was proposed. The solution process is elucidated including how to construct a suitable homotopy equation and how to choose an initial solution. Some examples are given to reveal the effectiveness and convenience of the method.

Keywords : Efficient method; fractional partial differential equations.
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1 Introduction

In recent years, considerable research and interest in fractional differential equations has been stimulated due to their numerous applications in many areas like physics, and engineering [1]. Many important phenomena in electromagnetics, acoustics, viscoelasticity, electrochemistry, corrosion and material science are well described by differential equations of fractional order [2-3].
Many new numerical techniques have been widely applied to fractional differential equations. Based on homotopy, which is a basic concept in topology, general analytical method namely the homotopy perturbation method (HPM) was established by He [4-7] in the year 1998, to obtain a series of solutions to the nonlinear differential equations. This simple method has been applied to solve Blasius equation [8], fluid mechanics equations [9], fractional KdV-Burgers equation [10], some

boundary value problems, and many other equations in subjects of different disciplines [11–24]. In this study, a new version of the HPM, which efficiently solves fractional differential equations, is being introduced.

2 Basic Definitions

In this section some basic definitions and properties of the fractional calculus theory used in this work will be discussed [25-26].

Definition 2.1. *A real function $f(x), x > 0$, in the space $C_\mu, \mu \in R$ if there exists a real number $p > \mu$,such that $f(x) = x^p f_1(x)$where $f_1(x) \in C[0, \infty]$ and it is said to be in the space C_μ^m if $f^{(m)} \in C_\mu, m \in N$.*

Definition 2.2. *The Riemann–Liouville fractional integral operator of order$\alpha \geq 0$, of a function $f \in C_\mu, \mu \geq -1$, is defined as:*

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \quad \alpha > 0,\, x > 0,\, J^0 f(x) = f(x).$$

The general and detailed properties of the operator J^α can be found in reference [26]. For this study, where $f \in C_\mu, \mu \geq -1$, $\alpha, \beta \geq 0$ and $\gamma > -1$:

$$\begin{aligned} &(1)\ J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x), \\ &(2)\ J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x), \\ &(3)\ J^\alpha x^\gamma = \tfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}. \end{aligned}$$

It is worth mentioning here that, the Riemann–Liouville derivative method has some disadvantages when used to model real-world phenomena with fractional differential equations. Therefore, a modified fractional differential operator, D_*^α, should be introduced to overcome those weaknesses in the previous models. Such modified fractional differential operators, D_*^α, were first proposed by Caputo [26], in his work on the theory of viscoelasticity.

Definition 2.3. *The fractional derivative of $f(x)$according to Caputo [26], is defined as:*

$$D_*^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt,$$

for: $m - 1 < \alpha \leq m$, $m \in N$, $x > 0$, $f \in C_{-1}^m$.

The following two properties of this operator will be used in what comes next.

Lemma 2.4. *If $m-1 < \alpha \leq m$, and $f \in C_\mu^m$, $\mu \geq -1$, then $D_*^\alpha J^\alpha f(x) = f(x)$,and*

$$J^\alpha D_*^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f(0^+) \frac{x^k}{k!}, x > 0.$$

Definition 2.5. *For m as the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order $\alpha > 0$, is defined as:*

$$D_{*t}^{\alpha} u(x,t) = \frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{\partial^m u(x,\tau)}{\partial \tau^m} d\tau, & m-1 < \alpha < m, \\ \frac{\partial^m u(x,t)}{\partial t^m}, & \alpha = m \in N. \end{cases}$$

For more information on the mathematical properties of fractional derivatives and integrals, one can consult the above mentioned references.

3 Theory of the Method

To illustrate the application and methodology of using the proposed new method, the following fractional differential equation will be considered:

$$A(u(X,t) - f(r) = 0, r \in \Omega, \tag{3.1}$$

$$B(u(X,t), \partial u/\partial n) = 0, r \in \Gamma, \tag{3.2}$$

where A is a general differential operator, $f(r)$is a known analytic function, $B$is a boundary condition, Γ is the boundary of the domainΩ,and $X = (x_1, x_2, \ldots x_n)$.
In general, the operator Acan be divided into two operators, $L$and N, where L is a linear operator, while $N$is a non-linear operator.
In this case, equation (3.1) can be re-written as follows:

$$L(u) + N(u) - f(r) = 0. \tag{3.3}$$

Using the homotopy technique, a homotopy $U(r,p) : \Omega \times [0,1] \to \mathbb{R}$ could be constructed, which satisfies:

$$H(U,p) = (1-p)[L(U) - u_0] + p[A(U) - f(r)] = 0, \quad p \in [0,1], \quad r \in \Omega, \tag{3.4}$$

or

$$H(U,p) = L(U) - u_0 + pu_0 + p[N(U) - f(r)] = 0. \tag{3.5}$$

Where$p \in [0,1]$, is called homotopy parameter, and $u_0$is an initial approximation for the solution of Eq.(3.1), which satisfies the boundary conditions.
Obviously from Eqs. (3.4) and (3.5), Eqs. (3.6) and (3.7) could be derived and written as:

$$H(U,0) = L(U) - u_0 = 0, \tag{3.6}$$

$$H(U,1) = A(U) - f(r) = 0. \tag{3.7}$$

It is assumed that the solution of Eq. (3.6) or Eq. (3.7) could be expressed as a series in$p$, as follows:

$$U = U_0 + pU_1 + p^2U_2 + \dots \tag{3.8}$$

Setting $p = 1$, produces the approximate solution of Eq. (3.1), which could be written in the following form:
$u = \lim_{p\to 1} U \ = U_0 + U_1 + U_2 + \dots$ Now Eq. (3.5) will be written in the following form:

$$L(U(X,t)) = u_0(X,t) + p[f(r(X,t)) - u_0(X,t) - N(U(X,t))]. \tag{3.9}$$

By applying the inverse operator,L^{-1}, to both sides of Eq. (3.9), Eq. (3.10) could be derived:

$$U(X,t) = L^{-1}(u_0(X,t)) + p\left(L^{-1}(f(r)) - L^{-1}(u_0(X,t) - L^{-1}(N(U(X,t))\right). \tag{3.10}$$

Suppose that the initial approximation of Eq. (3.1) has the form:

$$u_0(X,t) = \sum_{n=0}^{\infty} a_n(X)P_n(t) \tag{3.11}$$

where $a_1(X), a_2(X), a_3(X), \dots$are unknown coefficients, and $P_0(t), P_1(t), P_2(t), \dots$are specific functions dependent on the problem. Now by substituting Eqs. (3.8) and (3.11) into Eq. (3.10), we get:

$$\begin{aligned}&\textstyle\sum_{n=0}^{\infty} p^n U_n(X,t) = U(X,t) = L^{-1}(\sum_{n=0}^{\infty} a_n(X)P_n(t)) + \\ &p\left(L^{-1}(f(r)) - L^{-1}(\textstyle\sum_{n=0}^{\infty} a_n(X)P_n(t)) - L^{-1}(N(\sum_{n=0}^{\infty} p^n U_n(X,t))\right).\end{aligned} \tag{3.12}$$

Comparing the coefficients of terms with the identical powers of p, leads to:

$$\begin{aligned}
&p^0 \ : \ U_0(X,t) = L^{-1}(\textstyle\sum_{n=0}^{\infty} a_n(X)P_n(t)),\\
&p^1 \ : \ U_1(X,t) = L^{-1}(f(r)) - L^{-1}(\textstyle\sum_{n=0}^{\infty} a_n(X)P_n(t)) - L^{-1}N(U_0(X,t)),\\
&p^2 \ : \ U_2(X,t) = -L^{-1}N(U_0(X,t), U_1(X,t)),\\
&p^3 \ : \ U_3(X,t) = -L^{-1}N(U_0(X,t), U_1(X,t), U_2(X,t)),\\
&\vdots\\
&p^j \ : \ U_j(X,t) = -L^{-1}N(U_0(X,t), U_1(X,t), U_2(X,t), \dots, U_{j-1}(X,t)),\\
&\vdots
\end{aligned} \tag{3.13}$$

Now, if the above equations are solved in such a way that $U_1(X,t) = 0$, then Eq. (3.13) results in: $U_1(X,t) = U_2(X,t) = \cdots = 0$,
Therefore, the exact solution may be obtained as follows:

$$u(X,t) = U_0(X,t) = L^{-1}\left(\sum_{n=0}^{\infty} a_n(X)P_n(t)\right),$$

To show the capability of this method, it will be applied to some examples in the next section. The computations associated with the following examples were performed using MAPLE 15 software.

Example 3.1. *Let us consider the following time-fractional equation:*

$$\frac{\partial^\alpha u}{\partial t^\alpha} + x\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} = 2t^\alpha + 2x^2 + 2, \qquad 0 < \alpha \le 1, \tag{3.14}$$

with the initial condition:

$$u(x,0) = x^2. \tag{3.15}$$

To solve Eq. (3.14)*, the following homotopy should be constructed:*

$$(1-p)(\frac{\partial^\alpha U}{\partial t^\alpha}(x,t) - u_0(x,t)) + p\left(\frac{\partial^\alpha U}{\partial t^\alpha} + x\frac{\partial U}{\partial x} + \frac{\partial^2 U}{\partial x^2} - 2t^\alpha - 2x^2 - 2\right) = 0,$$

or

$$\frac{\partial^\alpha U}{\partial t^\alpha}(x,t) = u_0(x,t) - p\left(u_0(x,t) + x\frac{\partial U}{\partial x} + \frac{\partial^2 U}{\partial x^2} - 2t^\alpha - 2x^2 - 2\right). \tag{3.16}$$

Applying the inverse operator, J_t^α to both sides of the above equations, results in:

$$U(x,t) = U(x,0) + J_t^\alpha u_0(x,t) - J_t^\alpha\left(u_0(x,t) + x\frac{\partial U}{\partial x} + \frac{\partial^2 U}{\partial x^2} - 2t^\alpha - 2x^2 - 2\right). \tag{3.17}$$

Suppose the solution of Eq. (3.17) *has the form of Eq.* (3.8)*, then substituting Eq.* (3.18) *into Eq.* (3.17)*, and after collecting the terms with the same powers of p, and equating each coefficient of p to zero, all of that results in:*

$$\begin{aligned}
p^0 &: U_0(x,t) = U(x,0) + J_t^\alpha u_0(x,t),\\
p^1 &: U_1(x,t) = -J_t^\alpha\left(u_0(x,t) + x\tfrac{\partial U_0}{\partial x} + \tfrac{\partial^2 U_0}{\partial x^2} - 2t^\alpha - 2x^2 - 2\right),\\
p^2 &: U_2(x,t) = -J_t^\alpha\left(x\tfrac{\partial U_1}{\partial x} + \tfrac{\partial^2 U_1}{\partial x^2}\right),\\
&\vdots\\
p^j &: U_{j+1}(x,t) = -J_t^\alpha\left(x\tfrac{\partial U_j}{\partial x} + \tfrac{\partial^2 U_j}{\partial x^2}\right),\\
&\vdots
\end{aligned}$$

Assuming

$$u_0(x,t) = \sum_{n=0}^{\infty} a_n(x)P_n(t), P_k(t) = t^{\alpha k}, U(x,0) = u(x,0).$$

Solving the above equation for $U_1(x,t)$leads to the following result:

$$\begin{aligned}
&U_1(x,t) = -\tfrac{1}{\Gamma(\alpha+1)}a_0(x)t^\alpha + \left(-\tfrac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}a_1(x) - \tfrac{x}{\Gamma(2\alpha+1)}a_0'(x) - \tfrac{1}{\Gamma(2\alpha+1)}a_0''(x) + 2\tfrac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\right)t^{2\alpha}\\
&+\left(-\tfrac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}a_2(x) - \tfrac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}a_1'(x) - \tfrac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}a_1''(x)\right)t^{3\alpha}\\
&+\cdots
\end{aligned}$$

With vanishing $U_1(x,t)$, it could easily be shown that:

$$a_0(x) = 0, a_1(x) = 2, \quad a_2(x) = 0, \quad a_3(x) = 0, \cdots$$

Therefore, we get the solution of Eq. (3.14) *as:*

$$u(x,t) = U_0(x,t) = x^2 + \frac{2\Gamma(\alpha+1)}{\Gamma(2\alpha+1)} t^{2\alpha}.$$

which is an exact solution.

Example 3.2. *Consider the following fractional Fisher equation:*

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + u(1-u), \qquad 0 < \alpha \leq 1. \tag{3.18}$$

with a constant initial condition:

$$u(x,0) = \lambda.$$

For solving Eq. (3.18)*, the following homotopy should be constructed:*

$$\frac{\partial^\alpha U}{\partial t^\alpha}(x,t) = u_0(x,t) - p\left(u_0(x,t) + \frac{\partial^\alpha U}{\partial t^\alpha} - \frac{\partial^2 U}{\partial x^2} - U(1-U)\right). \tag{3.19}$$

Applying the inverse operator, J_t^α to both sides of the above equation, results in:

$$U(x,t) = U(x,0) + J_t^\alpha u_0(x,t) - J_t^\alpha \left(u_0(x,t) + \frac{\partial^\alpha U}{\partial t^\alpha} - \frac{\partial^2 U}{\partial x^2} - U(1-U)\right). \tag{3.20}$$

Suppose the solution of Eq. (3.20) *have the form shown in Eq.* (3.8)*, then substituting Eq.* (3.8) *into Eq.* (3.20) *and equating the coefficients of p with the same power, leads to:*

$$\begin{aligned}
p^0 &: U_0(x,t) = U(x,0) + J_t^\alpha u_0(x,t), \\
p^1 &: U_1(x,t) = -J_t^\alpha \left(u_0(x,t) - \tfrac{\partial^2 U_0}{\partial x^2} - U_0(1-U_0)\right), \\
p^2 &: U_2(x,t) = -J_t^\alpha \left(-\tfrac{\partial^2 U_1}{\partial x^2} - U_1 + 2U_0U_1\right), \\
&\vdots \\
p^j &: U_{j+1}(x,t) = -J_t^\alpha \left(-\tfrac{\partial^2 U_j}{\partial x^2} - U_j + \textstyle\sum_{k=0}^{j} U_k U_{j-k}\right), \\
&\vdots
\end{aligned}$$

Assuming $u_0(x,t) = \sum_{n=0}^{\infty} a_n(x) t^{\alpha k}$, $U(x,0) = u(x,0)$, and solving the above *equation for $U_1(x,t)$, leads to the following result:*

$$\begin{aligned}&U_1(x,t)=\left(-\tfrac{1}{\Gamma(\alpha+1)}a_0(x)+\tfrac{\lambda}{\Gamma(\alpha+1)}-\tfrac{\lambda^2}{\Gamma(\alpha+1)}\right)t^{\alpha}\\&+\left(-\tfrac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}a_1(x)+\tfrac{1}{\Gamma(2\alpha+1)}a_0''(x)+\tfrac{1}{\Gamma(2\alpha+1)}a_0(x)-2\lambda\tfrac{1}{\Gamma(2\alpha+1)}a_0(x)\right)t^{2\alpha}\\&+\begin{pmatrix}-\tfrac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}a_2(x)+\tfrac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}a_1''(x)+\tfrac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}a_1(x)\\-2\lambda\tfrac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}a_1(x)-\tfrac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)\Gamma(3\alpha+1)\Gamma(\alpha+1)}a_0^2(x)\end{pmatrix}t^{3\alpha}\\&+\cdots\end{aligned}$$

Furthermore, if it is assumed that $U_1(x,t)=0$, then:

$$a_0(x)=\lambda(1-\lambda),a_1(x)=\frac{\lambda(1-\lambda)(1-2\lambda)}{\Gamma(\alpha+1)},a_2(x)=-\frac{\lambda(1-\lambda)(1-2\lambda)^2}{\Gamma(2\alpha+1)}+\frac{(\lambda-2\lambda)^2}{(\Gamma(\alpha+1))^2},\cdots$$

Therefore, the solution of the fractional differential equation can be expressed as follows:

$$\begin{aligned}&u(x,t)=U_0(x,t)=\tfrac{\lambda(1-\lambda)}{\Gamma(\alpha+1)}t^{\alpha}+\tfrac{\lambda(1-\lambda)(1-2\lambda)}{\Gamma(2\alpha+1)}t^{2\alpha}\\&+\left(-\tfrac{\lambda(1-\lambda)(1-2\lambda)^2}{\Gamma(2\alpha+1)}+\tfrac{(\lambda-2\lambda)^2}{(\Gamma(\alpha+1))^2}\right)\tfrac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}t^{3\alpha}+\ldots,\end{aligned}$$

For the special case $\alpha=1$,the solution will be as follows:

$$\begin{aligned}u(x,t)&=\lambda(1-\lambda)t^{\alpha}+\frac{\lambda(1-\lambda)(1-2\lambda)}{2}t^2+\left(-\frac{\lambda(1-\lambda)(1-2\lambda)(1-6\lambda+6\lambda^2)}{6}\right)t^3+\ldots\\&=\frac{\lambda e^t}{1-\lambda+\lambda e^t},\end{aligned}$$

which is an exact solution.

Example 3.3. *Consider the following system of partial differential equations with fractional derivatives as follows:*

$$\left\{\begin{array}{l}\frac{\partial^{\alpha}u}{\partial t^{\alpha}}-v\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=1-x+y+t^{\alpha},\\ \frac{\partial^{\alpha}v}{\partial t^{\alpha}}-u\frac{\partial v}{\partial x}-\frac{\partial v}{\partial y}=1-x-y-t^{\alpha},\end{array}\right. \tag{3.21}$$

with initial conditions:

$$\begin{aligned}u(x,y,0)&=x+y-1,\\v(x,y,0)&=x-y+1.\end{aligned}$$

To solve Eq. (3.21)*, the following homotopy should be constructed:*

$$\begin{aligned}&\tfrac{\partial^{\alpha}U}{\partial t^{\alpha}}(x,y,t)=u_0(x,y,t)-p\left(u_0(x,y,t)-V\tfrac{\partial U}{\partial x}+\tfrac{\partial V}{\partial y}-1+x-y-t^{\alpha}\right),\\&\tfrac{\partial^{\alpha}V}{\partial t^{\alpha}}(x,y,t)=v_0(x,y,t)-p\left(v_0(x,y,t)-U\tfrac{\partial V}{\partial x}-\tfrac{\partial V}{\partial y}-1+x+y+t^{\alpha}\right).\end{aligned} \tag{3.22}$$

Applying the inverse operator, J_t^{α} to both sides of the above equations, gives:

$$\begin{aligned}&U(x,y,t)=U(x,y,0)+J_t^{\alpha}u_0(x,y,t)-pJ_t^{\alpha}\left(u_0(x,y,t)-V\tfrac{\partial U}{\partial x}+\tfrac{\partial U}{\partial y}-1+x-y-t^{\alpha}\right),\\&V(x,y,t)=V(x,y,0)+J_t^{\alpha}v_0(x,y,t)-pJ_t^{\alpha}\left(v_0(x,y,t)-U\tfrac{\partial V}{\partial x}-\tfrac{\partial V}{\partial y}-1+x+y+t^{\alpha}\right).\end{aligned} \tag{3.23}$$

Suppose the solutions of the system of equations, Eq. (3.23), *have the form as in Eq.* (3.8), *then substituting Eq.* (3.8) *into Eq.* (3.23), *and after collecting the same powers of* p, *and equating each coefficient of* p *to zero, all of that gives:*

$$p^0 \;:\; \begin{cases} U_0(x,y,t) = U(x,y,0) + J_t^\alpha u_0(x,y,t), \\ V_0(x,y,t) = V(x,y,0) + J_t^\alpha v_0(x,y,t), \end{cases}$$

$$p^1 \;:\; \begin{cases} U_1(x,t) = J_t^\alpha \left(-u_0(x,y,t) + V_0 \frac{\partial U_0}{\partial x} - \frac{\partial U_0}{\partial y} + 1 - x + y + t^\alpha\right), \\ V_1(x,t) = J_t^\alpha \left(-v_0(x,y,t) + U_0 \frac{\partial V_0}{\partial x} + \frac{\partial V_0}{\partial y} + 1 - x - y - t^\alpha\right), \end{cases}$$

$$p^2 \;:\; \begin{cases} U_1(x,y,t) = J_t^\alpha \left(V_0 \frac{\partial U_1}{\partial t} + V_1 \frac{\partial U_0}{\partial t} - \frac{\partial U_1}{\partial y}\right), \\ V_1(x,y,t) = J_t^\alpha \left(U_0 \frac{\partial V_1}{\partial x} + U_1 \frac{\partial V_0}{\partial x} + \frac{\partial V_1}{\partial y}\right), \end{cases}$$

$$p^j \;:\; \begin{cases} U_{j+1}(x,y,t) = J_t^\alpha \left(\sum_{k=0}^{j} V_k \frac{\partial U_{j-k}}{\partial t} - \frac{\partial U_j}{\partial y}\right), \\ V_{j+1}(x,y,t) = J_t^\alpha \left(\sum_{k=0}^{j} V_k \frac{\partial U_{j-k}}{\partial t} + \frac{\partial V_j}{\partial y}\right), \end{cases}$$

By assuming $\begin{cases} u_0(x,y,t) = \sum_{n=0}^{\infty} a_n(x,y) t^{\alpha k}, & U(x,y,0) = u(x,y,0), \\ v_0(x,y,t) = \sum_{n=0}^{\infty} b_n(x,y) t^{\alpha k}, & V(x,y,0) = v(x,y,0), \end{cases}$

and solving equations $U_1(x,t), V_1(x,t)$, *leads to the following results:*

$$\begin{aligned} U_1(x,y,t) &= \left(-\tfrac{1}{\Gamma(\alpha+1)} a_0(x,y) - \tfrac{1}{\Gamma(\alpha+1)}\right) t^\alpha \\ &+ \left(-\tfrac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)} a_1(x,y) + \tfrac{1}{\Gamma(2\alpha+1)} b_0(x,y) + \tfrac{x-y}{\Gamma(2\alpha+1)} a'_{0,x}(x,y) + \tfrac{1}{\Gamma(\alpha+1)}\right) t^{2\alpha} \\ &+ \cdots, \end{aligned}$$

$$\begin{aligned} V_1(x,y,t) &= \left(-\tfrac{1}{\Gamma(\alpha+1)} b_0(x,y) + \tfrac{1}{\Gamma(\alpha+1)}\right) t^\alpha \\ &+ \left(-\tfrac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)} b_1(x,y) + \tfrac{1}{\Gamma(2\alpha+1)} a_0(x,y) + \tfrac{x+y}{\Gamma(2\alpha+1)} b'_{0,x}(x,y) - \tfrac{1}{\Gamma(\alpha+1)}\right) t^{2\alpha} \\ &+ \cdots, \end{aligned}$$

By vanishing $U_1(x,y,t), V_1(x,y,t)$, *the coefficients* $a_n(x,y), b_n(x,y) (n = 1,2,3,\cdots)$, *can be determined from:*

$$\begin{aligned} &a_0(x,y) = 1, a_1(x,y) = 0, a_2(x,y) = 0, a_3(x,y) = 0, a_4(x,y) = 0, \cdots \\ &b_0(x,y) = -1, b_1(x,y) = 0, b_2(x,y) = 0, b_3(x,y) = 0, b_4(x,y) = 0, \cdots \end{aligned}$$

Therefore we get the solution of Eq. (14) *as:*

$$\begin{aligned} u(x,y,t) &= U_0(x,y,t) = x + y - 1 + \tfrac{1}{\Gamma(\alpha+1)} a_0(x,y) t^\alpha + \tfrac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)} a_1(x,y) t^{2\alpha} \\ &+ \tfrac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)} a_2(x,y) t^{3\alpha} + \tfrac{\Gamma(3\alpha+1)}{\Gamma(4\alpha+1)} a_3(x,y) t^{4\alpha} + \cdots = x + y + \tfrac{t^\alpha}{\Gamma(\alpha+1)} - 1, \end{aligned}$$

$$\begin{aligned} v(x,y,t) &= V_0(x,y,t) = x - y + 1 + \tfrac{1}{\Gamma(\alpha+1)} b_0(x,y) t^\alpha + \tfrac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)} b_1(x,y) t^{2\alpha} \\ &+ \tfrac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)} b_2(x,y) t^{3\alpha} + \tfrac{\Gamma(3\alpha+1)}{\Gamma(4\alpha+1)} b_3(x,y) t^{4\alpha} + \cdots = x - y - \tfrac{t^\alpha}{\Gamma(\alpha+1)} + 1, \end{aligned}$$

which is an exact solution.

4 Conclusion

In this manuscript, a novel algorithm for solving fractional differential equations was successfully developed and tested. The proposed method is simple and it finds exact solution to all equations using initial condition only. This method is also very powerful in finding solutions to various types of physical problems in many important practical applications. One of the other main advantages of this method is its fast convergence to the solution.

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