An Improved Approximate Solutions to Nonlinear PDEs Using The ADM and DTM

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Abstract : In this paper, we apply the Differential Transform Method (DTM) and the Adomian Decomposition Method (ADM) to three different types of nonlinear partial differential equations (PDEs) such as, General Equal Width Wave Equation (GEWE), General Regularized Long Wave Equation (GRLW), and Two-component KdV Evolutionary System of order two. The study outlines the significant features of the two methods. The results show that these methods are very efficient, convenient and can be applied to a large class of problems.

Keywords : Differential Transform Method, Adomian Decomposition Method, General Equal Width Wave Equation, General Regularized Long Wave Equation, Systems of partial differential equations, Approximate solutions.
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1 Introduction

The concept of Differential Transform Method (DTM) was introduced by Pukhov, who solved linear and non-linear initial problems in electric circuit analysis. Most of the applications that arise in mathematical physics and engineering fields can be described by partial differential equations (PDEs). In physics, for

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example, the heat flow and the wave propagation phenomena are well described by partial differential equations, see [11–21]. Partial differential equations have become a useful tool for describing these natural phenomena of science and engineering models. Therefore, it becomes increasingly important to be familiar with all traditional and recently developed methods for solving partial differential equations, and the implementation of these methods.

The Adomian decomposition method (ADM), see [1-5], proposed by George Adomian, has been applied to a wide class of linear and nonlinear PDEs, in physics, biology and chemical reactions. For nonlinear models, the method has shown reliable results in supplying analytical approximation that converges very rapidly to the exact solution. The aim of our study is to introduce the Differential Transform Method (DTM) and Adomian Decomposition Method (ADM) as an alternative to existing methods in solving different types of nonlinear PDEs such as, General Equal Width Wave Equation (GEWE), General Regularized Long Wave Equation (GRLW), and Two-component KdV Evolutionary System of order two. In sections 2 and 3, we give a brief description of the DTM and ADM. In section 4, we apply the DTM and ADM to give approximate solutions to the following: The general equal width wave equation (GEWE) in the form,

$$u_t + \epsilon u^p u_x - \nu u_{xxt} = 0, \tag{1.1}$$

where p is a positive integer, ϵ and ν are positive constants which require the boundary conditions $u \to 0$ as $x \to \pm\infty$, and x is the space variable, t is the time. The General Regularized Long Wave Equation (GRLW) in the form,

$$u_t + u_x + \epsilon u^p u_x - \nu u_{xxt} = 0. \tag{1.2}$$

and the non-linear two-component evolutionary system of a homogeneous (KdV) equations of order two in the form,

$$\begin{aligned} u_t &= -3v_{xx} \\ v_t &= u_{xx} + 4u^2 \end{aligned} \tag{1.3}$$

where the subscripts t and x denoting to the differentiation with respect to time and space respectively.

The rest of this paper is organized as follows: In Section 2, the differential transform method is introduced. Section 3 is devoted to the Adomian Decomposition method. In section 4, we compare the two methods by applying the two methods to three test problems to show the effectiveness of the DTM and ADM. Section 5 discussion and conclusion of this paper.

2   Analysis of the DTM

The basic definitions and fundamental operations of the one-dimensional differential transform are defined as follows, see [22]: Let $f(x)$ be an analytic function in the real numbers, and let x_0 be a real number. The function $f(x)$ is then represented by one series whose center is located at x_0. The differential transform of the function $f(x)$ is of the form:

$$F(k) = \frac{1}{k!} \left[\frac{d^k f(x)}{dx^k} \right]_{x=x_0}, \tag{2.1}$$

and the differential transform inverse of $F(k)$ is defined as

$$f(x) = \sum_{k=0}^{\infty} F(k)(x - x_0)^k. \tag{2.2}$$

In real applications, the function $f(x)$ is expressed by a finite series and Equation (2.2) can be written as:

$$f(x) = \sum_{k=0}^{N} F(k)(x - x_0)^k. \tag{2.3}$$

Some basic operations of the differential transformation obtained from equations (2.1) and (2.2) are given in the table below:

Table 1. Basic operations of the DTM [22–25]

Functional Form	Transformed form
f(x)=g(x)$\pm$ h(x)	F(k)=G(k)$\pm$H(k)
f(x)=α g(x)	F(k)=α G(k), where α is a constant.
f(x)=$\frac{d^n g(x)}{dx^n}$	F(k)=$\frac{(k+n)!}{k!} G(k+n)$.
f(x)=g(x).h(x)	$F(k) = \sum_{i=0}^{k} G(i).H(k-i)$.
f(x)=x^n	F(k)=δ(k-n), where δ(k-n)= $\begin{cases} 1, & \text{n=k;} \\ 0, & \text{n}\neq\text{k.} \end{cases}$
f(x)=u(x).v(x).w(x).	F(k)=$\sum_{i=0}^{k} \sum_{j=0}^{i} U(j) \, V(i-j) \, W(k-i)$
f(x)=$\int_{x_0}^{x}$ h(t)dt	F(k)=$\frac{H(k-1)}{k}$, where $k \geq 1$.

Note that from the above discussion, one can realize that the DTM is derived from the power series expansion.

Now, we illustrate the DTM by using the GEWE equation in standard form:

$$u_t + \epsilon u u_x - \nu u_{xxt} = 0, \tag{2.4}$$

subject to the initial condition

$$u(x,0) = 3sech^2\left(\frac{x}{2}\right), \tag{2.5}$$

where x is the space variable, t is the time.

Now we transform the nonlinear Equation (2.4) into nonlinear ODE by letting $\xi = x - ct$, where c is the velocity of the wave such that $u(x,t) = u(\xi)$ is the solution of Equation (2.4). Now Equation (2.4) becomes

$$-cu'(\xi) + \epsilon u(\xi)u'(\xi) + \nu cu'''(\xi) = 0, \tag{2.6}$$

subject to the initial condition

$$u(x,0) = \frac{3c}{\epsilon}sech^2\left(\frac{x}{2}\right). \tag{2.7}$$

Now integrate Equation (2.6) with respect to ξ to get

$$-cu(\xi) + \frac{1}{2}\epsilon u^2(\xi) + \nu cu''(\xi) = 0. \tag{2.8}$$

Applying the differential transform to Equation (2.8), and using Table 1, we get

$$\nu c(k+1)(k+2)U(k+2) - cU(k) + \frac{\epsilon}{2}\sum_{i=0}^{k} U(i)U(k-i) = 0, \tag{2.9}$$

which is equivalent to

$$U(k+2) = \frac{U(k)}{\nu(k+1)(k+2)} - \frac{\epsilon}{2\nu c(k+1)(k+2)}\sum_{i=0}^{k} U(i)U(k-i) = 0, \tag{2.10}$$

where $U(k)$ represent the DT of $u(\xi)$, $k \geq 0$, and $U(0)$ and $U(1)$ are:

$$U(0) = \frac{1}{0!}\left[\frac{d^0 u(x)}{dx^0}\right]_{x=0} = u(0) = \frac{3c}{\epsilon}, \tag{2.11}$$

and

$$U(1) = \frac{1}{1!}\left[\frac{du(x)}{dx}\right]_{x=0} = u'(0). \tag{2.12}$$

We set $U(1) = \beta$, and starting with $U(0)$ and $U(1)$, then $U(2)$ can be identified using Equation (2.11). By using $U(0)$, $U(1)$ and $U(2)$, $U(3)$ can be determined easily. Continuing in this manner, the N- differential transforms of $U(\xi)$ can be identified.

These differential transforms depend on the variable ξ, and the constants c and β. Now, applying the inverse transform of $U(k)$ using Equation (2.8) we get,

$$u(\xi) = \sum_{k=0}^{N} U(k)\xi^k. \tag{2.13}$$

Finally, the constants c and β will be determined using $u(x,0) = 3sech^2\left(\frac{x}{2}\right)$ for different values of x. Similarly, we can do the same thing to the GRLW and KdV systems.

3 Analysis of the ADM

It is well known that Adomian decomposition method suggests that the unknown linear function u may be represented by the decomposition series,

$$u = \sum_{n=0}^{\infty} u_n, \tag{3.1}$$

where the components u_n, $n \geq 0$ are to be determined in a recursive manner. However, the nonlinear terms $F(u)$, such as $u^2, u^3, u^4, sin(u), e^u, uu_x, u_x^2, etc.$ can be expressed by an infinite series of the so-called Adomian polynomials A_n given in the form,

$$F(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, u_2, ..., u_n), \tag{3.2}$$

where the A_n for the nonlinear term $F(u)$ can be evaluated by using the following expression,

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{i=0}^{n}\lambda^i u_i\right)\right]_{\lambda=0}, n = 0, 1, 2, ... \tag{3.3}$$

The general formula (3.3) can be simplified as follows. Assuming that the nonlinear function is F(u), therefore by using (3.3), Adomian polynomials, see [3] are given by:

$$
\begin{aligned}
A_0 &= F(u_0), \\
A_1 &= u_1 F'(u_0), \\
A_2 &= u_2 F'(u_0) + \frac{1}{2!}u_1^2 F''(u_0), \\
A_3 &= u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!}u_1^3 F'''(u_0), \\
A_4 &= u_4 F'(u_0) + (\frac{1}{2!}u_2^2 + u_1 u_3)F''(u_0) + \frac{1}{2!}u_1^2 u_2 F'''(u_0) + \frac{1}{4!}u_1^4 F^{(4)}(u_0).
\end{aligned}
\tag{3.4}
$$

Other polynomials can be generated in a similar manner. Two important observations can be made here. First, A_0 depends only on u_0, A_1 depends only on u_0 and u_1, A_2 depends only on u_0, u_1 and u_2, and so on. Second, substituting (3.4)

into (3.2) gives that

$$
\begin{aligned}
F(u) &= A_0 + A_1 + A_2 + A_3 + ... \\
&= F(u_0) + (u_1 + u_2 + u_3 + ...)F'(u_0) \\
&+ \frac{1}{2!}(u_1^2 + 2u_1u_2 + 2u_1u_3 + u_2^2 + ...)F''(u_0) + ... \\
&+ \frac{1}{3!}(u_1^3 + 3u_1^2u_2 + 3u_1^2u_3 + 6u_1u_2u_3 + ...)F'''(u_0) + ... \\
&= F(u_0) + (u - u_0)F'(u_0) + \frac{1}{2!}(u - u_0)^2F''(u_0) + ...
\end{aligned}
$$

The last expansion confirms the fact that the series in A_n polynomials is a Taylor series about a function u_0 and not about a point as is usually used. The Adomian polynomials given above in (3.4) clearly show that the sum of the subscripts of the components of u of each term of A_n is equal to n.

4   Numerical Examples

In this section we present, using the DTM and the ADM, the approximate solution for three different examples and implement the proposed method in solving these examples. Our results will be compared with the exact solutions.

4.1   Solving GEWE, GRLW, and KdV System Using DTM

First, we apply the DTM to all three test problems mentioned above.

Example 1. Consider the following nonlinear GEWE problem.

$$u_t + 0.5uu_x - u_{xxt} = 0, \tag{4.1}$$

subject to the initial condition

$$u(x,0) = 3sech^2\left(\frac{x}{2}\right), \tag{4.2}$$

where the exact solution is

$$u(x,t) = 3sech^2\left(\frac{x - 0.5t}{2}\right). \tag{4.3}$$

Now using the wave variable $\xi = x - ct$, equations (4.1–4.2) are converted to the ODE

$$cu''(\xi) - cu(\xi) + 0.25u^2(\xi) = 0, \tag{4.4}$$

subject to the initial condition

$$u(0) = 3. \tag{4.5}$$

Applying the differential transform to Equations (4.4–4.5) and by means of Table 1, we obtain the following recursive formula

$$U(k+2) = \frac{U(k)}{(k+1)(k+2)} - \frac{1}{4c(k+1)(k+2)} \sum_{i=0}^{k} U(i)U(k-i), k \geq 0. \quad (4.6)$$

and

$$U(0) = u(0) = 3, \quad U(1) = u'(0) = \beta, \quad (4.7)$$

where β is a constant to be determined later. Using $U(0)$, $U(1)$ we coded (4.6) in Mathematica, and obtain the following results:

$$U(2) = \frac{3}{2} - \frac{9}{8c}, \; U(3) = \left(\frac{\beta}{6} - \frac{\beta}{4c}\right), \; U(4) = \frac{27 + 2c\left(-27 - 2\beta^2 + 12c\right)}{192c^2}, U(5) = \frac{\beta\left(45 - 60c + 8c^2\right)}{960c^2}.$$
$$(4.8)$$

Continuing in this manner, the first 11-iterations can be identified eventually by using mathematica software. Hence, the approximate solution can be expressed as:

$$u_{appr}(x,t) = \sum_{i=0}^{10} U(i)(x - ct)^i. \quad (4.9)$$

Now, using the initial condition (4.2) and by the aid of Mathematica software, the constants c and β are

$$c = 0.515813, \qquad \beta = -0.0745867.$$

Substituting the values of c and β in Equation (4.9), the approximate solution is

$$\begin{aligned}
u_{appr}(x,t) = \; & 3 - 0.0745867(x - 0.515813t) - 0.681024(x - 0.515813t)^2 \\
& +0.023719(x - 0.515813t)^3 + 0.10806(x - 0.515813t)^4 \\
& -0.00472473(x - 0.515813t)^5 - 0.0143085(x - 0.515813t)^6 \\
& +0.000773468(x - 0.515813t)^7 + 0.00175039(x - 0.515813t)^8 \\
& -0.000112692(x - 0.515813t)^9 - 0.000203116(x - 0.515813t)^{10}.
\end{aligned}$$

Figure 1 below shows the comparison of the DTM approximate solution of order 10 and the exact solution in (4.3). It is clear from figure 1, the DTM approximation and the exact solution is in good agreement.
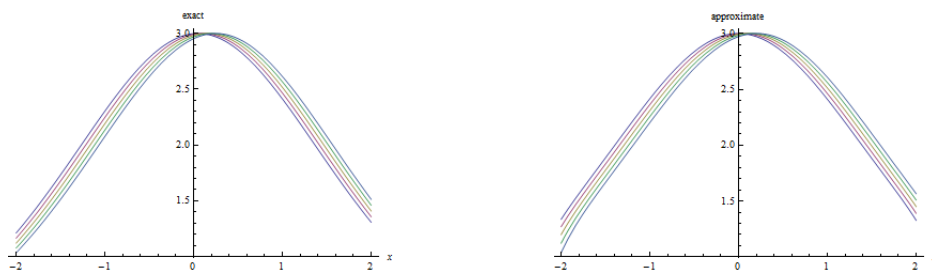
Figure 1: The exact, approximate solutions for Example 1

Example 2. Consider the following nonlinear GRLW problem

$$u_t + u_x + 0.5uu_x - u_{xxt} = 0, \qquad (4.10)$$

subject to the initial condition

$$u(x,0) = -3sec^2 \left(\frac{x}{2} \right), \qquad (4.11)$$

where the exact solution is

$$u(x,t) = -3sec^2 \left(\frac{x - 0.5t}{2} \right). \qquad (4.12)$$

Now using the wave variable $\xi = x - ct$, equations (4.10–4.11) are converted to the ODE

$$(1-c)u(\xi) + 0.25u^2(\xi) + cu''(\xi) = 0, \qquad (4.13)$$

subject to the initial condition

$$u(0) = -3. \qquad (4.14)$$

Applying the differential transform to Equations (4.13–4.14) and make a use of Table 1, we obtain the following recursive formula

$$U(k+2) = \frac{(c-1)U(k)}{c(k+1)(k+2)} - \frac{1}{4c(k+1)(k+2)} \sum_{i=0}^{k} U(i)U(k-i), \qquad (4.15)$$

and

$$U(0) = u(0) = -3, \quad U(1) = u'(0) = \beta, \qquad (4.16)$$

where β is a constant to be determined. Using $U(0)$, $U(1)$ we coded (4.15) in Mathematica, and obtain the following results:

$$U(2) = \left(-\frac{9}{8c} - \frac{3(c-1)}{2c} \right), \ U(3) = \left(\frac{\beta}{4c} + \frac{\beta(c-1)}{6c} \right), \ U(4) = \left(-\frac{\beta^2 + 6\left(\frac{9}{8c} + \frac{3(c-1)}{2c} \right)}{48c} + \frac{\left(\frac{9}{8c} + \frac{3(c-1)}{2c} \right)(1-c)}{12c} \right).$$

$$(4.17)$$

Continuing in this manner, the first 14-iterations can be identified eventually by using mathematica software. Hence, the approximate solution can be expressed as

$$u_{appr}(x,t) \quad = \quad \sum_{i=0}^{13} U(i)(x-ct)^i. \tag{4.18}$$

Now, using the initial condition (4.11), and by the aid of Mathematica software, the constants c and β are

$$c = 0.450128, \qquad \beta = -0.0637504.$$

Substituting the values of c and β in Equation (4.18), then the approximate solution is

$$\begin{aligned}
u_{appr}(x,t) \quad = \quad & -3 - 0.0637504(x - 0.450128t) - 0.666903(x - 0.450128t)^2 \\
& -0.0224273(x - 0.450128t)^3 - 0.117496(x - 0.450128t)^4 \\
& -0.00472827(x - 0.450128t)^5 - 0.0165539(x - 0.450128t)^6 \\
& -0.000831304(x - 0.450128t)^7 - 0.00218922(x - 0.450128t)^8 \\
& -0.000129954(x - 0.450128t)^9 - 0.000274757(x - 0.450128t)^{10} \\
& -0.0000188605(x - 0.450128t)^{11} - 0.0000333679(x - 0.450128t)^{12} \\
& -(2.5994582729727103)10^{-6}(x - 0.450128t)^{13}.
\end{aligned}$$

Figure 2 shows the comparison of the DTM approximate solution of order 13 and the exact solution in (4.12). It is clear from figure 2, the DTM approximation and the exact solution is in good agreement.
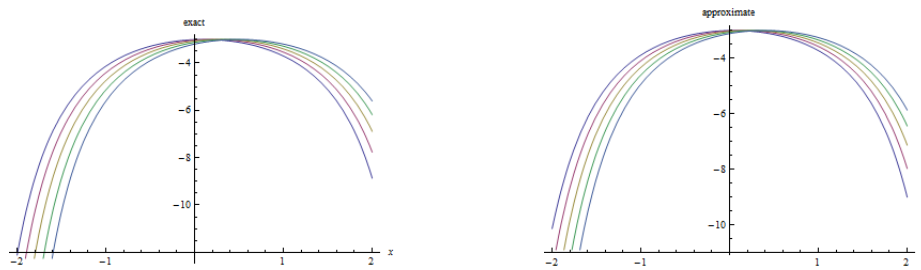

Figure 2: The exact, approximate solutions for Example 2

Example 3. We consider the non-linear two-component evolutionary system of homogeneous KdV equations of order two in the form

$$u_t = -3v_{xx}$$
$$v_t = u_{xx} + 4u^2,$$
(4.19)

subject to the initial conditions

$$u(x,0) = \frac{-3}{4(1+cos(x))}$$
$$v(x,0) = \frac{\sqrt{3}}{4} tan\left(\frac{x}{2}\right),$$
(4.20)

where the exact solutions are

$$u(x,t) = \frac{-3}{4(1+cos(x+\sqrt{3}t))}$$
$$v(x,t) = \frac{\sqrt{3}}{4} tan\left(\frac{x+\sqrt{3}t}{2}\right).$$
(4.21)

Here the subscripts t and x denoting to the differentiation with respect to time and space respectively. Thus by using $\xi = x - ct$, equations (4.19–4.20) are converted to ODE system

$$-cu'(\xi) = -3v''(\xi)$$
$$-cv'(\xi) = u''(\xi) + 4u^2(\xi).$$
(4.22)

From Equation (4.22), we have

$$v'(\xi) = \frac{c}{3}u(\xi),$$
(4.23)

and therefore,

$$c^2 u(\xi) + 12u^2(\xi) + 3u''(\xi) = 0,$$
(4.24)

subject to the initial condition

$$u(0) = -\frac{3}{8}.$$
(4.25)

Applying the differential transform to Equation (4.24–4.25), we get the recursive formula

$$U(k+2) = \frac{-c^2 U(k)}{3(k+1)(k+2)} - \frac{4}{(k+1)(k+2)} \sum_{i=0}^{k} U(i)U(k-i), k \geq 0 \qquad (4.26)$$

and

$$U(0) = u(0) = -\frac{3}{8}, \quad u(1) = U(1) = \beta,$$
(4.27)

where β is a constant to be determined. Using $U(0)$, $U(1)$ we coded (4.26) in Mathematica, and obtained the following:

$$U(2) = \left(\frac{c^2}{16} - \frac{9}{32} \right), \; U(3) = \left(\frac{\beta}{2} - \frac{\beta c^2}{18} \right), \; U(4) = \left(\frac{-81 - 384\beta^2 + 27c^2 - 2c^4}{1152} \right). \tag{4.28}$$

Continuing in this manner, the first 9-iterations can be identified eventually by using mathematica software. Hence, the approximate solutions can be expressed as

$$u(x,t) = \sum_{i=0}^{8} U(i)(x - ct)^i. \tag{4.29}$$

Now, using the initial condition (4.20) and by the aid of Mathematica software, the constants c and β are

$$c = -1.7320507758213959, \qquad \beta = (1.349735537423171)10^{-9}.$$

Substituting the values of c and β in Equation (4.29), the approximate solutions are

$$
\begin{aligned}
u_{appr}(x,t) \; = \; & -\frac{3}{8} + (1.349735537423171)10^{-9}(x + 1.73205t) - 0.09375(x + 1.73205t)^2 \\
& + (4.4991185405434545)10^{-10}(x + 1.73205t)^3 - 0.015625(x + 1.73205t)^4 \\
& + (9.560627259444544)10^{-11}(x + 1.73205t)^5 - 0.00221354(x + 1.73205t)^6 \\
& + (1.660389099082482)10^{-11}(x + 1.73205t)^7 \\
& - 0.000288318(x + 1.73205t)^8,
\end{aligned}
$$

and using Equation (4.23), we get

$$
\begin{aligned}
v_{appr}(x,t) \; = \; & 0.216506(x + 1.73205t) - (3.89635080791252)10^{-10}(x + 1.73205t)^2 \\
& + 0.0180422(x + 1.73205t)^3 - (6.493918132217263)10^{-11}(x + 1.73205t)^4 \\
& + 0.00180422(x + 1.73205t)^5 - (9.199717701144504)10^{-12}(x + 1.73205t)^6 \\
& + 0.00018257(x + 1.73205t)^7 - (1.1982825946796675)10^{-12}(x + 1.73205t)^8 \\
& + 0.0000184956(x + 1.73205t)^9.
\end{aligned}
$$

Figures 3 and 4 above shows the comparison of the DTM $u(x,t)$-approximate and $v(x,t)$-approximate solutions of order 8 and the exact solution in (4.21). It is clear from figure 3 and 4, the DTM approximation and the exact solution are in excellent agreement.
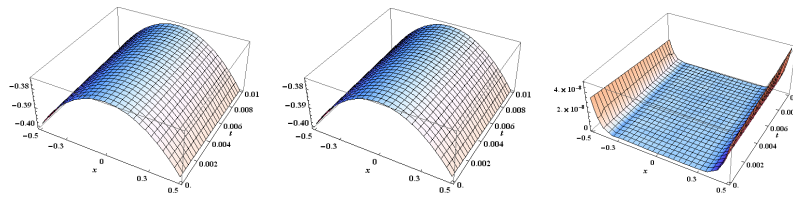
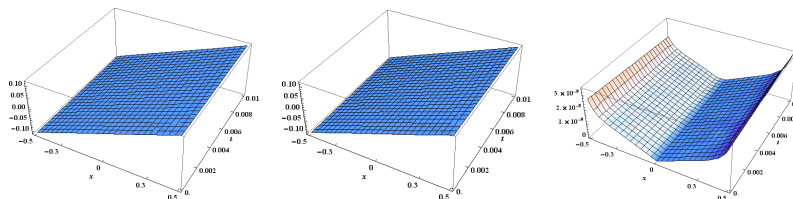Figure 3: The exact, approximate solutions and absolute error, respectively of u(x,t) for Example 3

Figure 4: The exact, approximate solutions and absolute error, respectively of v(x,t) for Example 3

4.2   Solving GEWE, GRLW, and KdV System Using ADM

In this section, we apply the ADM to the same previous examples that been considered by the DTM.

Example 1. Consider the following nonlinear GEWE problem

$$u_t + 0.5uu_x - u_{xxt} = 0, \tag{4.30}$$

subject to the initial condition

$$u(x,0) = 3sech^2\left(\frac{x}{2}\right), \tag{4.31}$$

which has the exact solution

$$u(x,t) = 3sech^2\left(\frac{x - 0.5t}{2}\right). \tag{4.32}$$

Applying the ADM, Equation (4.30) becomes

$$L_t\left(u(x,t)\right) = u_{xxt} - 0.5uu_x, \tag{4.33}$$

where L_t is defined by $L_t = \frac{\partial}{\partial t}$. Now the inverse operator L_t^{-1} is identified by

$$L_t^{-1}(.) = \int_0^t (.)\,dz. \tag{4.34}$$

Applying L_t^{-1} to both sides of (4.33) and using the initial condition we obtain

$$u(x,t) - u(x,0) = L_t^{-1} u_{xxt} - 0.5 L_t^{-1} u u_x. \tag{4.35}$$

Then

$$u(x,t) = 3sech^2 \left(\frac{x}{2} \right) + L_t^{-1} u_{xxt} - 0.5 L_t^{-1} u u_x. \tag{4.36}$$

Substituting

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t),$$

and the nonlinear term by

$$0.5 u u_x = 0.5 \sum_{n=0}^{\infty} A_n,$$

into Equation (4.36) gives

$$\sum_{n=0}^{\infty} u_n(x,t) = 3sech^2 \left(\frac{x}{2} \right) + L_t^{-1} \left(\sum_{n=0}^{\infty} (u_n(x,t))_{xxt} \right) - 0.5 L_t^{-1} \left(\sum_{n=0}^{\infty} A_n \right). \tag{4.37}$$

This gives the recursive relation

$$u_0(x,t) = 3sech^2 \left(\frac{x}{2} \right)$$

$$u_{k+1}(x,t) = L_t^{-1} \left((u_k)_{xxt} \right) - L_t^{-1} (A_k), \ \ k \geq 0. \tag{4.38}$$

The first two components are given by

$$u_0(x,t) = 3sech^2 \left(\frac{x}{2} \right)$$

$$u_1(x,t) = -L_t^{-1}(A_0) = -L_t^{-1} \left(-\frac{9}{2} sech^4 \left(\frac{x}{2} \right) tanh \left(\frac{x}{2} \right) \right), \tag{4.39}$$

where additional terms can be easily computed. The Adomian polynomials A_n for this form of nonlinearity are given by

$$A_0 = \frac{u_0 (u_0)_x}{2} = -\frac{9}{2} sech^4 \left(\frac{x}{2} \right) tanh \left(\frac{x}{2} \right)$$

$$A_1 = \frac{u_1 (u_0)_x}{2} + \frac{u_0 (u_1)_x}{2}$$

$$= -\frac{27}{4} t sech^6 \left(\frac{x}{2} \right) tanh^2 \left(\frac{x}{2} \right) + \frac{3}{2} sech^2 \left(\frac{x}{2} \right) \left(\frac{9}{4} t sech^6 \left(\frac{x}{2} \right) - 9 t sech^4 \left(\frac{x}{2} \right) tanh^2 \left(\frac{x}{2} \right) \right)$$

$$A_2 = \frac{u_2 (u_0)_x}{2} + \frac{u_1 (u_1)_x}{2} + \frac{u_0 (u_2)_x}{2}. \tag{4.40}$$

Combining the results obtained above, the approximate solution is given by

$$
\begin{aligned}
u_{appr}(x,t) \;=\;& 3\mathrm{sech}\,\left(\frac{x}{2}\right)^2 + \left(\frac{9}{2}sech\left(\frac{x}{2}\right)^4 tanh\left(\frac{x}{2}\right)\right)t + \left(\frac{27}{8}(3cosh(x)-4)sech\left(\frac{x}{2}\right)^8\right)t^2 \\
&+ \left(\frac{81}{8}(7cosh(x)-13)sech\left(\frac{x}{2}\right)^{10} tanh\left(\frac{x}{2}\right)\right)t^3.
\end{aligned}
$$

Figure 5 below shows the comparison of the ADM approximate solution of order 3 and the exact solution in (4.32). It is clear from figure 5, the ADM approximation and the exact solution are in excellent agreement.
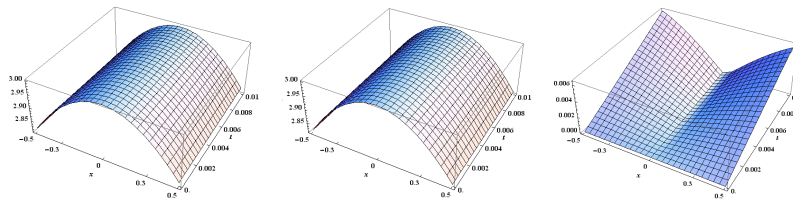

Figure 5: The exact, approximate solutions and absolute error, respectively for Example 1

Example 2. Consider the following nonlinear GRLW problem

$$
u_t + u_x + 0.5uu_x - u_{xxt} = 0, \tag{4.41}
$$

subject to the initial condition

$$
u(x,0) = -3sec^2\left(\frac{x}{2}\right), \tag{4.42}
$$

which has the exact solution

$$
u(x,t) = -3sec^2\left(\frac{x-0.5t}{2}\right). \tag{4.43}
$$

Applying the ADM, Equation (4.41) becomes

$$
L_t\left(u(x,t)\right) = u_{xxt} - 0.5uu_x - u_x, \tag{4.44}
$$

where L_t is defined by $L_t = \frac{\partial}{\partial t}$.

Applying L_t^{-1} to both sides of (4.44) and using the initial condition, we obtain

$$
u(x,t) - u(x,0) = L_t^{-1}u_{xxt} - 0.5L_t^{-1}uu_x - L_t^{-1}u_x. \tag{4.45}
$$

Then

$$u(x,t) = -3sec^2\left(\frac{x}{2}\right) + L_t^{-1}u_{xxt} - 0.5L_t^{-1}uu_x - L_t^{-1}u_x. \qquad (4.46)$$

Substituting

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t),$$

and the nonlinear term by

$$0.5uu_x = 0.5\sum_{n=0}^{\infty} A_n,$$

into Equation (4.46) gives

$$\sum_{n=0}^{\infty} u_n(x,t) = -3sec^2\left(\frac{x}{2}\right) + L_t^{-1}\left(\sum_{n=0}^{\infty}(u_n)_{xxt}\right) - L_t^{-1}\left(\sum_{n=0}^{\infty}(u_n)_x\right) - 0.5L_t^{-1}\left(\sum_{n=0}^{\infty} A_n\right).$$
$$(4.47)$$

This gives the recursive relation

$$u_0(x,t) = -3sec^2\left(\frac{x}{2}\right)$$
$$(4.48)$$
$$u_{k+1}(x,t) = L_t^{-1}\left((u_k)_{xxt}\right) - L_t^{-1}\left((u_k)_x\right) - L_t^{-1}(A_k), \ \ k \geq 0.$$

Thus, the first two components are given by

$$u_0(x,t) = -3sec^2\left(\frac{x}{2}\right)$$
$$(4.49)$$
$$u_1(x,t) = -L_t^{-1}(A_0) = -L_t^{-1}\left(\frac{9}{2}sec^4\left(\frac{x}{2}\right)tan\left(\frac{x}{2}\right)\right),$$

where additional terms can be easily computed. The Adomian polynomials A_n for this form of nonlinearity are given by

$$A_0 = \frac{u_0(u_0)_x}{2} = \frac{9}{2}sec^4\left(\frac{x}{2}\right)tan\left(\frac{x}{2}\right). \qquad (4.50)$$

Combining the results obtained above, the approximate solution is given by

$$\begin{aligned}
u_{appr}(x,t) &= -3\sec\left(\frac{x}{2}\right)^2 + 3sec\left(\frac{x}{2}\right)^2\left(\frac{2\sin(2x)}{(1+\cos(x))^2} - \frac{2\sin(x)}{(1+\cos(x))^2}\right)t \\
&+ 3\sec\left(\frac{x}{2}\right)^2\left(1 + \frac{42}{(1+cos(x))^2} - \frac{15\cos(x)}{(1+\cos(x))^2} - \frac{63\sec\left(\frac{x}{2}\right)^2}{2(1+\cos(x))^2}\right)t^2 \\
&+ 3\sec\left(\frac{x}{2}\right)^2\left(\frac{38\sin(x)}{(1+\cos(x))^2} - \frac{\cos(x)\sin(x)}{(1+\cos(x))^2}\right)t^3 \\
&- 3\sec\left(\frac{x}{2}\right)^2\left(\frac{9(59 - 42\cos(x) + 19\cos(2x))\sec\left(\frac{x}{2}\right)^4\tan\left(\frac{x}{2}\right)}{4(1+\cos(x))^2}\right)t^4.
\end{aligned}$$

Figure 6 below shows the comparison of the ADM approximate solution of order 3 and the exact solution in (4.43). It is clear from figure 6, the ADM approximation and the exact solution are in excellent agreement.
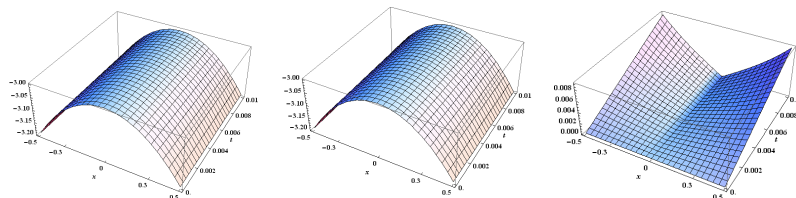

Figure 6: The exact, approximate solutions and absolute error, respectively for Example 2

Example 3. Consider the non-linear two-component evolutionary system of a homogeneous KdV equations of order two in the form

$$u_t = -3v_{xx}$$

$$v_t = u_{xx} + 4u^2,$$

(4.51)

subject to the initial conditions

$$u(x,0) = \frac{-3}{4(1+cos(x))}$$

$$v(x,0) = \frac{\sqrt{3}}{4} tan\left(\frac{x}{2}\right),$$

(4.52)

where the exact solutions are

$$u(x,t) = \frac{-3}{4(1+cos(x+\sqrt{3}t))}$$

$$v(x,t) = \frac{\sqrt{3}}{4} tan\left(\frac{x+\sqrt{3}t}{2}\right).$$

(4.53)

Applying the ADM, Equation (4.51) becomes

$$L_t\left(u(x,t)\right) + \frac{3}{4(1+cos(x))} = L_t\left(-3v_{xx}\right)$$

$$L_t\left(v(x,t)\right) - \frac{\sqrt{3}}{4} tan\left(\frac{x}{2}\right) = L_t\left(u_{xx} + 4u^2\right),$$

(4.54)

where L_t is defined by $L_t = \frac{\partial}{\partial t}$.

Applying L_t^{-1} to both sides of (4.54) and using the initial conditions we obtain that

$$u(x,t) = \frac{-3}{4(1+cos(x))} - L_t^{-1}\left(3v_{xx}\right)$$

$$v(x,t) = \frac{\sqrt{3}}{4} tan\left(\frac{x}{2}\right) + L_t^{-1}\left(u_{xx} + 4u^2\right).$$

(4.55)

Substituting

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t),$$

$$v(x,t) = \sum_{n=0}^{\infty} v_n(x,t),$$

and the nonlinear term by

$$4u^2 = 4 \sum_{n=0}^{\infty} A_n,$$

into Equation (4.55) gives

$$\sum_{n=0}^{\infty} u_n(x,t) = \frac{-3}{4(1+cos(x))} - 3L_t^{-1}\left(\sum_{n=0}^{\infty}(v_n)_{xx}\right)$$

$$\sum_{n=0}^{\infty} v_n(x,t) = \frac{\sqrt{3}}{4} tan\left(\frac{x}{2}\right) + L_t^{-1}\left(\sum_{n=0}^{\infty}(u_n(x,t))_{xx}\right) + 4L_t^{-1}\left(\sum_{n=0}^{\infty} A_n\right). \quad (4.56)$$

This gives the recursive relation

$$u_0(x,t) = \frac{-3}{4(1+cos(x))}$$

$$u_{k+1}(x,t) = -3L_t^{-1}\left((v_k)_{xx}\right), k \geq 0$$
(4.57)

and

$$v_0(x,t) = \frac{\sqrt{3}}{4} tan\left(\frac{x}{2}\right)$$

$$v_{k+1}(x,t) = L_t^{-1}\left((u_k)_{xx}\right) + 4L_t^{-1}(A_k), \; k \geq 0.$$
(4.58)

The first two components are given by

$$u_0(x,t) = \frac{-3}{4(1+cos(x))} \qquad u_1(x,t) = \frac{-3\sqrt{3}}{8} t sec^2\left(\frac{x}{2}\right) tan\left(\frac{x}{2}\right), \qquad (4.59)$$

where additional terms can be easily computed. The Adomian polynomials A_n for this form of nonlinearity are given by

$$A_0 = \frac{9}{16(1+cos(x))^2}, \qquad A_1 = \frac{9\sqrt{3}t sec^2\left(\frac{x}{2}\right) tan\left(\frac{x}{2}\right)}{16(1+cos(x))}, \qquad (4.60)$$

where additional terms can be easily computed. Accordingly, combining the results obtained above, the approximate solution of the system is given by

$$u_{appr}(x,t) = \frac{3}{35840(1+cos(x))}\left(13440t^2 - 3360t^4 + 336t^6 - 18t^8 - 8960\right)$$

$$+\frac{3sec^2\left(\frac{x}{2}\right)}{35840(1+cos(x))}\left(-20160t^2 + 25200t^4 + 1512t^6 + 4455t^8\right)$$

$$-\frac{3sec^4\left(\frac{x}{2}\right)}{35840(1+cos(x))}\left(25200t^4 + 61488t^6 + 56565t^8\right)$$

$$-\frac{3sec^8\left(\frac{x}{2}\right)}{35840(1+cos(x))}\left(122472t^6 + t^8\left(133650sec^2\left(\frac{x}{2}\right) + 292815\right)\right) + ...$$

and

$$v_{appr}(x,t) = \frac{sec^{10}\left(\frac{x}{2}\right)}{4587520}\left(470400t + 168000t^3 + 458640t^5 - 965880t^7\right)$$

$$+\frac{3tcos(x)sec^{10}\left(\frac{x}{2}\right)}{286720}\left(15680 + 4760t^2 + 4074t^4 + 25971t^6\right)$$

$$+\frac{3tcos(2x)sec^{10}\left(\frac{x}{2}\right)}{71680}\left(1960 + 280t^2 - 1302t^4 - 1347t^6\right)$$

$$+\frac{tcos(3x)sec^{10}\left(\frac{x}{2}\right)}{4587520}\left(107520 - 13440t^2 + 14112t^4 + 17136t^6\right)$$

$$+\frac{tcos(4x)sec^{10}\left(\frac{x}{2}\right)}{4587520}\left(13440 - 6720t^2 + 1008t^4 - 72t^6\right) + ...$$

Figures 7 and 8 shows the comparison of the ADM $u(x,t)$-approximate and $v(x,t)$-approximate solutions of order 8 and the exact solution in (4.53). It is clear from figure 7 and 8, the ADM approximation and the exact solution are in excellent agreement.
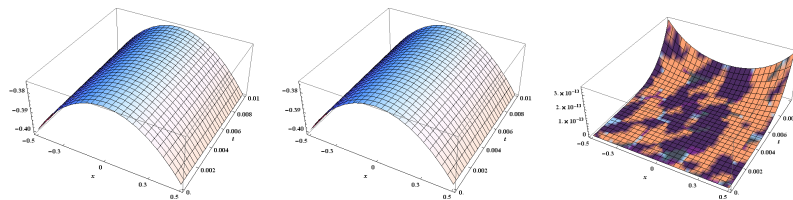

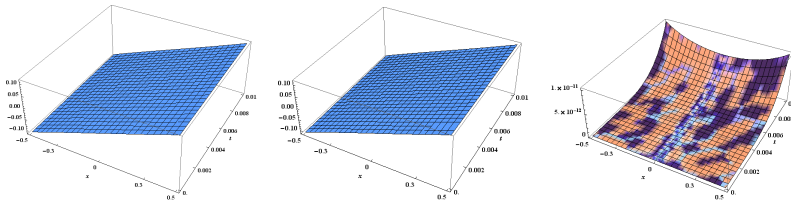Figure 7: The exact, approximate solutions and absolute error, respectively of u(x,t) for Example 3

Figure 8: The exact, approximate solutions and absolute error, respectively of v(x,t) for Example 3

5 Conclusion

In this paper, the Differential Transform Method (DTM) and the Adomian Decomposition Method (ADM) were proposed for solving the GEWE, GRLW, and the KdV system. We were be able successfully to find approximate solutions for nonlinear PDEs. The results we obtained were in excellent agreement with the exact solutions. The decomposition method introduces a significant improvement in the fields over existing techniques. Moreover, the decomposition method in away is easier, more convenient and more efficient. Also a comparative study has been conducted between the DTM and the ADM.

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