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On A Recent Characterization of Baire Class One Functions

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Abstract : Recently, a new characterization of Baire class one functions is given based on the countable decomposition of their set of points of discontinuity which improves a theorem by Henri Lebesgue. The proof makes use of the ϵ-δ characterization of Baire class one functions. In this note, we shall give another proof of this result that avoids the use of the ϵ-δ characterization. Furthermore, we use the theorem in finding applications involving the class of cliquish functions.

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1 Introduction

Let (X, d_X) and (Y, d_Y) be metric spaces. A function f from X into Y is said to be Baire class one if the inverse image of every open set U in Y under f is an F_σ set in X where an F_σ set is the union of a sequence of closed sets. Under the additional condition that Y is separable, B. Gagaeff proved the following equivalent proposition in 1932 ([1], pp. 375):

Theorem 1.1. *A function $f : X \to Y$ is Baire class one if and only if for every $\epsilon > 0$ there exists a sequence $Z_1, Z_2, \ldots$ of closed sets in X such that $X =$*

$Z_1 \cup Z_2 \cup \ldots$ *and* $\omega_f(Z_n) < \epsilon$ *for all natural numbers* n *where*

$$\omega_f(Z_n) = \sup\{d_Y(f(x), f(y)) : x, y \in Z_n\}.$$

However, it must be noted that Henri Lebesgue proved the real version of Theorem 1.1 much earlier in 1904 using an entirely different method. For this reason as well as for easy referencing, we call Theorem 1.1 the Lebesgue's Theorem.

Recently, P.Y. Lee, W.K. Tang and D. Zhao ([2]) showed that there is a characterization of Baire class one functions in terms of the usual ϵ-δ formulation as in the case of continuous functions but under the assumptions that X and Y are complete separable metric spaces. With the aid of Theorem 1.1 and the ϵ-δ characterization of Baire class one functions, J.P. Fenecios, E.A. Cabral and A.P. Racca jointly proved the following theorem in the real number line (See [3]):

Theorem 1.2. *A function* f *is Baire class one if and only if for each* $\epsilon > 0$ *there is a sequence of closed sets* $\{D_n\}$ *such that* $D_f = \bigcup_{n=1}^{\infty} D_n$ *and* $\omega_f(D_n) < \epsilon$ *for each* n *where* D_f *is the set of points of discontinuity of* f.

As we have remarked in [3], Theorem 1.2 may be viewed as an improvement of Lebesgue's theorem. In this study, we shall prove Theorem 1.2 using Theorem 1.1 alone. Although the proof is not hard and relies on no new tools we failed to see this until just recently. Quite surprisingly also, Henri Lebesgue and many others through reference [1] did not mentioned Theorem 1.2 and its proof. As much as we would like to imagine that this is a known result we are unable to find a reference. As far as the author knows, Theorem 1.2 with the proof that will be presented here is not yet existent in the literature. On the other hand, we give some applications of Theorem 1.2 involving the class of cliquish functions.

2 The Alternative Proof

Throughout, we shall assume that (X, d_X) and (Y, d_Y) are separable metric spaces. Let us denote by D_f the set of points of discontinuity of a function $f : X \to Y$ and $\omega_f(A)$ the oscillation or saltus of f in A. That is,

$$\omega_f(A) = \inf\{d_Y(f(x), f(y)) : x, y \in A\}.$$

Also, denote the open ball with center $x_0 \in X$ and radius $\delta > 0$ by $B(x_0, \delta)$, that is, $B(x_0, \delta) = \{y \in X : d_X(x_0, y) < \delta\}$.

We are now ready to give another version of the proof of Theorem 1.2.

Proof of Theorem 1.2:

Necessity. The necessity is the same as in ([3]).

Sufficiency. Observe that we may assume that X and Y are separable metric spaces. Let $\epsilon > 0$ be given. For each $x \notin D_f$ there is an open ball $B(x, \delta_x)$ with radius $\delta_x > 0$ such that

$$z \in B(x, \delta_x) \text{ implies } d_Y(f(x), f(z)) < \tfrac{\epsilon}{2}.$$

Since X is a separable metric space we can find a countable collection of open balls $\{B(\xi_i, \delta_{\xi_i})\}_{i=1}^{\infty}$ from the collection

$$\left\{B(x, \delta_x) : x \in X - D_f \ \text{ and } \ z \in B(x, \delta_x) \Longrightarrow d_Y(f(x), f(z)) < \frac{\epsilon}{2}\right\}$$

such that $X - D_f \subseteq \bigcup_{i=1}^{\infty} B(\xi_i, \delta_{\xi_i})$. By our assumption, there is a sequence of closed sets $\{D_n\}$ such that $D_f = \bigcup_{n=1}^{\infty} D_n$ and $\omega_f(D_n) < \epsilon$ for each n. Hence,

$$X = \left(\bigcup_{i=1}^{\infty} B(\xi_i, \delta_{\xi_i})\right) \cup \left(\bigcup_{n=1}^{\infty} D_n\right).$$

Let $H_1 = B(\xi_1, \delta_{\xi_1}), H_2 = D_1, H_3 = B(\xi_2, \delta_{\xi_2}), H_4 = D_2, \cdots$. Notice that for each j, we have $\omega_f(H_j) < \epsilon$ and $X = \bigcup_{j=1}^{\infty} H_j$. Observe further that each H_j is an F_σ set in X. Hence, for each j there is a sequence of closed sets $\left\{H_j^k\right\}_{k=1}^{\infty}$ in X such that $H_j = \bigcup_{k=1}^{\infty} H_j^k$. It is easy to see now that Theorem 1.1 holds and therefore f is Baire class one. □

3 Some Applications Involving the Class of Cliquish Functions

In this section, we shall give some applications of Theorem 1.2 involving the class of cliquish functions. It is well-known that Baire class one functions is a proper subset of the class of cliquish functions(See for instance [4]). We seek conditions that if being imposed on the latter class would allow us to determine the Baire class one functions among them.

Recall that f is cliquish at the point $x_0 \in X$ if for each $\epsilon > 0$ and $\delta > 0$ there exists an open set $U_{x_0}^{\delta} \subseteq B(x_0, \delta)$ not neccesarily containing x_0 such that

$$x, y \in U_{x_0}^{\delta} \Longrightarrow d_Y(f(x), f(y)) < \epsilon.$$

If f is cliquish at every point in X, we say that f is cliquish on X or we simply say f is cliquish.
Fix $\epsilon > 0$ and consider all numbers $\delta > 0$. Let

$$\mathcal{G}_\epsilon = \left\{U_x^\delta \text{ open in } X : U_x^\delta \subseteq B(x,\delta) \text{ and } \omega_f(U_x^\delta) < \epsilon\right\}$$

where x runs through all points in X for which f is cliquish there.

We are now ready to prove our first result.

Theorem 3.1. *Suppose f is cliquish. If for every $\epsilon > 0$ there is a subcollection $\mathcal{G}'_\epsilon$ of $\mathcal{G}_\epsilon$ such that $D_f \subseteq \bigcup_{G' \in \mathcal{G}'_\epsilon} G'$ then f is Baire class one.*

Proof: From Theorem 1.2 it is not hard to show that if there exists a sequence of closed sets $\{D_n\}$ in X such that $D_f \subseteq \bigcup_{n=1}^{\infty} D_n$ and $\omega_f(D_n) < \epsilon$ for each n then f is Baire class one. To complete the proof one needs only to observe that there is a countable collection $\{G'_n\}$ of $\mathcal{G}'_\epsilon$ such that $D_f \subseteq \bigcup_{n=1}^{\infty} G'_n$ and each G' in $\mathcal{G}'_\epsilon$ is an F_σ set in X. □

It must be noted that the converse of Theorem 3.1 does not hold. For instance take the function f defined by

$$f(x) = \begin{cases} 0, & x \neq 0; \\ 1, & x = 0. \end{cases}$$

Clearly, the function f is Baire class one but for any $0 < \epsilon < 1$ there is no subcollection $\mathcal{G}'_\epsilon$ of $\mathcal{G}_\epsilon$ that will cover $D_f = \{0\}$.

Next, we shall prove a convergence theorem involving a sequence of cliquish functions. The proof relies on the ϵ-δ characterization of Baire class one functions. For a detailed discussion on the ϵ-δ characterization one may consult reference [2] or [5].

Before we prove our result, let us recall first the following definition in [4]:

Definition 3.2. *The sequence $\{f_n\}_{n=1}^{\infty}$ of functions $f_n : X \to Y$ is said to be quasi-uniformly convergent to the limit function $f : X \to Y$ if for each $x \in X$ we have $\lim_{n\to\infty} f_n(x) = f(x)$ and for each $\epsilon > 0$ and each $m \in \mathbb{N} \cup \{0\}$ there exists a positive integer r such that for each $x \in X$ the inequality*

$$\min\left\{d_Y\left(f(x), f_{m+1}(x)\right), \ldots, d_Y\left(f(x), f_{m+r}(x)\right)\right\} < \epsilon$$

holds.

We are now ready to state and prove our theorem.

Theorem 3.3. *Let $\{f_n : X \to Y\}_{n=1}^{\infty}$ be a sequence of cliquish functions. Suppose the following hold:*

(i) For each n, f_n is Baire class one.
(ii) The sequence $\{f_n\}_{n=1}^{\infty}$ converges quasi-uniformly to $f : X \to Y$.
(iii) For each n and any $A \subseteq D_{f_n}$, $\omega_f(A) \leq \omega_{f_n}(A)$.

Then f is Baire class one.

Proof: We will use the ϵ-δ characterization of Baire class one functions to establish Theorem 3.3. Let $\epsilon > 0$ be given. Since the sequence $\{f_n\}_{n=1}^{\infty}$ converges quasi-uniformly to f then

$$\bigcap_{n=1}^{\infty} (X - D_{f_n}) \subset X - D_f$$

(See for instance [4, pp. 221]). It follows readily that $D_f \subset \bigcup_{n=1}^{\infty} D_{f_n}$. For each natural number n there exists a sequence $\left\{D_{f_n}^j\right\}_{j=1}^{\infty}$ of closed sets such that

$$D_{f_n} = \bigcup_{j=1}^{\infty} D_{f_n}^j \text{ and } \omega_{f_n}\left(D_{f_n}^j\right) < \epsilon \text{ for each } j.$$

Using the same argument as in the proof of Theorem 1.2, we can find a countable collection of open balls $\{B(\xi_i, \delta_{\xi_i})\}_{i=1}^{\infty}$ such that

$$X - D_f \subseteq \bigcup_{i=1}^{\infty} B(\xi_i, \delta_{\xi_i}) \quad \text{and} \quad x, y \in B(\xi_i, \delta_{\xi_i}) \Longrightarrow d_Y(f(x), f(y)) < \epsilon.$$

Hence, we can write the space X as

$$X = \left[\bigcup_{i=1}^{\infty} B(\xi_i, \delta_{\xi_i})\right] \cup \left[\bigcup_{n=1}^{\infty} \bigcup_{j=1}^{\infty} D_{f_n}^j\right].$$

One can find a sequence $\{E_k\}_{k=1}^{\infty}$ of F_σ sets in X such that $X = \bigcup_{k=1}^{\infty} E_k$, $E_s \cap E_t = \emptyset$ for $s \neq t$ and $E_k \subseteq B(\xi_i, \delta_{\xi_i})$ for some i or $E_k \subseteq D_{f_s}^t$ for some natural numbers s and t. It follows that there is a positive function $\delta : X \to \mathbb{R}^+$ such that for any $x \in E_n$ and $y \in E_m$, $m \neq n$ we have,

$$d_X(x, y) \geq \min\{\delta(x), \delta(y)\}.$$

For the justification of the previous two lines one may refer to [6] or [5]. Let us now proceed to show that f is Baire class one. Let $x, y \in X$ such that $d_X(x, y) < \min\{\delta(x), \delta(y)\}$. By the definition of the function $\delta(\cdot)$ there is a unique n such that $x, y \in E_n$. If $E_n \subseteq B(\xi_i, \delta_{\xi_i})$ for some i then clearly $d_Y(f(x), f(y)) < \epsilon$. On the other hand, if $E_n \subseteq D^t_{f_s}$ for some natural numbers s and t then by assumption (iii) we have,

$$\omega_f(E_n) \leq \omega_f\left(D^t_{f_s}\right) \leq \omega_{f_s}\left(D^t_{f_s}\right) < \epsilon.$$

Hence, f is Baire class one. □

Let us illustrate Theorem 3.3. Let $X = Y = \mathbb{R}$ and $\mathbb{Q} = \{r_n : n \in \mathbb{N}\}$ be the set of all rational numbers. Consider the sequence of cliquish functions $\{f_n : X \to Y\}$ such that for each n,

$$f_n(x) = \begin{cases} \frac{1}{m+1}, & x = r_m, \ \ m \leq n; \\ 0, & \text{otherwise.} \end{cases}$$

It easy to see that the given sequence converges to

$$f(x) = \begin{cases} \frac{1}{n+1}, & x = r_n; \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, one can verify that the hypotheses in Theorem 3.3 are all satisfied and hence f is Baire class one.

At the turn of the 19th century, several very interesting functions were introduced. For instance, the Riemann function and the Dirichlet function given by

$$f(x) = \begin{cases} \frac{1}{n}, & x = r_n; \\ 0, & \text{otherwise.} \end{cases}$$

where $\mathbb{Q} = \{r_n : n \in \mathbb{N}\}$ is an enumeration of the set of rational numbers and

$$f(x) = \begin{cases} \frac{1}{q}, & x = \frac{p}{q}, \ p, q \in \mathbb{Z} \text{ and } \gcd(p, q) = 1, \ q > 0; \\ 1, & x = 0; \\ 0, & \text{otherwise} \end{cases}$$

respectively, have interesting properties in the sense that both have dense set of points of continuity as well as dense set of points of discontinuity. It is a well-known fact that a function is cliquish if and only if it has a dense set of points of continuity. Hence, both functions are cliquish. Moreover, they belong to the first Baire class. On the other hand, it is easy to find a function which is not cliquish with dense set of points discontinuity but not Baire class one. Based on the examples above, it is tempting to conclude that every cliquish function with dense set of points of discontinuity is Baire class one. In fact, I attempted to prove it using Theorem 1.2. Alas, this conclusion does not always hold. Theorem 3.4 as well as its proof is due to an anonymous reviewer.

Theorem 3.4. *There is a cliquish function $f : X \to Y$ such that D_f is dense but not Baire class one.*

Proof: Take $X = [0,1]$ and $Y = \mathbb{R}$ under the usual metric. Let C be the Cantor set and let $X - C = \bigcup_{i=1}^{\infty} (a_i, b_i)$. Furthermore, let

$$H = \mathbb{Q} \cap \left(\bigcup_{i=1}^{\infty} (a_i, b_i) \right) = \{q_1, q_2, \ldots\}.$$

Let $f : X \to Y$ be defined as follows:

$$f(x) = \begin{cases} \frac{1}{i}, & x = q_i; \\ 0, & x \in (X - C) - \mathbb{Q}; \\ 1, & x = a_i \text{ or } x = b_i; \\ 2, & \text{otherwise.} \end{cases}$$

One can show that f is cliquish such that D_f is dense in X but f is not Baire class one. □

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