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# On the Strong and △-Convergence of S-Iteration Process for Generalized Nonexpansive Mappings on CAT(0) Space

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**Abstract :** In this paper, we give the strong and △-convergence theorems for the S-iteration process of generalized nonexpansive mappings on CAT(0) space which extend and improve many results in the literature.

**Keywords :** fixed point; CAT(0) space; generalized nonexpansive mapping; strong convergence; △-convergence; iterative process.
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## 1 Introduction

The notion of △-convergence in general metric space was introduced by Lim [1]. Kirk and Panyanak [2] specialized this concept to $CAT(0)$ space and showed that many Banach space results which involve weak convergence have precise analogs in this setting.

A metric space $X$ is a $CAT(0)$ *space* if it is geodesically connected and if every geodesic triangle in $X$ is at least as 'thin' as its comparison triangle in the Euclidean plane. It is well-known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a $CAT(0)$ space. The complex Hilbert

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ball with a hyperbolic metric is a $CAT(0)$ space (see [3]). Other examples include Pre-Hilbert spaces, R-trees (see [4]), Euclidean buildings (see [5]). For discussion of these spaces and of the fundamental role which plays in geometry, see Bridson and Haefliger [4], Burago et al. [6] and Gromov [7].

Fixed point theory in a $CAT(0)$ space has been first studied by Kirk (see [8, 9]). He showed that every nonexpansive mapping defined on a bounded closed convex subset of a complete $CAT(0)$ space always has a fixed point. Since then the fixed point theory in $CAT(0)$ space has been rapidly developed and much papers have appeared (see e.g. [2, 8–16]).

Recently, Kirk and Panyanak [2] used the concept of $\triangle$-convergence introduced by Lim [1] to prove on the $CAT(0)$ space analogs of some Banach space results which involve weak convergence. Further, Dhompongsa and Panyanak [12] obtained $\triangle$-convergence theorems for the Picard, Mann and Ishikawa iterations for nonexpansive mappings in the $CAT(0)$ space.

Agarwal et al. [17] introduced the S-iteration process in a Banach space;

$$\begin{cases} x_1 \in K \\ x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)\, x_n + \beta_n T x_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. Throughout the paper, $\mathbb{N}$ and $\mathbb{R}$ denote the set of natural numbers and the set of real numbers, respectively.

They showed that their process is independent of Mann and Ishikawa and converges faster than both of these (see [17, Proposition 3.1]).

Khan and Abbas [14] have modified S-iteration process in $CAT(0)$ space for nonexpansive mappings.

The purpose of this paper is to study the iterative scheme defined as follows:

Let $K$ be a nonempty, closed, convex subset of a complete $CAT(0)$ space $X$ and $T : K \to K$ be a generalized nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is a sequence generated iteratively by

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-a_n)Tx_n \oplus a_n T y_n \\ y_n = (1-b_n)x_n \oplus b_n T x_n, n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where and throughout the paper $\{a_n\}, \{b_n\}$ are the sequences such that $0 \leq a \leq a_n, b_n \leq b < 1$ for all $n \in \mathbb{N}$ and for some $a, b$.

In this paper, we study the S-iteration process for generalized nonexpansive mappings on the $CAT(0)$ space and generalize some results of Khan and Abbas [14]. This paper contains three section. In the Section 2, we first collect some known preliminaries and lemmas that will be used in the proofs of our main theorems. We give the main results which related to the strong and $\triangle$-convergence theorems of S-iteration process in $CAT(0)$ space, in Section 3. Under some suitable condition, we obtain the strong and $\triangle$-convergence theorems of $\{x_n\}$ to a fixed point of $T$ . It is worth mentioning that our results in $CAT(0)$ spaces can be applied to any $CAT(k)$ space with $k \leq 0$ since any $CAT(k)$ space is a $CAT(k^{'})$ space for every $k^{'} \geq k$ (see [4, p. 165]).

## 2 Preliminaries and Lemmas

Let us recall some definitons and known results in the existing literature on this concept.

Let $(X, d)$ be a metric space and $K$ its nonempty subset. Let $T : K \to K$ be a mapping. A point $x \in K$ is called a fixed point of $T$ if $Tx = x$. We will also denote by $F(T)$ the set of fixed points of $T$, that is, $F(T) = \{x \in K : Tx = x\}$. $T$ is said to be *nonexpansive* if

$$d(Tx, Ty) \leq d(x, y) \quad \text{for all} \quad x, y \in K.$$

Recently, Suzuki [18] introduced a condition on mappings, called *condition*$(C)$, which is weaker than nonexpansiveness. A mapping $T : K \to K$ is said to be quasi-nonexpansive if $F(T) \neq \emptyset$ and $d(Tx, p) \leq d(x, p)$ for all $x \in K$ and $p \in F(T)$.

**Definition 2.1.** *Let $T$ be a mapping on a subset $K$ of a metric space $(X, d)$. Then, $T : K \to K$ is said to satisfy condition$(C)$ (sometimes called generalized nonexpansive mapping) if*

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \text{ implies } d(Tx, Ty) \leq d(x, y),$$

*for all $x, y \in K$.*

**Proposition 2.2** ([18])**.** *Every nonexpansive mapping satisfies condition$(C)$.*

**Proposition 2.3** ([15])**.** *Assume that a mapping $T$ satisfies condition$(C)$ and has a fixed point. Then $T$ is quasi-nonexpansive.*

**Example 2.4.** *Define a mapping $T$ on $[0, 3]$ by*

$$T(x) = \begin{cases} 0 & \text{if } x \neq 3, \\ 1 & \text{if } x = 3. \end{cases}$$

*Then, $T$ satisfies condition$(C)$, but $T$ is not nonexpansive.*

Let $(X, d)$ be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or more briefly, a *geodesic* from $x$ to $y$) is a map $c$ from a closed interval $[0, l] \subset R$ to $X$ such that $c(0) = x, c(l) = y$, and $d(c(t), c(\acute{t})) = |t - \acute{t}|$, for all $t, t^{'} \in [0, l]$. In particular, $c$ is an isometry and $d(x, y) = l$. The image of $c$ is called a *geodesic* (or *metric*) *segment* joining $x$ and $y$. When it is unique, this geodesic is denoted by $[x, y]$. The space $(X, d)$ is said to be a *geodesic space* if every two points of $X$ are joined by a geodesic, and $X$ is said to be *uniquely geodesic* if there is exactly one geodesic joining $x$ to $y$, for each $x, y \in X$. A subset $Y \subset X$ is said to be *convex* if $Y$ includes every geodesic segment joining any two of its points.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic metric space $(X, d)$ consist of three points in $X$ (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A *comparison triangle* for geodesic triangle $\triangle(x_1, x_2, x_3)$ in $(X, d)$ is a triangle $\overline{\triangle}(x_1, x_2, x_3) = \triangle(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in the Euclidean plane $E^2$ such that $d_{E^2}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j)$, for $i, j \in \{1, 2, 3\}$.

A geodesic metric space is said to be a $CAT(0)$ space [4] if all geodesic triangles of appropriate size satisfy the following comparison axiom.

CAT(0): Let $\triangle$ be a geodesic triangle in $X$ and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then, $\triangle$ is said to satisfy the $CAT(0)$ *inequality* if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x, y) \leq d_{E^2}(\overline{x}, \overline{y}).$$

It is known that in a $CAT(0)$ space, the distance function is convex [4]. Complete $CAT(0)$ spaces are often called Hadamard spaces.

Finally, we observe that if $x, y_1, y_2$ are points of a $CAT(0)$ space and if $y_0$ is the midpoint of the segment $[y_1, y_2]$, which we will denote by $\frac{y_1 \oplus y_2}{2}$, then the $CAT(0)$ inequality implies

$$d\left(x, \frac{y_1 \oplus y_2}{2}\right)^2 \leq \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2. \tag{2.1}$$

Equality holds for the Euclidean metric. In fact (see [4, p. 163]), a geodesic metric space is a $CAT(0)$ space if and only if it satisfies inequality (2.1), (which is known as the $CN$ inequality of Bruhat and Tits [19]).

The following lemmas can be found in [12].

**Lemma 2.5.** *Let $X$ be a $CAT(0)$ space. Then*

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x, z) + td(y, z)$$

*for all $t \in [0, 1]$ and $x, y, z \in X$.*

**Lemma 2.6.** *Let $X$ be a $CAT(0)$ space. Then*

$$d((1-t)x \oplus ty, z)^2 \leq (1-t)d(x, z)^2 + td(y, z)^2 - t(1-t)d(x, y)^2,$$

*for all $t \in [0, 1]$ and $x, y, z \in X$.*

Now, we recall some definitions.

Let $X$ be a complete $CAT(0)$ space and $\{x_n\}$ be a bounded sequence in $X$. For $x \in X$, set

$$r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n).$$

The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf \ \{r(x, \{x_n\}) : x \in X\}$$

and the *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

**Definition 2.7** (See [2, Definition 3.1]). A sequence $\{x_n\}$ in a $CAT(0)$ space $X$ is said to be $\triangle-$convergent to $x \in X$ if $x$ is the unique asymptotic center of $\{u_n\}$, for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\triangle - \lim_n x_n = x$ and $x$ is called the $\triangle-$limit of $\{x_n\}$.

It is known that in a $CAT(0)$ space, $A(\{x_n\})$ consists of exactly one point [10]. Also, every $CAT(0)$ space has the *Opial* property, i.e., if $\{x_n\}$ is a sequence in $K$ and $\triangle - \lim_n x_n = x$ , then for each $y(\neq x) \in K$,

$$\limsup_{n\to\infty} \ d(x_n, x) < \limsup_{n\to\infty} \ d(x_n, y).$$

**Lemma 2.8** ([2]). *Every bounded sequence in a complete $CAT(0)$ space always has a $\triangle$-convergent subsequence.*

**Lemma 2.9** ([11]). *Let $K$ be a closed convex subset of a complete $CAT(0)$ space and $\{x_n\}$ be a bounded sequence in $K$. Then, the asymptotic center of $\{x_n\}$ is in $K$.*

**Lemma 2.10** ([15]). *Let $K$ be a closed convex subset of a complete $CAT(0)$ space $X$, and $T : K \to K$ be a generalized nonexpansive mapping. Then,*

$$d(x, Ty) \leq 3d(x, Tx) + d(x, y)$$

*for all $x, y \in K$.*

# 3 Main Results

Before proving the strong and $\triangle$-convergence theorems, we need the following lemmas.

**Lemma 3.1.** *Let $K$ be a nonempty, closed, convex subset of a complete $CAT(0)$ space $X$, $T : K \to K$ be a generalized nonexpansive mapping and $\{x_n\}$ be a sequence defined by the iteration process (1.2). If $F(T) \neq \emptyset$, then $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$.*

*Proof.* Set $y_n = (1 - b_n)x_n \oplus b_n T x_n$, $n \in \mathbb{N}$ . Since $T$ is a generalized nonexpansive mapping and $p \in F(T)$, we have $d(Ty_n, p) \leq d(y_n, p)$ and $d(Tx_n, p) \leq d(x_n, p)$ for all $n \in \mathbb{N}$. By combining these inequalities and Lemma 2.5, we get

$$\begin{aligned} d(x_{n+1}, p) &= d((1 - a_n)Tx_n \oplus a_n Ty_n, p) \\ &\leq (1 - a_n)d(Tx_n, p) + a_n d(Ty_n, p) \\ &\leq (1 - a_n)d(x_n, p) + a_n d(y_n, p). \end{aligned} \tag{3.1}$$

Also,

$$\begin{aligned} d(y_n,p) &= d((1-b_n)x_n \oplus b_nTx_n,p) \\ &\le (1-b_n)d(x_n,p)+b_nd(Tx_n,p) \\ &\le (1-b_n)d(x_n,p)+b_nd(x_n,p) \\ &= d(x_n,p). \end{aligned} \tag{3.2}$$

Using (3.1) and (3.2), we have

$$d(x_{n+1},p) \le d(x_n,p).$$

This implies $d(x_n,p)$ is decreasing and bounded below, and so $\lim_{n\to\infty} d(x_n,p)$ exists for all $p \in F(T)$. This completes the proof. □

**Lemma 3.2.** *Let $X, K, T, \{x_n\}$ satisfy the hypotheses of Lemma 3.1. Then, $F(T)$ is nonempty if and only if $\{x_n\}$ is bounded and* $\lim_{n\to\infty} d(x_n,Tx_n)=0$.

*Proof.* Suppose that $F(T)$ is nonempty and $p \in F(T)$. Then, by Lemma 3.1, $\lim_{n\to\infty} d(x_n,p)$ exists and $\{x_n\}$ is bounded. Set

$$\lim_{n\to\infty} d(x_n,p)=c \tag{3.3}$$

and $y_n=(1-b_n)x_n \oplus b_nTx_n$, for all $n \ge 1$. We first prove that $\lim_{n\to\infty} d(y_n,p)=c$. By (3.1), we have

$$d(x_{n+1},p) \le (1-a_n)\ d(x_n,p)+a_n\ d(y_n,p).$$

This gives that

$$a_n\ d(x_n,p) \le d(x_n,p)+a_n\ d(y_n,p)-d(x_{n+1},p)$$

or

$$\begin{aligned} d(x_n,p) &\le d(y_n,p)+\frac{1}{a_n}\left[d(x_n,p)-d(x_{n+1},p)\right] \\ &\le d(y_n,p)+\frac{1}{a}\left[d(x_n,p)-d(x_{n+1},p)\right]. \end{aligned}$$

This implies that

$$c \le \liminf_{n\to\infty}\ d(y_n,p). \tag{3.4}$$

By (3.2) and (3.3),

$$\limsup_{n\to\infty}\ d(y_n,p) \le c.$$

By combining this inequality and (3.4), we get

$$\lim_{n\to\infty}\ d(y_n,p)=c. \tag{3.5}$$

Next, by Lemma 2.6,

$$\begin{aligned} d(y_n,p)^2 &= d((1-b_n)x_n \oplus b_n T x_n, p)^2 \\ &\leq (1-b_n)d(x_n,p)^2 + b_n d(Tx_n,p)^2 - b_n(1-b_n)d(x_n,Tx_n)^2 \\ &\leq d(x_n,p)^2 - b_n(1-b_n)d(x_n,Tx_n)^2. \end{aligned}$$

Thus

$$b_n(1-b_n)d(x_n,Tx_n)^2 \leq d(x_n,p)^2 - d(y_n,p)^2$$

so that

$$\begin{aligned} d(x_n,Tx_n)^2 &\leq \frac{1}{b_n(1-b_n)}\left[d(x_n,p)^2 - d(y_n,p)^2\right] \\ &\leq \frac{1}{a(1-b)}\left[d(x_n,p)^2 - d(y_n,p)^2\right]. \end{aligned}$$

Using (3.3) and (3.5), we get

$$\limsup_{n\to\infty}\ d(x_n,Tx_n) \leq 0.$$

Hence,

$$\lim_{n\to\infty}\ d(x_n,Tx_n) = 0.$$

Conversely, suppose that $\{x_n\}$ is bounded and $\lim_{n\to\infty}\ d(x_n,Tx_n) = 0$. Let $A(\{x_n\}) = \{x\}$. Then, $x \in K$, by Lemma 2.9. Since $T$ is generalized nonexpansive, we have, by Lemma 2.10,

$$d(x_n,Tx) \leq 3d(x_n,Tx_n) + d(x_n,x),$$

which implies

$$\begin{aligned} \limsup_{n\to\infty}\ d(x_n,Tx) &\leq \limsup_{n\to\infty}\ [3d(x_n,Tx_n) + d(x_n,x)] \\ &= \limsup_{n\to\infty}\ d(x_n,x). \end{aligned}$$

By the uniqueness of asymptotic centers, we get $Tx = x$. Therefore, $x$ is a fixed point of $T$. This completes the proof. □

Now, we prove the $\triangle$-convergence theorem of S-iteration process in $CAT(0)$ space.

**Theorem 3.3.** *Let $X, K, T, \{x_n\}$ satisfy the hypotheses of Lemma 3.1 with $F(T) \neq \emptyset$. Then $\{x_n\}$, $\triangle$-converges to a fixed point of $T$.*

*Proof.* Lemma 3.2 guarantees that the sequence $\{x_n\}$ is bounded and

$$\lim_{n\to\infty} d\left(Tx_n, x_n\right) = 0.$$

Let $W_{\triangle}(x_n) = \cup A(\{u_n\})$, where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We claim that $W_{\triangle}(x_n) \subseteq F(T)$. Let $u \in W_{\triangle}(x_n)$. Then, there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemmas 2.8 and 2.9, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\triangle - \lim_{n\to\infty} v_n = v \in K$. Since $\lim_{n\to\infty} d(v_n, Tv_n) = 0$ and $T$ is generalized nonexpansive, then, by Lemma 2.10,

$$d(v_n, Tv) \leq 3\ d(v_n, Tv_n) + d(v_n, v).$$

By taking lim sup and using *Opial* property, we obtain $v \in F(T)$. By Lemma 3.1, $\lim_{n\to\infty} d(x_n, v)$ exists. Now, we claim that $u = v$. Assume on contrary, that $u \neq v$. Then, by the uniquess of asymptotic centers, we have

$$\begin{aligned}
\limsup_{n\to\infty}\ d(v_n, v) &< \limsup_{n\to\infty} d(v_n, u) \\
&\leq \limsup_{n\to\infty}\ d(u_n, u) \\
&< \limsup_{n\to\infty} d(u_n, v) \\
&= \limsup_{n\to\infty} d(x_n, v) \\
&= \limsup_{n\to\infty} d(v_n, v),
\end{aligned}$$

a contradiction. Thus $u = v \in F(T)$ and $W_{\triangle}(x_n) \subseteq F(T)$.

To show that $\{x_n\}$, $\triangle-$converges to a fixed point of $T$, we show that $W_{\triangle}(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$. By Lemmas 2.8 and 2.9, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\triangle-\lim_{n\to\infty} v_n = v \in K$. Let $A(\{u_n\}) = \{u\}$ and $A(\{x_n\}) = \{x\}$. We have already seen that $u = v$ and $v \in F(T)$. Finally, we claim that $x = v$. If not, then existence $\lim_{n\to\infty} d(x_n, v)$ and uniquess of asymptotic centers imply that

$$\begin{aligned}
\limsup_{n\to\infty} d(v_n, v) &< \limsup_{n\to\infty}\ d(v_n, x) \\
&\leq \limsup_{n\to\infty}\ d(x_n, x) \\
&< \limsup_{n\to\infty}\ d(x_n, v) \\
&= \limsup_{n\to\infty}\ d(v_n, v),
\end{aligned}$$

a contradiction and hence $x = v \in F(T)$. Therefore, $W_{\triangle}(x_n) = \{x\}$. □

Finally, we briefly discuss the strong convergence of S-iteration process in a $CAT(0)$ space setting.

We recall (see [20]), a mapping $T : K \to K$ is said to satisfy *Condition* $(I)$ if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that

$$d(x, Tx) \geq f(d(x, F(T))) \quad \text{for all} \quad x \in K,$$

where $d(x, F(T)) = \inf_{z\in F(T)} d(x, z)$.

**Theorem 3.4.** *Let $X, K, \{x_n\}$ satisfy the hypotheses of Lemma 3.1 and $T : K \to K$ be a generalized nonexpansive mapping satisfying Condition $(I)$ with $F(T) \neq \emptyset$. Then, $\{x_n\}$ converges strongly to a fixed point of $T$.*

*Proof.* By Lemma 3.1, $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$. Let this limit be $c$ , where $c \geq 0$. If $c = 0$, there is nothing to prove. Suppose that $c > 0$. Now, $d(x_{n+1}, p) \leq d(x_n, p)$ gives

$$\inf_{p\in F(T)} d(x_{n+1}, p) \leq \inf_{p\in F(T)} d(x_n, p),$$

which means that $d(x_{n+1}, F(T)) \leq d(x_n, F(T))$ and so $\lim_{n\to\infty} d(x_n, F(T))$ exists. Also, by Lemma 3.2, $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. It follows from Condition $(I)$ that

$$\lim_{n\to\infty} f\left(d\left(x_n, F(T)\right) \leq \lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

That is,

$$\lim_{n\to\infty} f\left(d\left(x_n, F(T)\right) = 0.$$

Since $f : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $f(0) = 0$, $f(t) > 0$ for all $t \in (0, \infty)$, we have

$$\lim_{n\to\infty} d\left(x_n, F(T)\right) = 0.$$

Next we show that $\{x_n\}$ is a Cauchy sequence in $K$. Let $\varepsilon > 0$ be arbitrarily chosen. Since $\lim_{n\to\infty} d\left(x_n, F(T)\right) = 0$, there exists a constant $n_0$ such that for all $n \geq n_0$, we have

$$d\left(x_n, F(T)\right) < \frac{\varepsilon}{4}.$$

In particular, $\inf \{d(x_{n_0}, p) : p \in F(T)\} < \frac{\varepsilon}{4}$. Thus there must exist $p^\star \in F(T)$ such that

$$d(x_{n_0}, p^\star) < \frac{\varepsilon}{2}.$$

Now, for all $m, n \geq n_0$, we have

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, p^\star) + d(x_n, p^\star) \\ &\leq 2d(x_{n_0}, p^\star) \\ &\leq 2\left(\frac{\varepsilon}{2}\right) \\ &= \varepsilon. \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in a closed subset $K$ of a complete $CAT(0)$ space $X$, therefore it must be convergent to a point in $K$. Let $\lim_{n\to\infty} x_n = q$. Now, $\lim_{n\to\infty} d\left(x_n, F(T)\right) = 0$ gives that since $T$ is quasi-nonexpansive, it is known by [21] that $F(T)$ is always closed, so $q \in F(T)$. Therefore $\{x_n\}$ converges strongly to a fixed point $q$ of $T$. This completes the proof. □

Now we give the following theorem which has different hypothesis from Theorem 3.4.

**Theorem 3.5.** *Let $X, K, T, \{x_n\}$ satisfy the hypotheses of Lemma 3.1 with $F(T) \neq \emptyset$ and $K$ be compact subset of a complete $CAT(0)$ space $X$. Then, $\{x_n\}$ converges strongly to a fixed point of $T$.*

*Proof.* Lemma 3.2 guarentees that $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(Tx_n, x_n) = 0$. Since $K$ is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to z \in K$. By Lemma 2.10, we have

$$d(x_{n_k}, Tz) \leq 3\ d(x_{n_k}, Tx_{n_k}) + d(x_{n_k}, z) \text{ for all } k \in \mathbb{N}.$$

Letting $k \to \infty$, we have $\{x_{n_k}\}$ converges to $Tz$. This implies $Tz = z$, that is $z \in F(T)$. By Lemma 3.1, we have $\lim_{n\to\infty} d(x_n, z)$ exists, thus $z$ is the strong limit of the sequence $\{x_n\}$. □

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# References

[1] T.C. Lim, Remarks on some fixed point theorems, Proc. Amer. Math. Soc. 60 (1976) 179–182.

[2] W.A. Kirk, B. Panyanak, A concept of convergence in geodesic spaces, Nonlinear Anal. 68 (2008) 3689–3696.

[3] K. Goebel, S. Reich, *Uniform Convexity, Hyperbolic Geometry and Nonexpansive Mappings*, Marcel Dekker, Inc., New York, 1984.

[4] M. Bridson, A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Springer-Verlag, Berlin, Heidelberg, 1999.

[5] K.S. Brown, *Buildings*, Springer-Verlag, New York, 1989.

[6] D. Burago, Y. Burago, S. Ivanov, *A Course in Metric Geometry*, in: Graduate Studies in Math., Vol. 33, Amer. Math. Soc., Providence, RI, 2001.

[7] M. Gromov, *Metric Structures for Riemannian and non-Riemannian Spaces*, in: Progress in Mathematics, Vol. 152, Birkhäuser, Boston, 1999.

[8] W.A. Kirk, Geodesic geometry and fixed point theory, in: Seminar of Mathematical Analysis (Malaga/Seville, 2002/2003), in: Colecc. Abierta, vol. 64, Univ. Sevilla Secr. Publ., Seville, 2003, pp. 195–225.

[9] W.A. Kirk, Geodesic geometry and fixed point theory II, in: International Conference on Fixed Point Theo. Appl., Yokohama Publ., Yokohama, 2004, pp. 113–142.

[10] S. Dhompongsa, W.A. Kirk, B. Sims, Fixed point of uniformly lipschitzian mappings, Nonlinear Anal. 65 (2006) 762–772.

[11] S. Dhompongsa, W.A. Kirk, B. Panyanak, Nonexpansive set-valued mappings in metric and Banach spaces, J. Nonlinear Convex Anal. 8 (2007) 35–45.

[12] S. Dhompongsa, B. Panyanak, On △-convergence theorems in CAT(0) spaces, Comput. Math. Appl. 56 (2008) 2572–2579.

[13] A. Razani, H. Salahifard, Invariant approximation for CAT(0) spaces, Nonlinear Anal. 72 (2010) 2421–2425.

[14] S.H. Khan, M. Abbas, Strong and △-convergence of some iterative schemes in CAT(0) spaces, Comput. Math. Appl. 61 (2011) 109–116.

[15] B. Nanjaras, B. Panyanak, W. Phuengrattana, Fixed point theorems and convergence theorems for Suzuki-generalized nonexpansive mappings in CAT(0) spaces, Nonlinear Analysis: Hybrid Systems 4 (2010) 25–31.

[16] W. Phuengrattana, Approximating fixed points of Suzuki-generalized nonexpansive mappings, Nonlinear Analysis: Hyprid Systems 5 (2011) 583–590.

[17] R.P. Agarwal, D. O'Regan, D.R. Sahu, Iterative construction of fixed points of nearly asymptotically nonexpansive mappings, J. Nonlinear Convex. Anal. 8 (1) (2007) 61–79.

[18] T. Suzuki, Fixed point theorems and convergence theorems for some generalized nonexpansive mappings, J. Math. Anal. Appl. 340 (2008) 1088–1095.

[19] F. Bruhat, J. Tits, Groupes r eductifs sur un corps local, Inst. Hautes Etudes Sci. Publ. Math. 41 (1972) 5–251.

[20] H.F. Senter, W.G. Dotson, Approximating fixed points of nonexpansive mappings, Proc. Amer. Math. Soc. 44 (1974) 375–380.

[21] P. Chaoha, A. Phon-on, A note on fixed point sets in CAT(0) spaces, J. Math. Anal. Appl. 320 (2006) 983–987.

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