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Some Limit Theorems for Independent Fuzzy Random Variables

H. Ahmadzade[†], M. Amini[†,1], S. M.ahmoud Taheri[‡], and A. Bozorgnia[†]

[†]Department of Statistics, Faculty of Mathematical Sciences
Ferdowsi University of Mashhad, Mashhad 1159-91775, Iran
e-mail : m-amini@ferdowsi.um.ac.ir (M. Amini)

[‡]Department of Basic Engineering Science
College of Engineering, University of Tehran, Tehran, Iran

Abstract : In this paper, based on two different approaches, some limit theorems are obtained for independent fuzzy random variables. Specially, as a direct extension of classical methods, we establish a strong convergence theorem for sums of independent fuzzy random variables based on the concept of variance. The main results are explained by using a couple of examples.

Keywords : fuzzy random variable; independence; law of large numbers; almost surely convergence; convergence in probability.
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1 Introduction

The concept of fuzzy random variable was developed by several authors as an extension of random sets or set valued random variables e.g. [12], [21], and [22]. Over the last years, fuzzy random variable has been extensively applied in the areas of probability and statistics and stochastic process. For some recent works on this topic, see, for example, [4, 16, 17, 20, 11]. There are many authors who have devoted their studies to limit theorems for fuzzy random variables. For the purposes of this study, we review some works on this topic. Klement et al. [13] have been established some limit theorems for independent identically distributed

[1]Corresponding author.

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fuzzy random variables based on embedding theorem as well as certain probability techniques in Banach space. Inoue [6] and Joo [7] obtained some strong laws of large numbers for independent tight fuzzy random sets. Joo et al. [8] and Joo [9] established a strong law of large numbers for stationary fuzzy random variables and weighted sums of level-wise independent fuzzy random variables, respectively. Hong and Kim [5] derived Marcinkiewcz type law of large numbers for independent fuzzy random variables. Kim [14] obtained Kolmogorov's strong law of large numbers for sums of independent and level-wise identically distributed fuzzy random variables. Also, Kim [15] derived a strong law of large numbers for random upper continuous fuzzy sets. Based on a certain distance in the Banach space, Li and Ogura [16] and Molchanov [18] established some strong laws of large numbers for independent fuzzy random variable. Fu and Zhang [4] studied strong laws of large numbers for arrays of rowwise independent compact sets and fuzzy random sets. It should be mentioned that, although the concept of variance has been found very convenient in studying limit theorems, but, as the authors know, it has not been developed the limit theorems for fuzzy random variables based on the concept of variance, except the work by Feng [2]. Based on a natural extension of the concept of variance, he extended the Kolmogorov's inequality to independent fuzzy random variables and obtained some limit theorems. His method is a direct application of classical methods in probability theory to fuzzy random variables. As an application, Parchami et al. [20] obtained a consistent confidence interval for fuzzy capability index. In this paper, using a certain metric on the space of fuzzy numbers, we state and prove some limit theorems for independent fuzzy random variables.

This paper is organized as follows. In Section 2, we consider some definitions, lemmas and theorems that needed to prove our results. A weak convergence theorem for an array of independent fuzzy random variables is obtained, in Section 3. Some strong convergence theorems for independent fuzzy random variables are investigated based on two different approaches, in Section 4. Also, we present some examples that satisfy the conditions which are needed for convergence of the sequence of independent fuzzy random variables. A brief conclusion is given in Section 5.

2 Preliminaries

In this section, we consider some elementary concepts of fuzzy set, fuzzy arithmetic and fuzzy random variables, based on [3, 19] . Suppose that $\mathbb{R}$ is the real line. Define $E = \{\tilde{u} : \mathbb{R} \to [0, 1]\}$, where $\tilde{u}$ satisfies the following arguments:

(i) $\tilde{u}$ is normal; (ii) $\tilde{u}$ is convex fuzzy set; (iii) $\tilde{u}$ is upper semicontinuous. For a $\tilde{u} \in E, [\tilde{u}]^r = \{x \in \mathbb{R} | \tilde{u}(x) \geq r, 0 < r \leq 1\}$ is called the r-level set of $\tilde{u}$.

We use the notations $\oplus$, $\ominus$ and $\odot$, and furthermore we have

i) $[\tilde{a} \oplus \tilde{b}]^r = [\tilde{a}^-(r) + \tilde{b}^-(r), \tilde{a}^+(r) + \tilde{b}^+(r)]$.

ii) If $\lambda > 0$ then $[\lambda \odot \tilde{a}]^r = [\lambda\tilde{a}^-(r), \lambda\tilde{a}^+(r)]$.

iii) If $\lambda < 0$ then $[\lambda \odot \tilde{a}]^r = [\lambda\tilde{a}^+(r), \lambda\tilde{a}^-(r)]$.

iv) $[\tilde{a} \ominus \tilde{b}]^r = [\tilde{a}^-(r) - \tilde{b}^+(r), \tilde{a}^+(r) - \tilde{b}^-(r)]$.
Let $\tilde{u}, \tilde{v} \in E$, and set

$$d_p(\tilde{u}, \tilde{v}) = (\int_0^1 h^p([\tilde{u}]^r, [\tilde{v}]^r) dr)^{\frac{1}{p}}, \quad 1 \le p < \infty,$$

$d_\infty(\tilde{u}, \tilde{v}) = sup_{0<r\le 1} h([\tilde{u}]^r, [\tilde{v}]^r)$, where h is Hausdorff metric i.e.
$h([\tilde{u}]^r, [\tilde{v}]^r) = max\{|u^-(r) - v^-(r)|, |u^+(r) - v^+(r)|\}$. Norm $||\tilde{u}||_p$ of a fuzzy number $\tilde{u} \in E$ is defined by $||\tilde{u}||_p = d_p(\tilde{u}, \tilde{0})$, where $\tilde{0}$ is the fuzzy number in E whose membership function equals 1 at 0 and zero otherwise. The norm $||.||_\infty$of $\tilde{u}$ is defined by $||\tilde{u}||_\infty = d_\infty(\tilde{u}, \tilde{0})$.
The operation $\langle .,. \rangle : E \times E \to [-\infty, \infty]$ is defined by

$$\langle \tilde{u}, \tilde{v} \rangle = \int_0^1 (\tilde{u}^-(r)\tilde{v}^-(r) + \tilde{u}^+(r)\tilde{v}^+(r)) dr.$$

If the indeterminacy of the form $\infty - \infty$ arises in the Lebesgue integral, then we say that $\langle \tilde{u}, \tilde{v} \rangle$ does not exist. It is easy to see that the operation $\langle .,. \rangle$ has the following properties:

$$\begin{aligned}
(i) &\quad \langle \tilde{u}, \tilde{u} \rangle \ge 0 \textit{ and } \langle \tilde{u}, \tilde{u} \rangle = 0 \Leftrightarrow \tilde{u} = \tilde{0}, \\
(ii) &\quad \langle \tilde{u}, \tilde{v} \rangle = \langle \tilde{v}, \tilde{u} \rangle, \\
(iii) &\quad \langle \tilde{u} + \tilde{v}, \tilde{w} \rangle = \langle \tilde{u}, \tilde{w} \rangle + \langle \tilde{v}, \tilde{w} \rangle, \\
(iv) &\quad \langle \lambda \tilde{u}, \tilde{v} \rangle = \lambda \langle \tilde{u}, \tilde{v} \rangle, \\
(v) &\quad |\langle \tilde{u}, \tilde{v} \rangle| < \sqrt{\langle \tilde{u}, \tilde{u} \rangle \langle \tilde{v}, \tilde{v} \rangle}.
\end{aligned}$$

For all $\tilde{u}, \tilde{v} \in E$, if $\langle \tilde{u}, \tilde{u} \rangle < \infty$ and $\langle \tilde{v}, \tilde{v} \rangle < \infty$, then the property (v) implies that $\langle \tilde{u}, \tilde{v} \rangle < \infty$. So, we can define

$$d_*(\tilde{u}, \tilde{v}) = \sqrt{\langle \tilde{u}, \tilde{u} \rangle - 2\langle \tilde{u}, \tilde{v} \rangle + \langle \tilde{v}, \tilde{v} \rangle}.$$

In fact, d_* is a metric in $\{\tilde{u} \in E | \langle \tilde{u}, \tilde{u} \rangle < \infty\}$.
Moreover, the norm $||\tilde{u}||_*$ of fuzzy number $\tilde{u} \in E$ is defined by $||\tilde{u}||_* = d_*(\tilde{u}, \tilde{0})$.

Let $(\Omega, \mathcal{A}, P)$ be a complete probability space. A fuzzy random variable (briefly: f.r.v.) is defined as a Borel measurable function $\tilde{X} : (\Omega, \mathcal{A}) \to (E, d_\infty)$. Let $\tilde{X}$ be a f.r.v. defined on the probability space $(\Omega, \mathcal{A}, P)$, then
$[\tilde{X}]^r = [X^-(r), X^+(r)], r \in (0, 1]$, is a random closed interval set, and $\tilde{X}^-(r)$ and $\tilde{X}^+(r)$ are real valued random variables. A f.r.v. $\tilde{X}$ is called integrably bounded if $E||\tilde{X}||_\infty < \infty$. The expectation $E\tilde{X}$ is defined as the unique fuzzy number which satisfies the property $[E\tilde{X}]^r = E[\tilde{X}]^r$, $0 < r \le 1$ [21, 25].

Definition 2.1. *([25]) Two f.r.v's $\tilde{X}$ and $\tilde{Y}$ are called independent if two σ-fields $\sigma(\tilde{X}) = \sigma(\{\tilde{X}^-(r), \tilde{X}^+(r) | r \in [0, 1]\})$ and $\sigma(\tilde{Y}) = \sigma(\{\tilde{Y}^-(r), \tilde{Y}^+(r) | r \in [0, 1]\})$ are independent.*

Definition 2.2. *A finite collection of f.r.v.'s* $\{\tilde{X}_k,\ 1 \leq k \leq n\}$ *is said to be independent if σ-fields* $\sigma(\{\tilde{X}_k^-(r), \tilde{X}_k^+(r)|r \in [0,1]\}|1 \leq k \leq n)$ *are independent. An infinite sequence* $\{\tilde{X}_n, n \geq 1\}$ *is called independent if every finite sub collection of it is independent.*

Definition 2.3. *([3]) Let* $\tilde{X}$ *and* $\tilde{Y}$ *be two f.r.v.'s in* L_2 $(L_2 = \{\tilde{X}|\tilde{X}$ *is f.r.v. and* $E||\tilde{X}||_2^2 < \infty\})$*, then*

$$Cov(\tilde{X}, \tilde{Y}) = \frac{1}{2}\int_0^1 (Cov(\tilde{X}^-(r), \tilde{Y}^-(r)) + Cov(\tilde{X}^+(r), \tilde{Y}^+(r)))dr.$$

Specially, the variance of $\tilde{X}$ *is defined by* $Var(\tilde{X}) = Cov(\tilde{X}, \tilde{X})$.

Theorem 2.4. *([3]) Let* $\tilde{X}$ *and* $\tilde{Y}$ *be two f.r.v.'s in* L_2 *and* $\tilde{u}, \tilde{v} \in E$ *and* $\lambda, k \in R$*. Then*
i) $Cov(\tilde{X}, \tilde{Y}) = \frac{1}{2}(E\langle \tilde{X}, \tilde{Y}\rangle - \langle E\tilde{X}, E\tilde{Y}\rangle)$
ii) $Var(\tilde{X}) = \frac{1}{2}Ed_*^2(\tilde{X}, E\tilde{X})$
iii) $Cov(\lambda\tilde{X} \oplus \tilde{u}, k\tilde{Y} \oplus \tilde{v}) = \lambda k Cov(\tilde{X}, \tilde{Y})$
iv) $Var(\lambda\tilde{X} \oplus u) = \lambda^2 Var(\tilde{X})$
v) $Var(\tilde{X} \oplus \tilde{Y}) = Var(\tilde{X}) + Var(\tilde{Y}) + 2Cov(\tilde{X}, \tilde{Y})$
vi) If $\tilde{X}$ *and* $\tilde{Y}$ *are independent, then* $Cov(\tilde{X}, \tilde{Y}) = 0$.

In order to establish strong and weak convergence, we need the following definitions.

Definition 2.5. *([26]) Let* $\tilde{X}$ *and* $\tilde{X}_n$ *be f.r.v.'s are defined on the same probability space* $(\Omega, \mathcal{A}, P)$*. i) We say that* $\{\tilde{X}_n\}$ *converges to* $\tilde{X}$ *in probability with respect to the metric* d_* *if, for all* $\epsilon > 0$, $lim_{n\to\infty} P(\omega : d_*(\tilde{X}_n(\omega), \tilde{X}(\omega)) > \epsilon) = 0$.
ii) We say that $\{\tilde{X}_n\}$ *converges to* $\tilde{X}$ *almost surely (briefly: a.s.) with respect to the metric* d_* *if* $P(\omega : lim_{n\to\infty} d_*(\tilde{X}_n(\omega), \tilde{X}(\omega)) = 0) = 1$.

Throughout this paper it is assumed that all of f.r.v.'s are defined on the probability space $(\Omega, \mathcal{A}, P)$.

3 A weak convergence theorem for an array of independent f.r.v.'s

In this section, based on Lemma 3.1, we establish a limit theorem for an array of independent f.r.v.'s.

Lemma 3.1. *([10]) Let* $\{\tilde{X}_k,\ 1 \leq k \leq n\}$ *be independent f.r.v.'s with* $E||\tilde{X}_k||_*^r < \infty$ *for* $k = 1, 2, ..., n$ *and* $1 \leq r \leq 2$*. Then,*

$$E|||\tilde{S}_n||_* - E||\tilde{S}_n||_*|^r \leq D_r \sum_{i=1}^n E||\tilde{X}_i||_*^r,$$

where $\tilde{S}_n = \oplus_{i=1}^n \tilde{X}_i$ *and* D_r *is a positive constant depending only on r; if* $r = 2$, *then it is possible to take* $D_2 = 4$.

The following theorem provides a weak convergence for f.r.v.'s as an extension of Theorem 1 in [23].

Theorem 3.2. *Let* $\{\tilde{X}_{n,k};\ 1 \le k \le n,\ n \ge 1\}$ *be an array of independent f.r.v.'s. Suppose that there exists a nonnegative random variable* X *with* $EX^r < \infty$ *for some* $0 < r < 1$ *such that for each n, k,* $P(||\tilde{X}_{n,k}||_* > \lambda) \le P(X > \lambda)$. *If* $\{a_{n,k};\ 1 \le k \le n,\ n \ge 1\}$ *is an array of nonnegative real numbers such that* $\sum_{k=1}^n a_{n,k}^r \le M$ *for all n and* $max_k a_{n,k} \to 0$ *as* $n \to \infty$, *then* $\tilde{S}_n = \oplus_{k=1}^n \{a_{n,k} \odot \tilde{X}_{n,k}\} \to \tilde{0}$ *in probability with respect to the metric* d_*.

Proof. Define $\tilde{Y}_{n,k} = a_{n,k} \odot \tilde{X}_{n,k} I\{||a_{n,k} \odot \tilde{X}_{n,k}||_* \le 1\}$, $\tilde{T}_n = \oplus_{k=1}^n \tilde{Y}_{n,k}$. It is easy to see that $||\tilde{S}_n||_* \le d_*(\tilde{S}_n, \tilde{T}_n) + ||\tilde{T}_n||_*$. To do this, it suffices to show that
a) $d_*(\tilde{T}_n, \tilde{S}_n) \to 0$, and b) $||\tilde{T}_n||_* \to 0$ in probability.
a) For each $\epsilon > 0$, we have

$$\begin{aligned} P(d_*(\tilde{S}_n, \tilde{T}_n) > \epsilon) &= P(\tilde{S}_n \ne \tilde{T}_n) \\ &\le \sum_{k=1}^n P(||\tilde{X}_{n,k}||_* > a_{n,k}^{-1}) \\ &\le \sum_{k=1}^n P(X > a_{n,k}^{-1}). \end{aligned}$$

By using Theorem 1 in [23], we have $\sum_{k=1}^n P(X > a_{n,k}^{-1}) \to 0$ as $n \to \infty$ and consequently $P(d_*(\tilde{S}_n, \tilde{T}_n) > \epsilon) \to 0$ as $n \to \infty$.
b) It suffices to show that $||\tilde{T}_n||_* - E||\tilde{T}_n||_* \to 0$ in probability, and $E||\tilde{T}_n||_* \to 0$. By Markov's inequality and Lemma 3.1, we can write

$$\begin{aligned} P(|||\tilde{T}_n||_* - E||\tilde{T}_n||_*| > \epsilon) &\le \frac{E|||\tilde{T}_n||_* - E||\tilde{T}_n||_*|^2}{\epsilon^2} \\ &\le \frac{\sum_{k=1}^n E||\tilde{Y}_{n,k}||_*^2}{\epsilon^2}. \end{aligned}$$

It suffices to show that $\sum_{k=1}^n E||\tilde{Y}_{n,k}||_*^2 \to 0$ as $n \to \infty$. By using integration by parts, we obtain

$$\begin{aligned} \int_{[x<T]} x^2 dP(||\tilde{X}_{n,k}||_* \le x) &\le -T^2 P(||\tilde{X}_{n,k}||_* > T) \\ &+ 2\int_0^T xP(X > x)dx \\ &\le 2\int_0^T xP(X > x)dx. \end{aligned}$$

But, for sufficiently large n,

$$\begin{aligned}\sum_{k=1}^{n} E||\tilde{Y}_{n,k}||_*^2 &\leq \sum_{k=1}^{n} Ea_{n,k}^2||\tilde{X}_{n,k}||_*^2 I_{\{||a_{n,k}\odot\tilde{X}_{n,k}||_*\leq 1\}}\\ &= \sum_{k=1}^{n} a_{n,k}^2 \int_{[x\leq a_{n,k}^{-1}]} x^2 dP(||\tilde{X}_{n,k}||_* \leq x)\\ &\leq 2\sum_{k=1}^{n} a_{n,k}^2 \int_0^{a_{n,k}^{-1}} xP(X > x)dx,\end{aligned}$$

Theorem 1 in [23] implies that $2\sum_{k=1}^{n} a_{n,k}^2 \int_0^{a_{n,k}^{-1}} xP(X > x)dx \to 0$ as $n \to \infty$. Therefore $\sum_{k=1}^{n} E||\tilde{Y}_{n,k}||_*^2 \to 0$ as $n \to \infty$. Now, we show that $E||\tilde{T}_n||_* \to 0$.
By sub additivity property of the norm $||.||_*$, i.e. $|| \oplus_{i=1}^{n} \tilde{X}_i||_* \leq \sum_{i=1}^{n} ||\tilde{X}_i||_*$, and Theorem 1 in [23], we have

$$\begin{aligned}E||\tilde{T}_n||_* &= E|| \oplus_{k=1}^{n} \tilde{Y}_{n,k}||_*\\ &\leq \sum_{k=1}^{n} a_{n,k}E||\tilde{X}_{n,k}||_* I_{\{||a_{n,k}\odot\tilde{X}_{n,k}||_*\leq 1\}}\\ &\leq \sum_{k=1}^{n} a_{n,k}^r\{a_{n,k}^{1-r} \int_0^{a_{n,k}^{-1}} xdP(||\tilde{X}_{n,k}||_* \leq x)\} \to 0.\end{aligned}$$

This completes the proof. □

Example 3.3. *Let* $\{\tilde{X}_{n,k};\ 1 \leq k \leq n,\ n \geq 1\}$ *be an array of independent f.r.v.'s with the following membership function*

$$\mu_{\tilde{X}_{n,k}(\omega)}(x) = \begin{cases} \dfrac{x - Y_{n,k}(\omega)}{Y_{n,k}(\omega)}, & Y_{n,k}(\omega) < x < 2Y_{n,k}(\omega),\\ 1, & x = 2Y_{n,k}(\omega),\\ \dfrac{3Y_{n,k}(\omega) - x}{Y_{n,k}(\omega)}, & 2Y_{n,k}(\omega) < x < 3Y_{n,k}(\omega),\\ 0, & otherwise,\end{cases}$$

where $\{Y_{n,k};\ 1 \leq k \leq n,\ n \geq 1\}$ *is an array of nonnegative independent real valued random variables. Note that* $||\tilde{X}_{n,k}||_* = \frac{\sqrt{78}}{3}Y_{n,k}$. *Now, let* X *be a nonnegative random variable with* $EX^r < \infty$ *for some* $0 < r < 1$*, such that for each n, k,* $P(\frac{\sqrt{78}}{3}Y_{n,k} > \lambda) \leq P(X > \lambda)$. *If* $\{a_{n,k};\ 1 \leq k \leq n,\ n \geq 1\}$ *is an array of nonnegative real numbers such that* $\sum_{k=1}^{n} a_{n,k}^r \leq M$ *for all n and* $max_k a_{n,k} \to 0$ *as* $n \to \infty$*, then, by Theorem 3.2,* $\tilde{S}_n = \oplus_{k=1}^{n}\{a_{n,k} \odot \tilde{X}_{n,k}\} \to \tilde{0}$ *in probability with respect to the metric* d_*.

4 Some strong convergence theorems for a sequence of independent f.r.v.'s

In this section, using two different approaches, we establish two strong convergence theorems for independent f.r.v.'s. The first approach is based on Lemma 3.1 and second one is based on the concept of variance and Lemma 4.4.

Theorem 4.1. *Let $\{\tilde{X}_n\}$ be a sequence of independent f.r.v.'s, $\{a_n\}$ a sequence of positive numbers with $a_n \uparrow \infty$, and ψ a nonnegative even function such that*

$$\frac{\psi(t)}{|t|} \uparrow, \quad \frac{\psi(t)}{t^2} \downarrow \quad as \quad |t| \uparrow . \tag{4.1}$$

If

$$\sum_{n=1}^{\infty} \sum_{i=1}^{n} E \frac{\psi(||\tilde{X}_i||_*)}{\psi(a_n)} < \infty, \tag{4.2}$$

and

$$\frac{1}{a_n} \sum_{i=1}^{n} E||\tilde{X}_i||_* \to 0, \tag{4.3}$$

then, $\oplus_{i=1}^{n} \{a_n^{-1} \odot \tilde{X}_i\}$ converges to $\tilde{0}$ a.s. with respect to the metric d_*.

Proof. For all $n \geq 1$ and $1 \leq i \leq n$, set

$$\tilde{Y}_{n,i} = \tilde{X}_i I\{||\tilde{X}_i||_* \leq a_n\} \text{ and } \tilde{Z}_{n,i} = \tilde{X}_i I\{||\tilde{X}_i||_* > a_n\}.$$

It is easy to see that $\frac{||\oplus_{i=1}^{n} \tilde{X}_i||_*}{a_n} \leq \frac{d_*(\oplus_{i=1}^{n} \tilde{Y}_{n,i}, \oplus_{i=1}^{n} \tilde{X}_i)}{a_n} + \frac{||\oplus_{i=1}^{n} \tilde{Y}_{n,i}||_*}{a_n}$. Then, it suffices to prove that $\frac{d_*(\oplus_{i=1}^{n} \tilde{Y}_{n,i}, \oplus_{i=1}^{n} \tilde{X}_i)}{a_n} \to 0$ a.s. and $\frac{||\oplus_{i=1}^{n} \tilde{Y}_{n,i}||_*}{a_n} \to 0$ a.s. By using (4.1) and (4.2), we can write

$$\begin{aligned}
\sum_{n=1}^{\infty} P\left(\frac{d_*(\oplus_{i=1}^{n} \tilde{Y}_{n,i}, \oplus_{i=1}^{n} \tilde{X}_i)}{a_n} > \epsilon\right) &\leq \sum_{n=1}^{\infty} \sum_{i=1}^{n} P(||\tilde{X}_i||_* > a_n) \\
&\leq \sum_{n=1}^{\infty} \sum_{i=1}^{n} E \frac{\psi(||\tilde{X}_i||_*)}{\psi(a_n)} I\{||\tilde{X}_i||_* > a_n\} \\
&\leq \sum_{n=1}^{\infty} \sum_{i=1}^{n} E \frac{\psi(||\tilde{X}_i||_*)}{\psi(a_n)} < \infty.
\end{aligned}$$

Now, Borel Cantelli Lemma implies that $\frac{d_*(\oplus_{i=1}^{n} \tilde{Y}_{n,i}, \oplus_{i=1}^{n} \tilde{X}_i)}{a_n} \to 0$ a.s. To show $\frac{||\oplus_{i=1}^{n} \tilde{Y}_{n,i}||_*}{a_n} \to 0$ a.s., it is sufficient to prove that $\frac{||\oplus_{i=1}^{n} \tilde{Y}_{n,i}||_* - E||\oplus_{i=1}^{n} \tilde{Y}_{n,i}||_*}{a_n} \to 0$

a.s. and $\frac{E||\oplus_{i=1}^{n}\tilde{Y}_{n,i}||_*}{a_n} \to 0$. By Lemma 3.1, and the relations (4.1) and (4.2), we have

$$\begin{aligned}\sum_{n=1}^{\infty} P(|\frac{|| \oplus_{i=1}^{n} \tilde{Y}_{n,i}||_* - E|| \oplus_{i=1}^{n} \tilde{Y}_{n,i}||_*}{a_n}| > \epsilon) &\leq \sum_{n=1}^{\infty} \frac{E||| \oplus_{i=1}^{n} \tilde{Y}_{n,i}||_* - E|| \oplus_{i=1}^{n} \tilde{Y}_{n,i}||_*|^2}{a_n^2 \epsilon^2} \\ &\leq \frac{4}{\epsilon^2} \sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{E||\tilde{Y}_{n,i}||_*^2}{a_n^2} \\ &\leq \frac{4}{\epsilon^2} \sum_{n=1}^{\infty} \sum_{i=1}^{n} E \frac{\psi(||\tilde{X}_i||_*)}{\psi(a_n)} < \infty.\end{aligned}$$

Then, Borel Cantelli Lemma implies that $\frac{||\oplus_{i=1}^{n}\tilde{Y}_{n,i}||_* - E||\oplus_{i=1}^{n}\tilde{Y}_{n,i}||_*}{a_n} \to 0$ a.s. It remains to show that $\frac{E||\oplus_{i=1}^{n}\tilde{Y}_{n,i}||_*}{a_n} \to 0$. By sub additivity of the norm $||.||_*$, and using the relations (4.1) and (4.2), we have

$$\begin{aligned}\frac{E|| \oplus_{i=1}^{n} \tilde{Z}_{n,i}||_*}{a_n} &\leq \frac{\sum_{i=1}^{n} E||\tilde{Z}_{n,i}||_*}{a_n} \\ &\leq \sum_{i=1}^{n} E \frac{\psi(||\tilde{X}_i||_*)}{\psi(a_n)} \to 0. \qquad (4.4)\end{aligned}$$

Now, the relations (4.4) and (4.3) imply that $\frac{E||\oplus_{i=1}^{n}\tilde{Y}_{n,i}||_*}{a_n} \to 0$. This completes the proof. □

By invoking the similar method, Sung [24] obtained Chung-Teicher type strong law of large numbers for a sequence Banach space valued random variables.

Remark 4.2. *The above theorem is an extension of Theorem 5.4.1 in [1] to the case of f.r.v.'s.*

Example 4.3. *Let $\tilde{u}$ be a fuzzy number with $||\tilde{u}||_* = 1$ for instance a fuzzy number with the following membership function*

$$\tilde{u}(x) = 1 - \frac{\sqrt{6}}{3}|x|, \quad -\frac{\sqrt{6}}{2} \leq x \leq \frac{\sqrt{6}}{2},$$

,$\psi(x) = |x|^p$, $1 \leq p \leq 2$ and $a_n = n^\beta$ and $\beta p > 3$ and $\beta > 1$. Let $\{\tilde{X}_n\}$ be a sequence of independent f.r.v.'s such that $P(\tilde{X}_n = n\tilde{u}) = \frac{1}{n}$, $P(\tilde{X}_n = \tilde{0}) = 1 - \frac{1}{n}$. Then, $\sum_{n=1}^{\infty}\sum_{i=1}^{n} E\frac{\psi(||\tilde{X}_i||_*)}{\psi(a_n)} < \infty$ and, therefore, by Theorem 4.1, $\oplus_{i=1}^{n}\{a_n^{-1} \odot \tilde{X}_i\}$ converges to $\tilde{0}$ a.s. with respect to the metric d_*.

To establish the next theorem, in which we employ a different approach, we need the following lemma.

Lemma 4.4. *If $\tilde{X}$ is a f.r.v., then*

$$||E\tilde{X}||_*^2 \leq E||\tilde{X}||_*^2.$$

Proof. Since $[E\tilde{X}]^r = E[\tilde{X}]^r, \forall r \in (0,1]$ [21], by applying Jensen's inequality and Fubini's theorem, we have

$$\begin{aligned} ||E\tilde{X}||_*^2 &= \int_0^1 \{(E\tilde{X}^-(r))^2 + (E\tilde{X}^+(r))^2\}dr \\ &\leq \int_0^1 \{E(\tilde{X}^-(r))^2 + E(\tilde{X}^+(r))^2\}dr \quad (\textit{since } \varphi(x) = x^2 \textit{ is a convex function}) \\ &= E\int_0^1 \{(\tilde{X}^-(r))^2 + (\tilde{X}^+(r))^2\}dr = E||\tilde{X}||_*^2. \end{aligned}$$

□

The following theorem is an extension of strong law of large numbers to independent f.r.v.'s, by using the concept of variance.

Theorem 4.5. *Let $\{\tilde{X}_n\}$ be a sequence of independent f.r.v.'s, $\{a_n\}$ a sequence of positive numbers with $a_n \uparrow \infty$, and ψ a nonnegative even function such that*

$$\frac{\psi(t)}{|t|} \uparrow, \quad \frac{\psi(t)}{t^2} \downarrow \quad as \quad |t| \uparrow. \tag{4.5}$$

If

$$\sum_{n=1}^{\infty} \sum_{i=1}^{n} E\frac{\psi(||\tilde{X}_i||_*)}{\psi(a_n)} < \infty, \tag{4.6}$$

and

$$\sum_{i=1}^{n} E\frac{\psi(||\tilde{X}_i||_*)}{\psi(a_n)} = o(n^{-1}). \tag{4.7}$$

then $\oplus_{i=1}^n \{a_n^{-1} \odot \tilde{X}_i\}$ converges to $\tilde{0}$ a.s. with respect to the metric d_*.

Proof. For all $n \geq 1$ and $1 \leq i \leq n$, set $\tilde{Y}_{n,i} = \tilde{X}_i I\{||\tilde{X}_i||_* < a_n\}$. It is easy to see that $\frac{||\oplus_{i=1}^n \tilde{X}_i||_*}{a_n} \leq \frac{d_*(\oplus_{i=1}^n \tilde{Y}_{n,i}, \oplus_{i=1}^n \tilde{X}_i)}{a_n} + \frac{||\oplus_{i=1}^n \tilde{Y}_{n,i}||_*}{a_n}$. Then, it is sufficient to show that $\frac{d_*(\oplus_{i=1}^n \tilde{Y}_{n,i}, \oplus_{i=1}^n \tilde{X}_i)}{a_n} \to 0$ and $\frac{||\oplus_{i=1}^n \tilde{Y}_{n,i}||_*}{a_n} \to 0$ a.s. By invoking (4.5) and (4.6) and using a similar way to the proof of Theorem 4.1, we have $\frac{d_*(\oplus_{i=1}^n \tilde{Y}_{n,i}, \oplus_{i=1}^n \tilde{X}_i)}{a_n} \to 0$ a.s. It remains to prove that $\frac{||\oplus_{i=1}^n \tilde{Y}_{n,i}||_*}{a_n} \to 0$ a.s. It is easy to see that $\frac{||\oplus_{i=1}^n \tilde{Y}_{n,i}||_*}{a_n} \leq \frac{d_*(\oplus_{i=1}^n \tilde{Y}_{n,i}, \oplus_{i=1}^n E\tilde{Y}_{n,i})}{a_n} + \frac{||\oplus_{i=1}^n E\tilde{Y}_{n,i}||_*}{a_n}$. It suffices to show that $\frac{d_*(\oplus_{i=1}^n \tilde{Y}_{n,i}, \oplus_{i=1}^n E\tilde{Y}_{n,i})}{a_n} \to 0$ a.s. and $\frac{||\oplus_{i=1}^n E\tilde{Y}_{n,i}||_*}{a_n} \to 0$. By Markov's inequality, (4.5) and (4.6), we have

$$\begin{aligned}\sum_{i=1}^{n} P(\frac{d_*(\oplus_{i=1}^{n}\tilde{Y}_{n,i}, \oplus_{i=1}^{n}E\tilde{Y}_{n,i})}{a_n} > \epsilon) &\leq \sum_{n=1}^{\infty}\sum_{i=1}^{n} 2\frac{Var(\tilde{Y}_{n,i})}{\epsilon^2 a_n^2}\\ &\leq \sum_{n=1}^{\infty}\sum_{i=1}^{n} \frac{2}{\epsilon^2} E\frac{\psi(||\tilde{X}_i||_*)}{\psi(a_n)} I\{||\tilde{X}_i||_* \leq a_n\}\\ &\leq \frac{2}{\epsilon^2}\sum_{n=1}^{\infty}\sum_{i=1}^{n} E\frac{\psi(||\tilde{X}_i||_*)}{\psi(a_n)} < \infty.\end{aligned}$$

Then Borel Cantelli Lemma implies that $\frac{d_*(\oplus_{i=1}^{n}\tilde{Y}_{n,i}, \oplus_{i=1}^{n}E\tilde{Y}_{n,i})}{a_n} \to 0$ a.s. Also, by sub additivity property of the norm $||.||_*$ and Lemma 4.4

$$\begin{aligned}\frac{||\oplus_{i=1}^{n} E\tilde{Y}_{n,i}||_*}{a_n} &\leq \sum_{i=1}^{n}\frac{||E\tilde{Y}_{n,i}||_*}{a_n}\\ &\leq \sum_{i=1}^{n} a_n^{-1}E^{\frac{1}{2}}||\tilde{Y}_{n,i}||_*^2.\end{aligned}$$

It suffices to show that $\sum_{i=1}^{n} a_n^{-1}E^{\frac{1}{2}}||\tilde{Y}_{n,i}||_*^2$ converges to 0. Recall the standard l_2 inequality

$$(\frac{1}{n}\sum_{i=1}^{n} x_i)^2 \leq \frac{1}{n}\sum_{i=1}^{n} x_i^2, \quad \forall x_i \in \mathbb{R}. \tag{4.8}$$

Now, using (4.5), (4.7), and (4.8), we can write

$$\begin{aligned}\{\sum_{i=1}^{n} a_n^{-1}E^{\frac{1}{2}}||\tilde{Y}_{n,i}||_*^2\}^2 &\leq n\sum_{i=1}^{n} a_n^{-2}E||\tilde{Y}_{n,i}||_*^2\\ &= n\sum_{i=1}^{n} a_n^{-2}E||\tilde{X}_i||_*^2 I\{||\tilde{X}_i||_* \leq a_n\}\\ &\leq n\sum_{i=1}^{n} E\frac{\psi(||\tilde{X}_i||_*)}{\psi(a_n)} I\{||\tilde{X}_i||_* \leq a_n\}\\ &\leq n\sum_{i=1}^{n} E\frac{\psi(||\tilde{X}_i||_*)}{\psi(a_n)} \to 0.\end{aligned}$$

This completes the proof. □

Remark 4.6. *Note that the conditions of Theorems 4.1 and 4.5 are different. The condition (3) of Theorem 4.1 is an alternative to condition (7) of Theorem 4.5. Of course, none of the two conditions do not imply another one. It is worth mentioning that the approach used in Theorem 4.5 is a novel approach, and has not been used until now.*

5 Conclusion

By using a certain metric on the space of fuzzy numbers and a definition of the variance for f.r.v.'s, some limit theorems for independent f.r.v.'s were proved. Two different approaches are studied to prove such limit theorems. The second approach, which is based on the concept of variance, is a novel approach. From a technical point of view, we can generalize the classical probabilistic results to f.r.v.'s based on the concept of variance, instead of using mathematical methods. The study of limit theorems for dependent f.r.v.'s , specially weak and strong laws of large numbers, for such random variables is a potential work for future research.

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