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Generalized Order and Generalized Type of Entire Functions of Several Complex Variables

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Abstract : In the present paper, we study the polynomial approximation of entire functions of several complex variables. The characterizations of generalized order and generalized type of entire functions of several complex variables have been obtained in terms of the approximation errors. .

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1 Introduction

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function and $M(r, f) = \max_{|z|=r} |f(z)|$ be its maximum modulus. The growth of $f(z)$ is measured in terms of its order ρ and type τ defined as under

$$\limsup_{r\to\infty} \frac{\ln\ln M(r, f)}{\ln r} \;=\; \rho, \tag{1.1}$$

$$\limsup_{r\to\infty} \frac{\ln M(r, f)}{r^{\rho}} \;=\; \tau, \tag{1.2}$$

for $0 < \rho < \infty$. Various workers have given different characterizations for entire functions of fast growth ($\rho = \infty$). M. N. Seremeta [5] defined the generalized order and generalized type with the help of general functions as follows.

Let L^o denote the class of functions h satisfying the following conditions

(i) $h(x)$ is defined on $[a, \infty)$ and is positive, strictly increasing, differentiable and tends to ∞ as $x \to \infty$,

(ii)
$$\lim_{x\to\infty} \frac{h\{(1 + 1/\psi(x))x\}}{h(x)} \;=\; 1,$$

for every function $\psi(x)$ such that $\psi(x) \to \infty$ as $x \to \infty$.

Let Λ denote the class of functions h satisfying condition (i) and

$$\lim_{x\to\infty} \frac{h(cx)}{h(x)} = 1$$

for every $c > 0$, that is, $h(x)$ is slowly increasing.

For the entire function $f(z)$ and functions $\alpha(x) \in \Lambda, \beta(x) \in L^o$, the generalized order of an entire function in terms of maximum modulus is defined as

$$\rho(\alpha,\ \beta,\ f) = \limsup_{r\to\infty} \frac{\alpha[\ln M(r,f)]}{\beta(\ln r)}. \tag{1.3}$$

Further, for $\alpha(x) \in L^o, \beta^{-1}(x) \in L^o, \gamma(x) \in L^o$, generalized type of an entire function f of finite generalized order ρ is defined as

$$\tau(\alpha,\ \beta,\ f) = \limsup_{r\to\infty} \frac{\alpha[\ln M(r,f)]}{\beta[(\gamma(r))^\rho]} \tag{1.4}$$

where $0 < \rho < \infty$ is a fixed number.

Let $f(z_1, z_2, \ldots, z_n)$ be an entire function, $z = (z_1, z_2, \ldots, z_n) \in C^n$. Let G be a full region in $\mathbb{R}^n_+$ (Positive hyper octant). Let $G_R \subset \mathbb{C}^n$ denote the region obtained from G by a similarity transformation about the origin, with ratio of similitude R. Let $d_k(G) = \sup_{z\in G} |z|^k$, where $|z|^k = |z_1|^{k_1}|z_2|^{k_2}\ldots|z_n|^{k_n}$, and let ∂G denote the boundary of the region G. Let

$$f(z) = f(z_1, z_2, \ldots, z_n) = \sum_{k_1,k_2,\ldots,k_n=0}^{\infty} a_{k_1\ldots k_n}\, z_1^{k_1} \ldots z_n^{k_n} = \sum_{\|k\|=0}^{\infty} a_k\, z^k,$$

$\|k\| = k_1 + k_2 + \cdots + k_n$, be the power series expansion of the function $f(z)$. Let $M_{f,G}(R) = \max_{z\in G_R} |f(z)|$. To characterize the growth of f, order(ρ_G) and type (σ_G) of f are defined as [2]

$$\rho_G = \limsup_{R\to\infty} \frac{\ln\ \ln M_{f,G}(R)}{\ln R},$$

$$\sigma_G = \limsup_{R\to\infty} \frac{\ln M_{f,G}(R)}{R^{\rho_G}}.$$

For the entire function, $f(z) = \sum_{\|k\|=0}^{\infty} a_k\, z^k$, A. A. Gol'dberg [3, Th .1] obtained the order and type in terms of the coefficients of its Taylor expansion as

$$\rho_G = \limsup_{\|k\|\to\infty} \frac{\|k\|\ \ln\|k\|}{-\ln|a_k|}. \tag{1.5}$$

$$(e\,\rho_G\,\sigma_G)^{1/\rho_G} = \limsup_{\|k\|\to\infty} \left\{\|k\|^{1/\rho_G}\, [|a_k|\ d_k(G)]^{1/\|k\|}\right\}, \quad (0 < \rho_G < \infty) \tag{1.6}$$

where $d_k(G) = \max_{r \in G} r^k; \quad r^k = r_1^{k_1} r_2^{k_2} \dots r_n^{k_n}$.

For an entire function of several complex variables $f(z) = \sum_{\|k\|=0}^{\infty} a_k z^k$, and functions $\alpha(x) \in \Lambda, \beta(x) \in L^o$, Seremeta [5, Th .1'] proved that

$$\rho = \limsup_{R\to\infty} \frac{\alpha[\ln M_{f,G}(R)]}{\beta(\ln R)} = \limsup_{\|k\|\to\infty} \frac{\alpha(\|k\|)}{\beta[-\frac{1}{\|k\|}\ln(|a_k|\ d_k(G))]}. \tag{1.7}$$

Further, for $\alpha(x) \in L^o, \beta^{-1}(x) \in L^o, \gamma(x) \in L^o$, Seremeta [5, Th .2'] proved that

$$\sigma = \limsup_{R\to\infty} \frac{\alpha[\ln M_{f,G}(R)]}{\beta[(\gamma(R))^{\rho}]} = \limsup_{\|k\|\to\infty} \frac{\alpha(\frac{\|k\|}{\rho})}{\beta[(\gamma\{e^{1/\rho}[|a_k|\ d_k(G)]^{-1/\|k\|}\})^{\rho}]} \tag{1.8}$$

where $0 < \rho < \infty$ is a fixed number.

For the entire function $f(z)$, we define

$$\|f\|_{L^p} = \left\{\frac{1}{A}\int\int_{z\ \in G} |f(z)|^p\ d\sigma_1 d\sigma_2 \dots d\sigma_n\right\}^{1/p} < \infty,$$

where $z = (z_1, z_2, \dots, z_n),\ d\sigma_j = dx_j\ dy_j,\ z_j = x_j + iy_j,\ j = 1, 2, ..., n.$
and A is the area of G.

Let $P_m = \left\{q : q = \sum_{\|k\|\le m} a_k\ z^k\right\}$ be the class of polynomials of degree at most m. Then we define error of an entire function f on a region G as

$$E_{\|k\|}(f) = E_{\|k\|}(f, G) = \inf\left\{\|f - q\|_{L^p} : q \in P_{\|k\|}\right\},\ p > 0.$$

To the best of our knowledge, polynomial approximation and growth characteristics for entire functions of several complex variables on a full region G in R_+^n have not been obtained so far. Similarly, the characterization of generalized order and generalized type for approximating entire functions in a region in terms of approximation errors have not been studied so far.

In this paper, we have made an attempt to bridge this gap. We have characterized the generalized order and generalized type of the entire functions of several complex variables in terms of the approximation errors.

Before proving main results we state a lemma.

Lemma 1.1 *Let* $H(z) = \sum_{\|k\|=m} a_k\ z^k$ *be a polynomial of degree m, where* $\|k\| = m = k_1 + k_2 + ... + k_n$. *Let* $M_{H,G}(1) = \max_{z\ \in\ G} |H(z)|$. *Then*

$$1 \le M_{H,G}(1) \max_{\|k\|\ =\ m} \{|a_k|\ d_k(G)\} \le (m+1)^n.$$

The result can be obtained on the lines similar to those used by A. A. Gol'dberg [3].

2 Main Results

First we prove the following

Theorem 2.1 *Let $\alpha(x) \in L^o$, and $\beta(x) \in \Lambda$. Set $F(x;c) = \beta^{-1}[c\ \alpha(x)]$. If $dF(x;c)/d\ln x = O(1)$ as $x \to \infty$ for all c, $0 < c < \infty$, then*

$$\limsup_{R \to \infty} \frac{\alpha[\ln M_{f,G}(R)]}{\beta\ (\ln R)} = \limsup_{\|k\| \to \infty} \frac{\alpha(\|k\|)}{\beta(-\frac{1}{\|k\|}\ln(E_{\|k\|}(f)d_k(G)))}.$$

Proof. Let $f(z) = \sum_{\|k\|=0}^{\infty} a_k\ z^k$ be an entire function and $q = \sum_{\|k\| \le m} a_k\ z^k$ be the partial sum of f. Therefore from the definition of error, we have

$$E_{\|k\|}(f) \le \|f - q\|_{L^p} = \| \sum_{\|j\|=\|k\|+1}^{\infty} a_j\ z^j \|_{L^p} \le \sum_{\|j\|=\|k\|+1}^{\infty} |a_j|\ r^j, \tag{2.1}$$

where r is a fixed number and $r \in (1, \infty)$. From (1.7), we have

$$|a_k| d_k(G) \le e^{-\|k\|\ F(\|k\|;\frac{1}{\rho})}.$$

By using above inequality and (2.1), we get

$$\begin{aligned} E_{\|k\|}(f) &\le \frac{1}{d_k(G)} \sum_{\|j\|=\|k\|+1}^{\infty} e^{-\|j\|\ F(\|j\|;\frac{1}{\rho})}\ |z|^j \\ &\le \frac{1}{d_k(G)}\ e^{-(\|k\|+1)\ F(\|k\|+1;\frac{1}{\rho})}\ r^{\|k\|+1} \left[1 - \frac{r}{e^{F(\|k\|+1;\frac{1}{\rho})}}\right]^{-1}. \end{aligned}$$

By setting $r = 1 + \frac{1}{\|k\|}$ in the above inequality, we get

$$E_{\|k\|}(f)\ d_k(G) \le e^{-(\|k\|+1)\ F((\|k\|+1);\frac{1}{\rho})}\ (1 + \frac{1}{\|k\|})^{(\|k\|+1)} \left[1 - \frac{(1+\frac{1}{\|k\|})}{e^{(\|k\|+1)\ F((\|k\|+1);\frac{1}{\rho})}}\right]^{-1}.$$

As $\|k\| \to \infty$, the above inequality becomes

$$\begin{aligned} \frac{1}{E_{\|k\|}(f)\ d_k(G)} &\ge O(1)\ e^{(\|k\|+1)\ F((\|k\|+1);\frac{1}{\rho})-1}. \\ -\ln E_{\|k\|}(f)\ d_k(G) &\ge (\|k\|+1)\ F((\|k\|+1);\frac{1}{\rho}) + O(1) \\ &\ge (\|k\|+1)\ \beta^{-1}\left[\frac{1}{\rho}\ \alpha(\|k\|+1)\right] + O(1). \end{aligned}$$

Now proceeding to limits and since $\alpha(x) \in L^o$, and $\beta(x) \in \Lambda$, we obtain

$$\limsup_{\|k\| \to \infty} \frac{\alpha(\|k\|)}{\beta(-\frac{1}{\|k\|}\ln(E_{\|k\|}(f)d_k(G)))} \le \rho. \tag{2.2}$$

Conversely, let

$$\limsup_{\|k\| \to \infty} \frac{\alpha(\|k\|)}{\beta(-\frac{1}{\|k\|} \ln(E_{\|k\|}(f)\, d_k(G)))} = \eta.$$

Suppose $\eta < \infty$. Then for any $\epsilon > 0$ there exists N' such that for all k with $\|k\| = m > N'$, we have

$$E_{\|k\|}(f)\, d_k(G) \leq \exp\{-\|k\| F(\|k\|\,;\, 1/\overline{\eta})\} \tag{2.3}$$

where $\overline{\eta} = \eta + \epsilon$. The inequality

$$\sqrt[\|k\|]{R^{\|k\|}\, E_{\|k\|}\, d_k(G)} \leq R\, e^{-F(\|k\|\,;\,1/\overline{\eta})} \leq \frac{1}{2} \tag{2.4}$$

is fulfilled beginning with some $\|k\| = m = m(R) = E[\alpha^{-1}[\overline{\eta}\, \beta(\ln R + \ln 2)]]$, where $E[Q]$ denotes the integer part of Q. Then

$$\sum_{\|k\| = m(R)+1}^{\infty} E_{\|k\|}(f)\, d_k(G)\, R^{\|k\|} \leq \sum_{\|k\| = m(R)+1}^{\infty} \frac{1}{2^{\|k\|}} \leq 1. \tag{2.5}$$

Now

$$M_{f,G}(R) \leq \sum_{\|k\| = 0}^{\infty} E_{\|k\|}(f)\, d_k(G)\, R^{\|k\|} = \sum_{\|k\| = 0}^{m_0} E_{\|k\|}(f)\, d_k(G)\, R^{\|k\|} + \sum_{\|k\| = m_0+1}^{m_1(R)} E_{\|k\|}(f)\, d_k(G)\, R^{\|k\|} + \sum_{\|k\| = m_1(R)+1}^{\infty} E_{\|k\|}(f)\, d_k(G)\, R^{\|k\|}. \tag{2.6}$$

By applying the lemma and (2.5), the above inequality becomes

$$M_{f,G}(R) \leq (1 + \|k\|)^n + m_1(R) \max_{m_0 \leq \|k\| \leq m_1(R)} (E_k\, d_k(G) R^{\|k\|}) + \sum 2^{-\|k\|}.$$

From (2.4), we have

$$2\,R \leq \exp\{F(\|k\|\,;\, 1/\overline{\eta})\}.$$

Now, we express k in terms of R. Thus

$$\ln 2 + \ln R \leq F(\|k\|\,;\, 1/\overline{\eta}) = \beta^{-1}[\frac{1}{\overline{\eta}}\, \alpha(\|k\|)]$$

where $m_1(R) = m(R) + 1$, and $m_0 = \max\{N', m_1(R)\}$. Now

$$\max_{m_0 \leq \|k\| \leq m_1(R)} (E_{\|k\|}(f)\, R^{\|k\|}) \leq \max_{m_0 \leq \|k\| \leq m_1(R)} \psi(\|k\|) \leq \exp\{A\, \alpha^{-1}[\overline{\eta}\, \beta(\ln R + A)]\}$$

where $\psi(\|k\|) \;=\; R^{\|k\|}\;\exp\{-\|k\|F(\|k\|\;;\;1/\overline{\eta})\}$. From (2.6), we have

$$M_{f,G}(R)(1+o(1)) \;\leq\; \exp\{(A+o(1))\;\alpha^{-1}[\overline{\eta}\;\beta(\ln R\;+\;A)]\}.$$

Then we have

$$\frac{\alpha[(A+o(1))^{-1}\ln M_{f,G}(R)]}{\beta(\ln R\;+\;A)} \;\leq\; \overline{\eta} \;=\; \eta+\epsilon.$$

Now proceeding to limits and using the properties of $\alpha(x)$ and $\beta(x)$, since ϵ is arbitrary, we obtain

$$\rho \;=\; \limsup_{R\,\rightarrow\,\infty}\frac{\alpha(\ln M_{f,G}(R))}{\beta(\ln R)} \;\leq\; \eta. \tag{2.7}$$

From (2.2) and (2.7), we obtain the required result. □

Now we prove

Theorem 2.2 *Let $\alpha(x), \beta^{-1}(x)$ and $\gamma(x) \in L^0$. Let ρ be a fixed number, $0<\rho<\infty$. Set $F(x;\;\sigma,\;\rho) = \gamma^{-1}\{[\beta^{-1}(\sigma\;\alpha(x))]^{1/\rho}\}$. Suppose that for all $\sigma, 0<\sigma<\infty$, F satisfies*

$$d\;\ln F(x\sigma,\rho)/d\ln x = O(1) \quad asx \rightarrow \infty;$$

then the following equation holds:

$$\limsup_{R\rightarrow\infty}\frac{\alpha(\ln M_{f,G}(R))}{\beta[(\gamma(R))^{\rho}]} = \limsup_{\|k\|\rightarrow\infty}\frac{\alpha(\frac{\|k\|}{\rho})}{\beta\{[\gamma(e^{1/\rho}[E_{\|k\|}(f)\;d_k(G)]^{-1/\|k\|})]^{\rho}\}}.$$

Proof. From (1.8), we have

$$|a_k|\;d_k(G) \;\leq\; e^{\|k\|/\rho}\;\left[F(\frac{\|k\|}{\rho}\;;\;\frac{1}{\sigma}\;,\;\rho)\right]^{-\|k\|}.$$

By using above inequality and (2.1), we get

$$E_{\|k\|}(f) \;\leq\; \frac{1}{d_k(G)}\;\left(\frac{r\;e^{1/\rho}}{[F(\frac{\|k\|+1}{\rho}\;;\;\frac{1}{\sigma}\;,\;\rho)]}\right)^{(\|k\|+1)}\;\left[1-\frac{r\;e^{1/\rho}}{[F(\frac{\|k\|+1}{\rho}\;;\;\frac{1}{\sigma}\;,\;\rho)]}\right]^{-1}.$$

By setting $r = 1+\frac{1}{\|k\|}$ in the above inequality, we get

$$E_{\|k\|}(f)\;d_k(G) \;\leq\; \left(\frac{(1+\frac{1}{\|k\|})\;e^{1/\rho}}{[F(\frac{\|k\|+1}{\rho}\;;\;\frac{1}{\sigma}\;,\;\rho)]}\right)^{(\|k\|+1)}\;\left[1-\frac{(1+\frac{1}{\|k\|})\;e^{1/\rho}}{[F(\frac{\|k\|+1}{\rho}\;;\;\frac{1}{\sigma}\;,\;\rho)]}\right]^{-1}.$$

As $\|k\| \to \infty$, the above inequality becomes

$$e^{1/\rho}\,[E_{\|k\|}(f)\,d_k(G)]^{-\frac{1}{\|k\|}} \;\geq\; O(1)\,\gamma^{-1}\left\{\left[\beta^{-1}\left(\frac{1}{\overline{\sigma}}\,\alpha\left(\frac{\|k\|}{\rho}\right)\right)\right]^{1/\rho}\right\}.$$

Now proceeding to limits and since $\alpha(x),\ \beta^{-1}(x),\ \gamma(x) \in L^0$, we obtain

$$\limsup_{\|k\| \to \infty} \frac{\alpha(\frac{\|k\|}{\rho})}{\beta\{[\gamma(e^{1/\rho}\,\{E_{\|k\|}(f)\,d_k(G)\}^{-1/\|k\|})]^{\rho}\}} \;\leq\; \sigma. \tag{2.8}$$

Conversely, let

$$\limsup_{\|k\| \to \infty} \frac{\alpha(\frac{\|k\|}{\rho})}{\beta\{[\gamma(e^{1/\rho}[E_{\|k\|}(f)\,d_k(G)]^{-1/\|k\|})]^{\rho}\}} \;=\; \tau.$$

Suppose $\tau \;<\; \infty$. Then for every $\epsilon \;>\; 0$ there exists M' such that for all k with $\|k\| \;= m \;\geq\; M'$, we have

$$E_{\|k\|}(f)\,d_k(G) \;\leq\; \frac{\exp\,(\frac{\|k\|}{\rho})}{[F(\|k\|/\rho\,;1/\overline{\tau}\,,\rho)]^{\|k\|}}$$

where $\overline{\tau} \;=\; \tau+\epsilon$. The inequality

$$\sqrt[\|k\|]{E_{\|k\|}(f)\,d_k(G)\,R^{\|k\|}} \;\leq\; \frac{e^{1/\rho}\,R}{F(\|k\|/\rho\,;1/\overline{\tau}\,,\rho)} \;\leq\; \frac{1}{2} \tag{2.9}$$

is fulfilled for all $\|k\|$ beginning with some

$$\|k\| \;= m \;=\; m(R) \;=\; E[\rho\,\alpha^{-1}\{\overline{\tau}\,\beta\,[(\gamma\,(2e^{1/\rho}\,R))^{\rho}]\}].$$

Then

$$\sum_{\|k\|=m(R)+1}^{\infty} E_{\|k\|}(f)\,d_k(G)\,R^{\|k\|} \;\leq\; \sum_{\|k\|=m(R)+1}^{\infty} \frac{1}{2^{\|k\|}} \;\leq\; 1. \tag{2.10}$$

Hence

$$M_{f,G}(R) \;\leq\; \sum_{\|k\|\,=\,0}^{\infty} E_{\|k\|}\,d_k(G)\,R^{\|k\|} \;=\; \sum_{\|k\|\,=\,0}^{m_0} E_{\|k\|}(f)\,d_k(G)\,R^{\|k\|} \;+$$

$$\sum_{\|k\|\,=\,m_0+1}^{m_1(R)} E_{\|k\|}(f)\,d_k(G)\,R^{\|k\|} \;+\; \sum_{\|k\|\,=\,m_1(R)+1}^{\infty} E_{\|k\|}(f)\,d_k(G)\,R^{\|k\|} \tag{2.11}$$

By applying the lemma and (2.10), the above inequality becomes

$$M_{f,G}(R) \;\leq\; (1+\|k\|)^n \;+\; m_1(R) \max_{m_0\,\leq\,\|k\|\,\leq m_1(R)} (E_{\|k\|}(f)\,d_k(G)R^{\|k\|}) \;+\; \sum 2^{-\|k\|},$$

where $m_1(R) = m(R) + 1$, and $m_0 = \max\{M', m_1(R)\}$. Hence

$$\max_{m_0 \le \|k\| \le m_1(R)} (E_{\|k\|}(f)\, R^{\|k\|}) \le \max_{m_0 \le \|k\| \le m_1(R)} \chi(\|k\|)$$
$$\le \exp\{A\,\rho\,\alpha^{-1}\{\overline{\tau}\,\beta\,[(\gamma(Re^{\frac{1}{\rho}-A}))^{\rho}]\}\},$$

where

$$\chi(\|k\|) = (R\,e^{1/\rho})^{\|k\|}\,[F(\|k\|/\rho;\, 1/\overline{\tau},\, \rho)]^{-\|k\|}.$$

From (2.11)

$$M_{f,G}(R) \le \exp\{(A\,\rho + o(1))\,\alpha^{-1}\{\overline{\tau}\,\beta\,[(\gamma\,(R\,e^{\frac{1}{\rho}+A}))^{\rho}]\}\},$$

or

$$\frac{\alpha\left[(A\,\rho + o(1))^{-1}\ln M_{f,G}(R)\right]}{\beta\,[(\gamma\,(R\,e^{\frac{1}{\rho}+A}))^{\rho}]} \le \overline{\tau} = \tau + \epsilon.$$

Since $\alpha(x) \in L^0$, $\beta^{-1}(x) \in L^0$, $\gamma(x) \in L^0$, proceeding to limits and since ϵ is arbitrary, we obtain

$$\sigma = \limsup_{R\to\infty} \frac{\alpha(\ln M_{f,G}(R))}{\beta[(\gamma(R))^{\rho}]} \le \tau = \limsup_{\|k\|\to\infty} \frac{\alpha(\frac{\|k\|}{\rho})}{\beta\{[\gamma(e^{1/\rho}[E_{\|k\|}(f)\,d_k(G)]^{-1/\|k\|})]^{\rho}\}}. \quad (2.12)$$

From (2.8) and (2.12), we obtain the required result. □

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