Thai Journal of Mathematics
Volume 12 (2014) Number 3 : 525–536

http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

On the Conharmonic Curvature Tensor of Kenmotsu Manifolds

Nader Asghari and Abolfazl Taleshian[1]

Department of Mathematics, University of Mazandaran
P.O. Box 47416-1467, Mazandaran, Iran
e-mail : `nasgharigm2009@yahoo.com (N. Asghari)`
`taleshian@umz.ac.ir (A. Taleshian)`

Abstract : The object of the present paper is to study Kenmotsu manifold admitting a conharmonic curvature tensor.

Keywords : Kenmotsu manifold; conharmonic curvature tensor;
η-Einstein manifold.
2010 Mathematics Subject Classification : 53C25; 53C15; 53D10.

1 Introduction

In [1], Tanno classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension. For such a manifold M, the sectional curvature of plane sections containing ξ is a contact, say c. If $c > 0$, M is homogeneous Sasakian manifold of constant sectional curvature. If $c = 0$, M is the product of a line or circle with a Kaehler manifold of constant holomorphic sectional curvature. If $c < 0$, M is a warped product space $R \times_f C^n$. In 1971, Kenmotsu studied a class of contact Riemannian manifolds satisfying some special conditions [2]. We call it Kenmotsu manifold. Recently, Kenmotsu manifolds have been studied by many authors such as De and Pathak [3], Jun et al. [4], Prakasha [5] and many others.

In this paper, we studied the properties of Kenmotsu manifold equipped with conharmonic curvature tensor. We prove that conharmonically flat Kenmotsu manifold is an η-Einstein manifold. Also, we study Kenmotsu manifolds in with

[1]Corresponding author.

$\bar{C}(\xi, X).R = 0$, where $\bar{C}$ is conharmonic curvature tensor. In this case, we show that manifold is locally isometric to the Hyperbolic $H^n(-1)$. Then we study Kenmotsu manifold with the irrotational conharmonic curvature tensor. Also, we study Kenmotsu manifolds in with $\tilde{C}(\xi, U).\bar{C} = 0$, where $\tilde{C}$ is concircular curvature tensor. In this case, we show that a Kenmotsu manifold is an η-Einstein manifold. Next, we prove that φ-conharmonically flat Kenmotsu manifold is an η-Einstein manifold. Moreover we show that a Kenmotsu manifold satisfying $C(\xi, Y).\bar{C} = 0$, is an η-Einstein manifold, where C is Weyl conformal curvature tensor. Finally we investigate conharmonic φ-recurrent and locally conharmonic φ-symmetric Kenmotsu manifold.

2 Preliminaries

If on an add dimensional differentiable manifold M_n, $n = 2m + 1$, of differentiability class C^{r+1}, there exist a vector real linear function φ, a 1-form η, the associated vector field ξ and the Riemannian metric g satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \tag{2.1}$$

$$\eta(\varphi X) = 0, \tag{2.2}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

for all vector fields X and Y, then (M_n, g) is said to be an almost contact metric manifold and structure (φ, η, ξ, g), is called an almost contact metric structure to M_n [1, 2, 5]. In view of above relations we get

$$\eta(\xi) = 1, g(X, \xi) = \eta(X), \varphi(\xi) = 0. \tag{2.4}$$

If moreover,

$$(\nabla_X \varphi)Y = -g(X, \varphi Y)\xi - \eta(Y)\varphi X, \tag{2.5}$$

and

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.6}$$

for all vector fields X, Y. Where ∇ denotes the operator of covariant differentiation with respect to the Riemannian metric g, then $(M_n, \varphi, \xi, \eta, g)$ is called a Kenmotsu manifold [6].

Also, the following relations hold in Kenmotsu manifold [3–5, 7–10]:

$$\begin{aligned} g(R(X,Y)Z, \xi) &= \eta(R(X,Y)Z) \\ &= g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \end{aligned} \tag{2.7}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.8}$$

$$R(\xi, X)Y = \eta(Y)X - g(X,Y)\xi, \tag{2.9}$$

$$R(\xi, X)\xi = X - \eta(X)\xi, \tag{2.10}$$

$$S(X, \xi) = -(n-1)\eta(X), \tag{2.11}$$

$$Q\xi = -(n-1)\xi, \tag{2.12}$$

$$S(\varphi X, \varphi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y), \tag{2.13}$$

for any vector fields X, Y, Z, where $R(X,Y)Z$ is the curvature tensor, and S is the *Ricci* tensor.

A Kenmotsu manifold (M_n, g) is said to be η-Einstein if its *Ricci* tensor S is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{2.14}$$

for any vector fields X, Y where a, b are functions on (M_n, g). If $b = 0$, then η-Einstein manifold becomes to Einstein manifold. Kenmotsu [2], proved that if (M_n, g) is an η-Einstein manifold, then $a + b = -(n-1)$.

In view of (2.4) and (2.14), we have

$$QX = aX + b\eta(X)\xi, \tag{2.15}$$

where Q is the Ricci operator defined by

$$S(X,Y) = g(QX,Y). \tag{2.16}$$

Again, contracting (2.15) with respect to X and using (2.4), we have

$$r = na + b, \tag{2.17}$$

where r is the scalar curvature.

Now, substituting $X = \xi$ and $Y = \xi$ in (2.14) and then using (2.4) and (2.9), we obtain

$$a + b = -(n-1). \tag{2.18}$$

Equations (2.17) and (2.18) gives

$$a = \left(\frac{r}{n-1} + 1\right), b = -\left(\frac{r}{n-1} + n\right). \tag{2.19}$$

Definition 2.1. The *conharmonic* curvature tensor $\bar{C}$ of type $(1,3)$ on Kenmotsu manifold M of dimensional n is defined by

$$\begin{aligned}\bar{C}(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}[&S(Y,Z)X - S(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY],\end{aligned} \tag{2.20}$$

for any vector fields X, Y, Z, on M. The manifold is said to be conharmonically flat if $\bar{C}$ vanishes identically on M.

Definition 2.2. The *concircular* curvature tensor $\tilde{C}$ on Kenmotsu manifold M of dimensional n is defined by

$$\tilde{C}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{2.21}$$

for any vector fields X, Y, Z, where R is the curvature tensor and r is the scalar curvature.

Definition 2.3. The Weyl conformal curvature tensor C on Kenmotsu manifold M of dimensional n is defined b

$$C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y], \quad (2.22)$$

for all vector fields X, Y, Z on M.

3 Main Results

In this section, we prove the following theorems:

Theorem 3.1. *An n-dimensional conharmonically flat Kenmotsu manifold is an η-Einstein manifold.*

Proof. If $\bar{C} = 0$ then we get from (2.20) that

$$R(X,Y)Z = \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]. \quad (3.1)$$

Putting $Z = \xi$ in (3.1) and using (2.4), (2.8) and (2.11) we obtain

$$\eta(X)Y - \eta(Y)X = \frac{1}{n-2}[-(n-1)\eta(Y)X + (n-1)\eta(X)Y + \eta(Y)QX - \eta(X)QY]. \quad (3.2)$$

Taking $Y = \xi$ in (3.2) and using (2.4) we get

$$\eta(X)\xi - X = \frac{1}{n-2}[-(n-1)X + (n-1)\eta(X)\xi + QX + (n-1)\eta(X)\xi]. \quad (3.3)$$

Therefore with simplify of the above equation we get

$$QX = X - n\eta(X)\xi. \quad (3.4)$$

Similarly, We obtain

$$QY = Y - n\eta(Y)\xi. \quad (3.5)$$

Now, putting (3.4) and (3.5) in (3.1) we obtain

$$R(X,Y)Z = \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)X - ng(Y,Z)\eta(X)\xi - g(X,Z)Y + ng(X,Z)\eta(Y)\xi], \quad (3.6)$$

putting $X = \xi$ and using (2.9) and (2.11) we get

$$\eta(Z)Y - g(Y,Z)\xi = \frac{1}{n-2}[S(Y,Z)\xi + (n-1)\eta(Z)Y + g(Y,Z)\xi - ng(Y,Z)\xi - \eta(Z)Y + n\eta(Z)\eta(Y)\xi].$$

With simplify of the above equation we obtain

$$S(Y,Z) = g(Y,Z) - n\eta(Y)\eta(Z).$$

Therefore, in view of (2.14), manifold is an η-Einstein. □

Theorem 3.2. *Let M be an n-dimensional η-Einstein Kenmotsu manifold. Then M satisfies in condition $\bar{C}(\xi, X).R = 0$, if and only if M is locally isometric to the Hyperbolic $H^n(-1)$.*

Proof. If M is an η-Einstein Kenmotsu manifold then in view of (2.4), (2.9), (2.14), (2.15) and (2.19), (2.20) becomes

$$\bar{C}(\xi, Y)Z = \frac{r}{(n-1)(n-2)}[\eta(Y)Z - g(Y,Z)\xi].$$

Since $\bar{C}(\xi, X).R = 0$, we have

$$(\bar{C}(\xi, X).R)(Y,Z)U = 0,$$

this implies that

$$0 = \bar{C}(\xi, X)R(Y,Z)U - R(\bar{C}(\xi, X)Y, Z)U - R(Y, \bar{C}(\xi, X)Z)U - R(Y,Z)\bar{C}(\xi, X)U. \tag{3.7}$$

In view of (3.7) and using (2.4), (2.7), (2.8) and (2.9) we obtain

$$0 = \left\{\frac{r}{(n-1)(n-2)}\right\}[\eta(R(Y,Z)U)X - `R(Y,Z,U,X)\xi - \eta(Y)R(X,Z)U + g(X,Y)R(\xi,Z)U - \eta(Z)R(Y,X)U + g(X,Z)R(Y,\xi)U - \eta(U)R(Y,Z)X + g(X,U)R(Y,Z)\xi],$$

where

$$`R(X,Y,Z,U) = g(R(X,Y,Z),U).$$

Taking the inner product of the last equation with ξ we get

$$0 = \left\{\frac{r}{(n-1)(n-2)}\right\}[\eta(R(Y,Z)U)\eta(X) - `R(Y,Z,U,X) - \eta(Y)\eta(R(X,Z)U) + g(X,Y)\eta(R(\xi,Z)U) - \eta(Z)\eta(R(Y,X)U) + g(X,Z)\eta(R(Y,\xi)U) - \eta(U)\eta(R(Y,Z)X) + g(X,U)\eta(R(Y,Z)\xi)].$$

With simplify of the above equation we obtain

$$\left\{\frac{r}{(n-1)(n-2)}\right\}[-`R(Y,Z,U,X) - g(X,Y)g(Z,U) + g(X,Z)g(Y,U)] = 0.$$

Finally we obtain

$$`R(Y,Z,U,X) = g(X,Z)g(Y,U) - g(X,Y)g(Z,U),$$

this implies that

$$R(Y,Z)U = -[g(Z,U)Y - g(Y,U)Z].$$

The above equation implies that M is of constant curvature -1 and consequently it is locally isometric with the Hyperbolic $H^n(-1)$. This the completes the proof of the theorem. □

Theorem 3.3. *If the conharmonic curvature tensor $\bar{C}$ on a Kenmotsu manifold is irrotational, then $\bar{C}$ given by*

$$\bar{C}(X,Y)Z = \frac{r}{(n-1)(n-2)}[g(X,Z)Y - g(Y,Z)X], \tag{3.8}$$

where r is the scalar curvature.

Proof. The rotation ($Curl$) of conharmonic curvature tensor $\bar{C}$ on a Riemannian manifold is given by

$$\begin{aligned} Rot\bar{C} &= (\nabla_U\bar{C})(X,Y)Z + (\nabla_X\bar{C})(U,Y)Z \\ &\quad + (\nabla_Y\bar{C})(X,U)Z - (\nabla_Z\bar{C})(X,Y)U, \end{aligned} \tag{3.9}$$

where ∇ denotes the Rimannian connection. By virtue of second Bianchi identity, we have

$$(\nabla_U\bar{C})(X,Y)Z + (\nabla_X\bar{C})(U,Y)Z + (\nabla_Y\bar{C})(X,U)Z = 0. \tag{3.10}$$

Therefore in view of (3.9), (3.10) reduces to

$$Rot\bar{C} = -(\nabla_Z\bar{C})(X,Y)U. \tag{3.11}$$

Now, if the conharmonic curvature tensor is irrotational, then $Curl\bar{C} = 0$ and so by (3.11) we obtain

$$(\nabla_Z\bar{C})(X,Y)U = 0,$$

this implies that

$$\nabla_Z\bar{C}(X,Y)U = \bar{C}(\nabla_Z X,Y)U + \bar{C}(X,\nabla_Z Y)U + \bar{C}(X,Y)\nabla_Z U.$$

Putting $U = \xi$ in the above equation we get

$$\nabla_Z\bar{C}(X,Y)\xi = \bar{C}(\nabla_Z X,Y)\xi + \bar{C}(X,\nabla_Z Y)\xi + \bar{C}(X,Y)\nabla_Z\xi. \tag{3.12}$$

Replacing $Z = \xi$ in (2.20) and using (2.4), (2.8), (2.5) and (2.19) we obtain

$$\bar{C}(X,Y)\xi = \frac{r}{(n-1)(n-2)}[\eta(X)Y - \eta(Y)X]. \tag{3.13}$$

Using (3.13) in (3.12) we get

$$\bar{C}(X,Y)Z = \frac{r}{(n-1)(n-2)}[g(X,Z)Y - g(Y,Z)X].$$

The proof is complete. □

Theorem 3.4. *Let M be an n-dimensional Kenmotsu manifold. Then M satisfies in condition $\tilde{C}(\xi,U).\bar{C} = 0$, if and only if either M has scalar curvature $r = n(1-n)$ or M is an η-Einstein manifold.*

Proof. Since $\tilde{C}(\xi,U).\bar{C} = 0$ we have

$$\tilde{C}(\xi,U).\bar{C}(X,Y)W = 0,$$

this implies that

$$[\tilde{C}(\xi,U),\bar{C}(X,Y)]W - \bar{C}(\tilde{C}(\xi,U)X,Y)W - \bar{C}(X,\tilde{C}(\xi,U)Y)W = 0,$$

in view of (2.21) we get

$$\begin{aligned} 0 = \left(-1 - \frac{r}{n(n-1)}\right)&[-\eta(\bar{C}(X,Y)W)U + \bar{C}(X,Y,W,U)\xi \\ &+ \eta(X)\bar{C}(U,Y)W - g(U,X)\bar{C}(\xi,Y)W \\ &+ \eta(Y)\bar{C}(X,U)W - g(U,Y)\bar{C}(X,\xi)W \\ &+ \eta(W)\bar{C}(X,Y)U - g(U,W)\bar{C}(X,Y)\xi]. \end{aligned}$$

Therefore M has scalar curvature $r = n(1-n)$ or

$$\begin{aligned} 0 = &-\eta(\bar{C}(X,Y)W)U + \bar{C}(X,Y,W,U)\xi + \eta(X)\bar{C}(U,Y)W \\ &- g(U,X)\bar{C}(\xi,Y)W + \eta(Y)\bar{C}(X,U)W - g(U,Y)\bar{C}(X,\xi)W \\ &+ \eta(W)\bar{C}(X,Y)U - g(U,W)\bar{C}(X,Y)\xi. \end{aligned}$$

Taking the inner product of the last equation with ξ we get

$$\begin{aligned} 0 = &-\eta(\bar{C}(X,Y)W)\eta(U) + \bar{C}(X,Y,W,U) \\ &+ \eta(X)\eta(\bar{C}(U,Y)W) - g(U,X)\eta(\bar{C}(\xi,Y)W) \\ &+ \eta(Y)\eta(\bar{C}(X,U)W) - g(U,Y)\eta(\bar{C}(X,\xi)W) \\ &+ \eta(W)\eta(\bar{C}(X,Y)U) - g(U,W)\eta(\bar{C}(X,Y)\xi). \end{aligned}$$

Finally, with simplify we get

$$\bar{C}(X,Y,W,U) = 0,$$

which implies that M is conharmonically flat. Thus in view of Theorem 3.1, M is an η-Einstein manifold. The converse is trivial. □

Definition 3.5. An n-dimensional, $(n > 3)$, Kenmotsu manifold satisfying the condition

$$\varphi^2\bar{C}(\varphi X, \varphi Y)\varphi Z = 0, \tag{3.14}$$

is called φ-conharmonically flat manifold.

Theorem 3.6. *Let M be an n-dimensional, $(n > 3)$, φ-conharmonically flat Kenmotsu manifold. Then M is an η-Einstein manifold.*

Proof. If M is φ- conharmonically flat Kenmotsu manifold then we get from (3.14) that

$$\varphi^2\bar{C}(\varphi X, \varphi Y)\varphi Z = 0,$$

this implies that

$$g(\bar{C}(\varphi X, \varphi Y)\varphi Z, \varphi W) = 0,$$

for any vector fields X, Y, Z, W on M. Using (2.20) we obtain

$$\begin{aligned} g(R(\varphi X, \varphi Y)\varphi Z, \varphi W) = \frac{1}{n-2}[&S(\varphi Y, \varphi Z)g(\varphi X, \varphi W) \\ &- S(\varphi X, \varphi Z)g(\varphi Y, \varphi W) \\ &+ g(\varphi Y, \varphi Z)S(\varphi X, \varphi W) \\ &- g(\varphi X, \varphi Z)S(\varphi Y, \varphi W)]. \end{aligned} \tag{3.15}$$

Let $\{e_1, \ldots, e_{n-1}, \xi\}$ be a local orthonormal basis of vector fields in M. Using that $\{\varphi e_1, \ldots, \varphi e_{n-1}, \xi\}$ is also a local orthonormal basis, if we put $X = W = e_i$ in (3.15) and sum up with respect to i, then

$$\begin{aligned} \sum_{i=1}^{n-1} g(R(\varphi e_i, \varphi Y), \varphi Z, \varphi e_i) = \frac{1}{n-2}\sum_{i=1}^{n-1}[&S(\varphi Y, \varphi Z)g(\varphi e_i, \varphi e_i) \\ &- S(\varphi e_i, \varphi Z)g(\varphi Y, \varphi e_i) \\ &+ g(\varphi Y, \varphi Z)S(\varphi e_i, \varphi e_i) \\ &- g(\varphi e_i, \varphi Z)S(\varphi Y, \varphi e_i)]. \end{aligned} \tag{3.16}$$

It can be easily verify that

$$\sum_{i=1}^{n-1} g(R(\varphi e_i, \varphi Y)\varphi Z, \varphi e_i) = S(\varphi Y, \varphi Z) + g(\varphi Y, \varphi Z), \tag{3.17}$$

$$\sum_{i=1}^{n-1} S(\varphi e_i, \varphi e_i) = r - (n-1), \tag{3.18}$$

$$\sum_{i=1}^{n-1} g(\varphi e_i, \varphi e_i) = n-1, \tag{3.19}$$

$$\sum_{i=1}^{n-1} g(\varphi Y, \varphi e_i)S(\varphi e_i, \varphi Z) = S(\varphi Y, \varphi Z). \tag{3.20}$$

So by virtue of - the equation can be written as

$$S(\varphi Y,\varphi Z)+g(\varphi Y,\varphi Z)=\frac{1}{n-2}[(n-3)S(\varphi Y,\varphi Z)+(r-(n-1))g(\varphi Y,\varphi Z)],$$

this implies that

$$S(\varphi Y,\varphi Z)=(r-(2n-3))g(\varphi Y,\varphi Z).$$

In view of and we get

$$S(Y,Z)+(n-1)\eta(Y)\eta(Z)=(r-(2n-3))g(Y,Z)-(r-(2n-3))\eta(Y)\eta(Z).$$

Finally we obtain

$$S(Y,Z)=(r-(2n-3))g(Y,Z)-(r-n+2)\eta(Y)\eta(Z).$$

Therefore, in view of , M is an η-Einstein manifold. The proof is complete. □

Theorem 3.7. *Let M be an n-dimensional Kenmotsu manifold. Then M satisfies in condition $C(\xi,Y).\bar{C}=0$, if and only if either M has scalar curvature $r=(1-n)$ or M is an η-Einstein manifold.*

Proof. Since $C(\xi,Y).\bar{C}=0$ we have

$$C(\xi,Y).\bar{C}(Z,U)V=0,$$

this implies that

$$[C(\xi,Y),\bar{C}(Z,U)]V-\bar{C}(C(\xi,Y)Z,U)V-\bar{C}(Z,C(\xi,Y)U)V=0,$$

in view of (2.22) we get

$$\begin{aligned}0=\frac{n-1+r}{(n-1)(n-2)}&[-\eta(\bar{C}(X,Y)W)U+\bar{C}(X,Y,W,U)\xi\\&+\eta(X)\bar{C}(U,Y)W-g(U,X)\bar{C}(\xi,Y)W\\&+\eta(Y)\bar{C}(X,U)W-g(U,Y)\bar{C}(X,\xi)W\\&+\eta(W)\bar{C}(X,Y)U-g(U,W)\bar{C}(X,Y)\xi].\end{aligned}$$

Therefore M has scalar curvature $r=1-n$ or

$$\begin{aligned}0=&-\eta(\bar{C}(X,Y)W)U+\bar{C}(X,Y,W,U)\xi+\eta(X)\bar{C}(U,Y)W\\&-g(U,X)\bar{C}(\xi,Y)W+\eta(Y)\bar{C}(X,U)W-g(U,Y)\bar{C}(X,\xi)W\\&+\eta(W)\bar{C}(X,Y)U-g(U,W)\bar{C}(X,Y)\xi.\end{aligned}$$

Taking the inner product of the last equation with ξ we get

$$\begin{aligned}0=&-\eta(\bar{C}(X,Y)W)\eta(U)+\bar{C}(X,Y,W,U)\\&+\eta(X)\eta(\bar{C}(U,Y)W)-g(U,X)\eta(\bar{C}(\xi,Y)W)\\&+\eta(Y)\eta(\bar{C}(X,U)W)-g(U,Y)\eta(\bar{C}(X,\xi)W)\\&+\eta(W)\eta(\bar{C}(X,Y)U)-g(U,W)\eta(\bar{C}(X,Y)\xi).\end{aligned}$$

Finally, with simplify we get

$$\bar{C}(X,Y,W,U)=0.$$

Therefore in view of Theorem 3.1, M is an η-Einstein manifold. The converse is trivial. This the completes the proof of the theorem. □

Definition 3.8. A Kenmotsu manifold is said to be conharmonic φ-recurrent Manifold if there exist a non-zero 1-form A such that

$$\varphi^2((\nabla_W\bar{C})(X,Y)Z)=A(W)\bar{C}(X,Y)Z, \tag{3.21}$$

for arbitrary vector fields X,Y,Z,W. If the 1-form A vanishes, then the manifold reduces to the locally conharmonic φ-symmetric manifold.

Theorem 3.9. *A conharmonic φ-recurrent Kenmotsu manifold is an Einstein manifold.*

Proof. Let us consider a conharmonic φ-recurrent Kenmotsu manifold. Then by virtue of (2.20) and Definition 3.8, we have

$$-((\nabla_W\bar{C})(X,Y)Z)+\eta((\nabla_W\bar{C})(X,Y)Z)\xi=A(W)\bar{C}(X,Y)Z, \tag{3.22}$$

from which it follows that

$$-g((\nabla_W\bar{C}(X,Y)Z))+\eta((\nabla_W\bar{C})(X,Y)Z)\eta(U)=A(W)g(\bar{C}(X,Y)Z,U).$$

Let $\{e_i\}$, $i=1,\dots,n$, be a locally orthonormal basis of the tangent space at any point of the manifold. Let us put $X=U=e_i$ in (3.23), where, $1\le i\le n$, we get

$$\begin{aligned}&-(\nabla_WS)(Y,Z)-\frac{1}{n-2}[\sum(\nabla_WS)(e_i,Z)g(Y,e_i)\\&-\sum(\nabla_WS)(e_i,Z)g(e_i,\xi)\eta(Y)]\\&=A(W)\left[-\frac{r}{n-2}g(Y,Z)\right].\end{aligned} \tag{3.23}$$

substitute Z by ξ in (3.23), following due to (2.1), (2.4) and (2.11) we obtain

$$\begin{aligned}&-(\nabla_WS)(Y,\xi)-\frac{1}{n-2}[\sum(\nabla_WS)(e_i,\xi)g(Y,e_i)\\&-\sum(\nabla_WS)(e_i,\xi)g(e_i,\xi)\eta(Y)]\\&=-\left(\frac{r}{n-2}\right)\eta(Y)A(W).\end{aligned} \tag{3.24}$$

On the other hand since $(\nabla_W\eta)(Y))=g(W,Y)-\eta(Y)\eta(W)$ and in view of (2.6) and (2.11) we have

$$(\nabla_WS)(Y,\xi)=-(n-1)g(Y,W)-S(Y,W). \tag{3.25}$$

By substituting $(\nabla_W S)(Y,\xi)$ from (3.25) in (3.24)it results

$$\frac{n-1}{n-2}S(Y,W)+\frac{(n-1)^2}{n-2}g(Y,W)=-\left(\frac{r}{n-2}\right)\eta(Y)A(W). \tag{3.26}$$

Replacing Y by φY and W by φW in (3.26), using (2.3) and (2.13) we get

$$S(Y,W)=-(n-1)g(Y,W).$$

This completes the proof. □

Theorem 3.10. *A locally conharmonic φ-symmetric Kenmotsu manifold is a manifold of constant curvature.*

Proof. From Definition 3.8 we have

$$\varphi^2((\nabla_W\bar{C})(X,Y)Z)=0.$$

in view (2.1) it follows

$$-(\nabla_W\bar{C})(X,Y)Z+\eta((\nabla_W\bar{C})(X,Y)Z)=0. \tag{3.27}$$

Substituting ξ with Z in (3.27)

$$-(\nabla_W\bar{C})(X,Y)\xi+\eta((\nabla_W\bar{C})(X,Y)\xi)=0. \tag{3.28}$$

On the other hand we have

$$\begin{aligned}(\nabla_W\bar{C})(X,Y)\xi=(\nabla_W R)(X,Y)\xi-\frac{1}{n-2}[&-(n-1)g(Y,W)X\\ &-S(Y,W)X+(n-1)g(X,W)Y+S(X,W)Y],\end{aligned}$$

and

$$\eta((\nabla_W\bar{C})(X,Y)\xi)=0.$$

Therefore we get from (3.28)

$$\begin{aligned}&-R(X,Y)W-\frac{1}{n-2}[-(n-1)g(Y,W)X\\ &-S(Y,W)X+(n-1)g(X,W)Y+S(X,W)Y]=0.\end{aligned} \tag{3.29}$$

From which it follows that

$$\begin{aligned}g(R(X,Y)W,U)=-\frac{1}{n-2}[&-(n-1)g(Y,W)g(X,U)-S(Y,W)g(X,U)\\ &+(n-1)g(X,W)g(Y,U)+S(X,W)S(Y,U)].\end{aligned} \tag{3.30}$$

Let $\{e_i\}$, $i=1,\dots,n$, be an orthonormal basis of the tangent space at any point of manifold. Let us put $Y=W=e_i$ in (3.30), where, $1\le i\le n$, we get

$$S(X,U)=(n-1)g(X,U)+\frac{r}{n-1}g(X,U). \tag{3.31}$$

Using (3.31) in (3.30) we obtain

$$R(X,Y)W = \frac{2(n-1)^2 + r}{(n-1)(n-2)}[g(Y,W)X - g(X,W)Y].$$

This proves the theorem. □

Acknowledgement : The authors are highly thankful to the referee for his valuable suggestions in the improvement of the paper.

References

[1] S. Tanno, The automorphism groups of almost contact Rimannian manifolds, Tohoku Math. J. 21 (1969) 21–38.

[2] K. Kenmotsu, A class of almost contact Rimannian manifolds, Tohoku Math. J. 24 (1972) 93–103.

[3] U.C. De, G. Pathak, On 3-dimensional Kenmotsu manifolds, Indian J. Pure Applied Math. 35 (2004) 159–165.

[4] J.B. Jun, U.C. De, G. Pathak, On Kenmotsu manifolds, J. Korean Math. Soc. 42 (2005) 435–445.

[5] D.G. Prakasha, On ϕ-symmetric Kenmotsu manifolds with respect to quarter symmetric metric connection, Int. Electron. J. Geom. 4 (1) (2011) 88–96.

[6] H.B. Karadağ, M. Karadağ, Null generalized slant helices in Lorentzian space, Differential Geometry-Dynamical Systems 10 (2008) 178–185.

[7] C.B. Bagewadi and Venkatesha, On pseudo projectivw φ-recurrent Kenmotsu manifolds, Soochow J. Math. 32 (2006) 1–7.

[8] U.C. De, A. Yildiz, A.F. Yaliniz, On $\varphi-$recurrent Kenmotsu manifolds, Turk. J. Math. 33 (2009) 17–25.

[9] C. Özgür, M.M. Tripathi, On the quasi-conformal curvature tensor of a Kenmotsu manifold, Mathematica Pannonica 17 (2) (2006) 221–229.

[10] A. Yildiz, U.C. De, On type of Kenmotsu manifolds, Differential Geometry-Dynamical Systems 12 (2010) 289–298.

(Received 25 July 2012)
(Accepted 17 April 2013)