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A System of Multi-Valued Variational Inclusions Involving P-Accretive Mappings in Real Uniformly Smooth Banach Spaces

Faizan A. Khan[†,1], **Saleh A. Al-Mezel**[†], **Kaleem R. Kazmi**[‡]

[†]Department of Mathematics, Faculty of Science
University of Tabuk, Tabuk-71491, Kingdom of Saudia Arabia
e-mail : faizan_math@yahoo.com (F.A. Khan)
almezel@ut.edu.sa (S.A. Al-Mezel)

[‡]Department of Mathematics, Faculty of Science
Aligarh Muslim University, Aligarh-202002, India
e-mail : krkazmi@gmail.com (K.R. Kazmi)

Abstract : In this paper, we consider a class of accretive mappings called P-accretive mappings in real Banach spaces. We prove that the proximal-point mapping of the P-accretive mapping is single-valued and Lipschitz continuous. Further, we consider a system of multi-valued variational inclusions involving P-accretive mappings in real uniformly smooth Banach spaces. Using proximal-point mapping method, we prove the existence of solution and discuss the convergence analysis of iterative algorithm for the system of multi-valued variational inclusions. The theorems presented in this paper extend and improve many known results in the literature.

Keywords : system of multi-valued variational inclusions; P-accretive mappings; proximal-point mapping method; iterative algorithm; convergence analysis.

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[1]Corresponding author.

1 Introduction

Variational inclusions, as the generalization of variational inequalities, have been widely studied in recent years. One of the most interesting and important problems in the theory of variational inclusions is the development of an efficient and implementable iterative algorithm. Variational inclusions include variational, quasi-variational, variational-like inequalities as special cases. Various kinds of iterative methods have been studied to find the approximate solutions for variational inclusions. Among these methods, the proximal-point mapping method for solving variational inclusions has been widely used by many authors. For details, we refer to see [1–12] and the references therein.

In 2001, Huang and Fang [13] were the first to introduce the generalized m-accretive mapping and have given the definition of the proximal-point mapping for the generalized m-accretive mapping in Banach spaces. Since then a number of researchers investigated several classes of generalized m-accretive mappings such as H-accretive, (H, η)-accretive, and (A, η)-accretive mappings, see for example [3, 5, 11, 13–18]. In recent past, the methods based on different classes of proximal-point mappings have been developed to study the existence of solutions and to discuss convergence analysis of iterative algorithms for various classes of variational inclusions, see for example [1–12]. Recently, by using proximal-point mapping method, Chang et al. [19], Chidume et al. [14], Ding and Luo [2], Ding and Yao [20], Fang and Huang [3], Kazmi and Khan [5], Noor [21, 22], Verma [23], Zeng et al. [18] and Zou and Huang [11] introduced and studied a class of P-accretive mappings and discussed the existence of solutions and convergence analysis of iterative algorithms for various classes of variational inclusions (inequalities) in the setting of Hilbert and Banach spaces.

Very recently, by using proximal-point mapping method, Ding and Feng [1], Fang et al. [15], Feng and Ding [24], Kazmi and Bhat [4], Kazmi and Khan [6], Kazmi et al. [7, 8], Noor [9], Peng and Zou [10] and Zou and Huang [12] introduced and studied a class of P-accretive mappings and discussed the existence of solutions and convergence analysis of iterative algorithms for various classes of system of variational inclusions (inequalities) in the setting of Hilbert and Banach spaces.

Inspired by recent research work in this direction, we consider a class of accretive mappings called P-accretive mappings, a natural generalization of accretive (monotone) mappings studied in [3, 5, 11, 13–18] in Banach spaces. We prove that the proximal-point mapping of the P-accretive mapping is single-valued and Lipschitz continuous. Further, we consider a system of multi-valued variational inclusions involving P-accretive mappings in real uniformly smooth Banach spaces. Using proximal-point mapping method, we prove the existence of solution and discuss the convergence analysis of iterative algorithm for the system of multi-valued variational inclusions. The results presented in this paper generalize and improve some known results given in [1, 4, 6–10, 12, 15, 24, 25].

2 Preliminaries

Let E be a real Banach space equipped with norm $\|\cdot\|$; E^* be the topological dual space of E; $\langle\cdot,\cdot\rangle$ be the dual pair between E and E^*; $CB(E)$ be the family of all nonempty closed and bounded subsets of E; $C(E)$ be the family of all nonempty compact subsets of E; 2^E be the power set of E. Let $H(\cdot,\cdot)$ be the Hausdorff metric on $CB(E)$ defined by

$$H(A,B)=\max\left\{\sup_{x\in A}\inf_{y\in B}d(x,y),\ \sup_{y\in B}\inf_{x\in A}d(x,y)\right\};\qquad A,B\in CB(E);$$

and $J:E\to 2^{E^*}$ be the normalized duality mapping defined by

$$J(x)=\{f\in E^*:\ \ \langle x,\ f\rangle\ =\ \|x\|^2,\|x\|\ =\ \|f\|\},\quad x\in E.$$

First, we recall and define the following concepts and results.

Definition 2.1 ([17, 26])**.** A Banach space E is called smooth if, for every $x\in E$ with $\|x\|=1$, there exists a unique $f\in E^*$ such that $\|f\|=f(x)=1$. The modulus of smoothness of E is the function $\rho_E:[0,\infty)\to[0,\infty)$, defined by

$$\rho_E(\tau)=\sup\left\{\frac{(\|x+y\|+\|x-y\|)}{2}-1:\ x,y\in E,\ \|x\|=1,\ \|y\|=\tau\right\}.$$

Definition 2.2 ([26])**.** The Banach space E is said to be uniformly smooth if

$$\lim_{\tau\to 0}\frac{\rho_E(\tau)}{\tau}=0.$$

We note that if E is smooth then the normalized duality mapping J is single-valued and if $E\equiv H$, a Hilbert space, then J is the identity map on H.

Lemma 2.3 ([14, 27])**.** *Let E be an uniformly smooth Banach space and let $J:E\to E^*$ be the normalized duality mapping. Then for all $x,y\in E$, we have*

(i) $\|x+y\|^2\ \le\ \|x\|^2+2\langle y,\ J(x+y)\rangle$;

(ii) $\langle x-y,\ J(x)-J(y)\rangle\ \le\ 2d^2\rho_E(4\|x-y\|/d)$, *where* $d=\sqrt{(\|x\|^2+\|y\|^2)/2}$.

Definition 2.4 ([5, 28])**.** A multi-valued mapping $T:E\to CB(E)$ is said to be ξ-$\mathcal{H}$-Lipschitz continuous if there exists a constant $\xi>0$ such that

$$\mathcal{H}(T(x),T(y))\ \le\ \xi\|x-y\|,\quad \forall\, x,y\in E,$$

where $\mathcal{H}(\cdot,\cdot)$ is the Hausdorff metric on $CB(E)$.

Lemma 2.5 ([28, 29])**.**

(a) *Let E be a real Banach space. Let $A : E \to CB(E)$ and let $\epsilon > 0$ be any real number, then for every $x, y \in E$ and $u_1 \in A(x)$, there exists $u_2 \in A(y)$ such that*
$$||u_1 - u_2|| \leq \mathcal{H}\big(A(x), A(y)\big) + \epsilon||x - y||;$$

(b) *Let $A : E \to CB(E)$ and let $\delta > 0$ be any real number, then for every $x, y \in E$ and $u_1 \in A(x)$, there exists $u_2 \in A(y)$ such that*
$$||u_1 - u_2|| \leq \delta\mathcal{H}\big(A(x), A(y)\big).$$

We note that if $G : E \to C(E)$ then Lemma 2.5(a)-(b) is true for $\epsilon = 0$ and $\delta = 1$, respectively.

3 P-Proximal-Point Mappings

The following results give some properties of P-accretive mappings.

Definition 3.1 ([29])**.** A mapping $A : E \to E$ is said to be

(i) accretive if
$$\langle A(x) - A(y),\ J(x - y)\rangle \ \geq\ 0\ , \quad \forall\ x, y \in E;$$

(ii) strictly accretive if
$$\langle A(x) - A(y),\ J(x - y)\rangle > 0, \quad \forall\ x, y \in E;$$
and the equality holds only when $x = y$.

(iii) ξ-strongly accretive if there exists a constant $\xi > 0$ such that
$$\langle A(x) - A(y),\ J(x - y)\rangle \ \geq\ \xi\|x - y\|^2, \quad \forall\ x, y \in E;$$

(iv) δ-Lipschitz continuous if there exists a constant $\delta > 0$ such that
$$\|A(x) - A(y)\| \ \leq\ \delta\|x - y\|, \quad \forall\ x, y \in E.$$

Definition 3.2 ([14])**.** A multi-valued mapping $M : E \to 2^E$ is said to be

(i) accretive if
$$\langle u - v,\ J(x - y)\rangle \ \geq\ 0, \quad \forall x, y \in E, u \in M(x), v \in M(y);$$

(ii) ξ-strongly accretive if there exists a constant $\xi > 0$ such that
$$\langle u - v,\ J(x - y)\rangle \ \geq\ \xi\|x - y\|^2, \quad \forall x, y \in E, u \in M(x), v \in M(y);$$

(iii) m-accretive if M is accretive and $(I+\rho M)(E)=E$ for any fixed $\rho>0$, where I is identity mapping on E.

The following definition and results are given in [3], (see also [5]).

Definition 3.3 ([3, 5])**.** Let $P: E \to E$ be a nonlinear mapping. Then a multi-valued mapping $M: E \to 2^E$ is said to be P-accretive, if M is accretive and $(P+\rho M)(E)=E$ for any $\rho>0$.

Theorem 3.4 ([3, 5])**.** *Let $P: E \to E$ be a strictly accretive mapping and let $M: E \to 2^E$ be a P-accretive multi-valued mapping. Then*

(a) $\langle u-v, J(x-y)\rangle \geq 0, \quad \forall (v,y) \in Graph(M)$ implies $(u,x) \in Graph(M)$, where $Graph(M) := \{(u,x) \in E \times E : u \in M(x)\}$;

(b) the mapping $(P+\rho M)^{-1}$ is single-valued for all $\rho>0$.

By Theorem 3.4, we can define P-proximal point mapping for a P-accretive mapping M as follows:

$$J^M_{P,\rho}(z) = (P+\rho M)^{-1}(z), \quad \forall z \in E, \tag{3.1}$$

where $\rho>0$ is a constant and $P: E \to E$ is a strictly accretive mapping.

Theorem 3.5 ([3, 5])**.** *Let $P: E \to E$ be a δ-strongly-accretive mapping and $M: E \to 2^E$ be P-accretive mapping. Then the P-proximal point mapping $J^M_{P,\rho}: E \to E$ is $\frac{1}{\delta}$-Lipschitz continuous, that is,*

$$\|J^M_{P,\rho}(x) - J^M_{P,\rho}(y)\| \leq \frac{1}{\delta}\|x-y\|, \quad \forall x,y \in E.$$

4 System of Multi-Valued Variational Inclusions

Throughout rest of the paper unless otherwise stated, we assume that, for each $i=1,2$, E_i is a real uniformly smooth Banach space with norm $\|\cdot\|_i$, and denote the duality pairing between E_i and its dual E_i^* by $\langle\cdot,\cdot\rangle_i$.

Let $N_i: E_1 \times E_2 \to E_i$, $P_i, g_i: E_i \to E_i$ be nonlinear mappings and $A, C: E_1 \to CB(E_1)$, $B, D: E_2 \to CB(E_2)$ be multi-valued mappings. Let $M_1: E_1 \times E_1 \to 2^{E_1}$ and $M_2: E_2 \times E_2 \to 2^{E_2}$ be P_1-accretive and P_2-accretive mappings, respectively, such that $(g_1(x), g_2(y)) \in \text{domain}(M_1(\cdot,x), M_2(\cdot,y))$ for all $(x,y) \in E_1 \times E_2$. We consider the following system of multi-valued variational inclusions (in short, SMVI):

Find $(x,y) \in E_1 \times E_2$, $u \in A(x)$, $v \in B(y)$, $w \in C(x)$, $z \in D(y)$ such that

$$\begin{cases} N_1(u,v) \ + M_1(g_1(x),x) \ni \theta_1; \\ \\ N_2(w,z) + M_2(g_2(y),y) \ni \theta_2, \end{cases} \tag{4.1}$$

where θ_1 and θ_2 are zero vectors of E_1 and E_2, respectively.

We remark that for suitable choices of the mappings $g_1, g_2, A, B, C, D, M_1, M_2, N_1, N_2, P_1, P_2$, and the spaces E_1, E_2, one can obtain many other known systems of variational inclusions (inequalities) from SMVI(4.1), see for example [1, 4, 6–10, 12, 15, 24, 25].

Assume that $dom(P_i) \cap g_i(E) \neq \emptyset$ for each $i = 1, 2$.

We need the following concepts and results:

Definition 4.1. Let $A, B : E \to CB(E)$ be multi-valued mappings. A mapping $N : E \times E \to E$ is said to be

(i) α-strongly-accretive with respect to A in the first argument if there exists a constant $\alpha > 0$ such that

$$\langle N(u_1, v_1) - N(u_2, v_1), J(x_1 - x_2)\rangle \geq \alpha\|x_1 - x_2\|^2,$$

for all $x_1,\ x_2,\ y_1 \in E,\ u_1 \in A(x_1),\ u_2 \in A(x_2),\ v_1 \in B(y_1)$;

(ii) (β, γ)-mixed Lipschitz continuous if there exist constants $\beta, \gamma > 0$ such that

$$\|N(x_1, y_1) - N(x_2, y_2)\| \leq \beta\|x_1 - x_2\| + \gamma\|y_1 - y_2\|,$$

for all $x_1,\ x_2,\ y_1,\ y_2 \in E$.

Remark 4.2. *The concept of (β, γ)-mixed Lipschitz continuity of mapping N is more general than the Lipschitz continuity of mapping N in first and second argument.*

The following lemma, which will be used in the sequel, is an immediate consequence of the definitions of $J^{M_1(\cdot,x)}_{P_1,\rho_1},\ J^{M_2(\cdot,y)}_{P_2,\rho_2}$.

Lemma 4.3. *For any given $(x, y) \in E_1 \times E_2,\ u \in A(x), v \in B(y), w \in C(x), z \in D(y)$, (x, y, u, v, w, z) is a solution of SMVI(4.1) if and only if (x, y, u, v, w, z) satisfies*

$$g_1(x) = J^{M_1(\cdot,x)}_{P_1,\rho_1}[P_1 \circ g_1(x) - \rho_1 N_1(u, v)],$$

$$g_2(y) = J^{M_2(\cdot,y)}_{P_2,\rho_2}[P_2 \circ g_2(y) - \rho_2 N_2(w, z)],$$

where $\rho_1, \rho_2 > 0$ are constants; $J^{M_1(\cdot,x)}_{P_1,\rho_1} \equiv (P_1 + \rho_1 M_1(\cdot, x))^{-1}$; $J^{M_2(\cdot,y)}_{P_2,\rho_2} \equiv (P_2 + \rho_2 M_2(\cdot, y))^{-1}$, and $P_1 \circ g_1$ denotes P_1 composition g_1.

5 Iterative Algorithm

Using Lemma 2.5 and Lemma 4.3, we suggest and analyze the following iterative algorithm for finding the approximate solution of SMVI(4.1):

Iterative Algorithm 5.1. For given $(x_0, y_0) \in E_1 \times E_2,\ u_0 \in A(x_0), v_0 \in B(y_0), w_0 \in C(x_0), z_0 \in D(y_0)$, compute approximate solution $(x_n, y_n, u_n, v_n, w_n, z_n)$ given by iterative schemes:

$$g_1(x_{n+1}) = J^{M_1(.,x_n)}_{P_1,\rho_1}[P_1 \circ g_1(x_n) - \rho_1 N_1(u_n, v_n)], \tag{5.1}$$

$$g_2(y_{n+1}) = J_{P_2,\rho_2}^{M_2(.,y_n)}[P_2 \circ g_2(y_n) - \rho_2 N_2(w_n, z_n)], \tag{5.2}$$

$$u_n \in A(x_n) : \ ||u_{n+1} - u_n|| \le \left(1 + (1+n)^{-1}\right) \ \mathcal{H}_1(A(x_{n+1}), A(x_n)), \tag{5.3}$$

$$v_n \in B(y_n) : \ ||v_{n+1} - v_n|| \le \left(1 + (1+n)^{-1}\right) \ \mathcal{H}_2(B(y_{n+1}), B(y_n)), \tag{5.4}$$

$$w_n \in C(x_n) : \ ||w_{n+1} - w_n|| \le \left(1 + (1+n)^{-1}\right) \ \mathcal{H}_1(C(x_{n+1}), C(x_n)), \tag{5.5}$$

$$z_n \in D(y_n) \ : \ ||z_{n+1} - z_n|| \le \left(1 + (1+n)^{-1}\right) \ \mathcal{H}_2(D(y_{n+1}), D(y_n)), \tag{5.6}$$

where $n = 0, 1, 2, \ldots;\ \rho_1, \rho_2 > 0$ are constants.

6 Main Result

Now, we prove the existence of a solution and discuss the convergence analysis of Iterative Algorithm 5.1 for SMVI(4.1).

Theorem 6.1. *For each $i = 1, 2$, let E_i be real uniformly smooth Banach space with $\rho_{E_i}(t) \le c_i t^2$ for some $c_i > 0$; let the multi-valued mappings $A, C : E_1 \to CB(E_1)$ be μ_1-$\mathcal{H}_1$-Lipschitz, μ_2-$\mathcal{H}_1$-Lipschitz continuous and $B, D : E_2 \to CB(E_2)$ be η_1-$\mathcal{H}_2$-Lipschitz, η_2-$\mathcal{H}_2$-Lipschitz continuous, respectively; let the mappings $N_1 : E_1 \times E_2 \to E_1$ be α_1-strongly accretive with respect to $P_1 \circ g_1$ in the first argument and (β_1, γ_1)-mixed Lipschitz continuous and $N_2 : E_1 \times E_2 \to E_2$ be α_2-strongly accretive with respect to $P_2 \circ g_2$ in the second argument and (β_2, γ_2)-mixed Lipschitz continuous; let the mappings $P_i,\ P_i \circ g_i,\ (g_i - I_i) : E_i \to E_i$ be such that P_i be δ_i-strongly accretive, $P_i \circ g_i$ be ξ_i-Lipschitz continuous and $(g_i - I_i)$ be k_i-strongly accretive, respectively, where $I_i : E_i \to E_i$ is an identity mapping; let $M_1 : E_1 \times E_1 \to 2^{E_1}$ and $M_2 : E_2 \times E_2 \to 2^{E_2}$ be such that for each fixed $(x, y) \in E_1 \times E_2$, $M_1(\cdot, x)$ and $M_2(\cdot, y)$ are P_1-accretive and P_2-accretive mappings, respectively. Suppose that there are constants $\lambda_1, \lambda_2 > 0$ such that*

$$\|J_{P_1,\rho_1}^{M_1(.,x_1)}(x) - J_{P_1,\rho_1}^{M_1(\cdot,x_2)}(x)\|_1 \le \lambda_1 \|x_1 - x_2\|_1, \quad \forall x, x_1, x_2 \in E_1, \tag{6.1}$$

$$\|J_{P_2,\rho_2}^{M_2(.,y_1)}(y) - J_{P_2,\rho_2}^{M_2(\cdot,y_2)}(y)\|_2 \le \lambda_2 \|y_1 - y_2\|_2, \quad \forall y, y_1, y_2 \in E_2, \tag{6.2}$$

and $\rho_1, \rho_2 > 0$ satisfy the following condition:

$$\begin{cases} L_1\Big[R_1\left(\sqrt{\xi_1^2 - 2\rho_1\alpha_1 + 64c_1\rho_1^2\beta_1^2\mu_1^2}\right) + \lambda_1\Big] + \rho_2\beta_2\mu_2 R_2 L_2 < 1; \\ L_2\Big[R_2\left(\sqrt{\xi_2^2 - 2\rho_2\alpha_2 + 64c_2\rho_2^2\gamma_2^2\eta_2^2}\right) + \lambda_2\Big] + \rho_1\gamma_1\eta_1 R_1 L_1 < 1; \\ L_1 := \frac{1}{\sqrt{2k_1+1}}; \quad R_1 := \frac{1}{\delta_1}; \quad L_2 := \frac{1}{\sqrt{2k_2+1}}; \quad R_2 := \frac{1}{\delta_2}. \end{cases} \tag{6.3}$$

Then iterative sequence $\{(x_n, y_n, u_n, v_n, w_n, z_n)\}$ generated by Iterative Algorithm 5.1 converges strongly to (x, y, u, v, w, z), a solution of SMVI(4.1).

Proof. Since for each $i = 1, 2$, it follows from Theorem 3.5 that for $(x, y) \in E_1 \times E_2$, the proximal point mappings $J^{M_1(\cdot,x)}_{P_1,\rho_1}$ and $J^{M_2(\cdot,y)}_{P_2,\rho_2}$ are $\frac{1}{\delta_1}$-Lipschitz continuous and $\frac{1}{\delta_2}$-Lipschitz continuous, respectively. Now, since $(g_i - I_i)$ is k_i-strongly accretive, we have the following estimate:

$$\begin{aligned}
&\|x_{n+1} - x_n\|_1^2 \\
&\quad = \|g_1(x_{n+1}) - g_1(x_n) + x_{n+1} - x_n - (g_1(x_{n+1}) - g_1(x_n))\|_1^2 \\
&\quad \le \|g_1(x_{n+1}) - g_1(x_n)\|_1^2 - 2\langle (g_1 - I_1)(x_{n+1}) - (g_1 - I_1)(x_n), J_1(x_{n+1} - x_n)\rangle_1 \\
&\quad \le \|g_1(x_{n+1}) - g_1(x_n)\|_1^2 - 2k_1\|x_{n+1} - x_n\|_1^2
\end{aligned}$$

which implies that

$$\|x_{n+1} - x_n\|_1 \ \le \frac{1}{\sqrt{2k_1 + 1}} \ \|g_1(x_{n+1}) - g_1(x_n)\|_1. \tag{6.4}$$

Similarly, we have

$$\|y_{n+1} - y_n\|_2 \ \le \frac{1}{\sqrt{2k_2 + 1}} \ \|g_2(y_{n+1}) - g_2(y_n)\|_2. \tag{6.5}$$

Now, by using (5.1) and (6.1), we have

$$\begin{aligned}
&\|g_1(x_{n+1}) - g_1(x_n)\|_1 \\
&\quad = \|J^{M_1(\cdot,x_n)}_{P_1,\rho_1}(P_1 \circ g_1(x_n) - \rho_1 N_1(u_n, v_n)) - J^{M_1(\cdot,x_{n-1})}_{P_1,\rho_1}(P_1 \circ g_1(x_{n-1}) \\
&\qquad - \rho_1 N_1(u_{n-1}, v_{n-1}))\|_1 \\
&\quad \le \|J^{M_1(.,x_n)}_{P_1,\rho_1}(P_1 \circ g_1(x_n) - \rho_1 N_1(u_n, v_n)) - J^{M_1(.,x_n)}_{P_1,\rho_1}(P_1 \circ g_1(x_{n-1}) \\
&\qquad - \rho_1 N_1(u_{n-1}, v_{n-1}))\|_1 + \|J^{M_1(\cdot,x_n)}_{P_1,\rho_1}(P_1 \circ g_1(x_{n-1}) - \rho_1 N_1(u_{n-1}, v_{n-1})) \\
&\qquad - J^{M_1(\cdot,x_{n-1})}_{P_1,\rho_1}(P_1 \circ g_1(x_{n-1}) - \rho_1 N_1(u_{n-1}, v_{n-1}))\|_1 \\
&\quad \le \frac{1}{\delta_1}\big(\|P_1 \circ g_1(x_n) - P_1 \circ g_1(x_{n-1}) - \rho_1[N_1(u_n, v_n) - N_1(u_{n-1}, v_n)]\|_1 \\
&\qquad + \rho_1\|N_1(u_{n-1}, v_n) - N_1(u_{n-1}, v_{n-1})\|_1\big) + \lambda_1\|x_n - x_{n-1}\|_1.
\end{aligned} \tag{6.6}$$

Further, using α_1-strongly accretivity with respect to $P_1 \circ g_1$ in the first argument and (β_1, γ_1)-mixed Lipschitz continuity of $N_1(\cdot, \cdot)$; μ_1-$\mathcal{H}_1$-Lipschitz continuity of

A; η_1-$\mathcal{H}_2$-Lipschitz continuity of B and Lemma 2.3, it follows that

$$\begin{aligned}
&\|P_1 \circ g_1(x_n) - P_1 \circ g_1(x_{n-1}) - \rho_1(N_1(u_n, v_n) - N_1(u_{n-1}, v_n))\|_1^2 \\
&\quad \le \|P_1 \circ g_1(x_n) - P_1 \circ g_1(x_{n-1})\|_1^2 \\
&\qquad - 2\rho_1\langle N_1(u_n, v_n) - N_1(u_{n-1}, v_n), J_1(P_1 \circ g_1(x_n) - P_1 \circ g_1(x_{n-1}))\rangle_1 \\
&\qquad - 2\rho_1\langle N_1(u_n, v_n) - N_1(u_{n-1}, v_n), J_1(P_1 \circ g_1(x_n) - P_1 \circ g_1(x_{n-1}) \\
&\qquad - \rho_1(N_1(u_n, v_n) - N_1(u_{n-1}, v_n)) - J_1(P_1 \circ g_1(x_n) - P_1 \circ g_1(x_{n-1}))\rangle_1 \\
&\quad \le \|P_1 \circ g_1(x_n) - P_1 \circ g_1(x_{n-1})\|_1^2 \\
&\qquad - 2\rho_1\alpha_1\langle N_1(u_n, v_n) - N_1(u_{n-1}, v_n), J_1(P_1 \circ g_1(x_n) - P_1 \circ g_1(x_{n-1}))\rangle_1 \\
&\qquad + 64c_1\rho_1^2\|N_1(u_n, v_n) - N_1(u_{n-1}, v_n)\|_1^2 \\
&\quad \le (\xi_1^2 - 2\rho_1\alpha_1 + 64c_1\rho_1^2\beta_1^2\mu_1^2(1 + (1 + n)^{-1})^2)\|x_n - x_{n-1}\|_1^2, \qquad (6.7)
\end{aligned}$$

and

$$\|N_1(u_{n-1}, v_n) - N_1(u_{n-1}, v_{n-1})\|_1 \;\le\; \gamma_1\eta_1(1 + (1 + n)^{-1})\|y_n - y_{n-1}\|_2. \qquad (6.8)$$

From (6.4) and (6.6)-(6.8), we have

$$\begin{aligned}
&\|x_{n+1} - x_n\|_1 \\
&\quad \le \frac{1}{\sqrt{2k_1 + 1}}\Bigg[\left(\frac{1}{\delta_1}\left(\sqrt{\xi_1^2 - 2\rho_1\alpha_1 + 64c_1\rho_1^2\beta_1^2\mu_1^2(1 + (1 + n)^{-1})^2}\right) + \lambda_1\right) \\
&\qquad \times \|x_n - x_{n-1}\|_1 + \frac{\rho_1\gamma_1\eta_1}{\delta_1}(1 + (1 + n)^{-1})\|y_n - y_{n-1}\|_2^2\Bigg]. \qquad (6.9)
\end{aligned}$$

Also, by using (5.2) and (6.2), we have

$$\begin{aligned}
&\|g_2(y_{n+1}) - g_2(y_n)\|_2 \\
&\quad = \|J_{P_2,\rho_2}^{M_2(\cdot,y_n)}(P_2 \circ g_2(y_n) - \rho_1 N_2(w_n, z_n)) - J_{P_2,\rho_2}^{M_2(\cdot,y_{n-1})}(P_2 \circ g_2(y_{n-1}) \\
&\qquad - \rho_2 N_2(w_{n-1}, z_{n-1}))\|_2 \\
&\quad \le \|J_{P_2,\rho_2}^{M_2(\cdot,y_n)}(P_2 \circ g_2(y_n) - \rho_2 N_2(w_n, z_n)) - J_{P_2,\rho_2}^{M_2(\cdot,y_n)}(P_2 \circ g_2(y_{n-1}) \\
&\qquad - \rho_2 N_2(w_{n-1}, z_{n-1}))\|_2 + \|J_{P_2,\rho_2}^{M_2(\cdot,y_n)}(P_2 \circ g_2(y_{n-1}) \\
&\qquad - \rho_2 N_2(w_{n-1}, z_{n-1})) - J_{P_2,\rho_2}^{M_2(\cdot,y_{n-1})}(P_2 \circ g_2(y_{n-1}) - \rho_2 N_2(w_{n-1}, z_{n-1}))\|_2 \\
&\quad \le \frac{1}{\delta_2}\big(\|P_2 \circ g_2(y_n) - P_2 \circ g_2(y_{n-1}) - \rho_2[N_2(w_n, z_n) - N_2(w_n, z_{n-1})]\|_2 \\
&\qquad + \rho_2\|N_2(w_n, z_{n-1}) - N_2(w_{n-1}, z_{n-1})\|_2\big) + \lambda_2\|y_n - y_{n-1}\|_2. \qquad (6.10)
\end{aligned}$$

Further, using α_2-strongly accretivity with respect to $P_2 \circ g_2$ in the second argument and (β_2, γ_2)-mixed Lipschitz continuity of $N_2(\cdot, \cdot)$; μ_2-$\mathcal{H}_1$-Lipschitz continuity of C; η_2-$\mathcal{H}_2$-Lipschitz continuity of D and Lemma 2.3, it follows that

$$\begin{aligned}
&\|P_2\circ g_2(y_n)-P_2\circ g_2(y_{n-1})-\rho_2(N_2(w_n,z_n)-N_2(w_n,z_{n-1}))\|_2^2\\
&\quad\le \|P_2\circ g_2(y_n)-P_2\circ g_2(y_{n-1})\|_2^2-2\rho_2\langle N_2(w_n,z_n)-N_2(w_n,z_{n-1}),\\
&\qquad J_2(P_2\circ g_2(y_n)-P_2\circ g_2(y_{n-1}))\rangle_2-2\rho_2\langle N_2(w_n,z_n)-N_2(w_n,z_{n-1}),\\
&\qquad J_2(P_2\circ g_2(y_n)-P_2\circ g_2(y_{n-1})-\rho_2(N_2(w_n,z_n)-N_2(w_n,z_{n-1}))\\
&\qquad -J_2(P_2\circ g_2(y_n)-P_2\circ g_2(y_{n-1}))\rangle_2\\
&\quad\le \|P_2\circ g_2(y_n)-P_2\circ g_2(y_{n-1})\|_2^2-2\rho_2\langle N_2(w_n,z_n)-N_2(w_n,z_{n-1}),\\
&\qquad J_2(P_2\circ g_2(y_n)-P_2\circ g_2(y_{n-1}))\rangle_2+64c_2\rho_2^2\|N_2(w_n,z_n)-N_2(w_n,z_{n-1})\|_2^2\\
&\quad\le (\xi_2^2-2\rho_2\alpha_2+64c_2\rho_2^2\gamma_2^2\eta_2^2(1+(1+n)^{-1})^2)\|y_n-y_{n-1}\|_2^2, \qquad (6.11)
\end{aligned}$$

and

$$\|N_2(w_n,z_{n-1})-N_2(w_{n-1},z_{n-1})\|_2 \;\le\; \beta_2\mu_2(1+(1+n)^{-1})\|x_n-x_{n-1}\|_1. \quad (6.12)$$

From (6.5) and (6.10)-(6.12), we have

$$\begin{aligned}
&\|y_{n+1}-y_n\|_2\\
&\quad\le \frac{1}{\sqrt{2k_2+1}}\Big[\left(\frac{1}{\delta_2}\left(\sqrt{\xi_2^2-2\rho_2\alpha_2+64c_2\rho_2^2\gamma_2^2\eta_2^2(1+(1+n)^{-1})^2}\right)+\lambda_2\right)\\
&\qquad\times\|y_n-y_{n-1}\|_2+\frac{\rho_2\beta_2\mu_2}{\delta_2}(1+(1+n)^{-1})\|x_n-x_{n-1}\|_1^2\Big]. \qquad (6.13)
\end{aligned}$$

From (6.9) and (6.13), we have

$$\begin{aligned}
\|x_{n+1}-x_n\|_1+\|y_{n+1}-y_n\|_2 &= k_1^n\|x_n-x_{n-1}\|_1+k_2^n\|y_n-y_{n-1}\|_2\\
&\le \theta^n(\|x_n-x_{n-1}\|_1+\|y_n-y_{n-1}\|_2), \qquad (6.14)
\end{aligned}$$

where $\theta^n=\max\{k_1^n,k_2^n\}$,

$$\begin{cases}
k_1^n := L_1\Big[R_1\left(\sqrt{\xi_1^2-2\rho_1\alpha_1+64c_1\rho_1^2\beta_1^2\mu_1^2(L^n)^2}\right)+\lambda_1\Big]+\rho_2\beta_2\mu_2R_2L_2L^n;\\
k_2^n := L_2\Big[R_2\left(\sqrt{\xi_2^2-2\rho_2\alpha_2+64c_2\rho_2^2\gamma_2^2\eta_2^2(L^n)^2}\right)+\lambda_2\Big]+\rho_1\gamma_1\eta_1R_1L_1L^n; \quad (6.15)\\
L_1:=\frac{1}{\sqrt{2k_1+1}};\quad R_1:=\frac{1}{\delta_1};\quad L_2:=\frac{1}{\sqrt{2k_2+1}};\quad R_2:=\frac{1}{\delta_2};\quad L^n:=(1+(1+n)^{-1}).
\end{cases}$$

Letting $\theta^n\to\theta$ as $n\to\infty$ ($k_1^n\to k_1, k_2^n\to k_2$ as $n\to\infty$), where $\theta = \max\{k_1,k_2\}$;

$$\begin{cases}
k_1 := L_1\Big[R_1\left(\sqrt{\xi_1^2-2\rho_1\alpha_1+64c_1\rho_1^2\beta_1^2\mu_1^2}\right)+\lambda_1\Big]+\rho_2\beta_2\mu_2R_2L_2;\\
k_2 := L_2\Big[R_2\left(\sqrt{\xi_2^2-2\rho_2\alpha_2+64c_2\rho_2^2\gamma_2^2\eta_2^2}\right)+\lambda_2\Big]+\rho_1\gamma_1\eta_1R_1L_1; \qquad (6.16)\\
L_1:=\frac{1}{\sqrt{2k_1+1}};\quad R_1:=\frac{1}{\delta_1};\quad L_2:=\frac{1}{\sqrt{2k_2+1}};\quad R_2:=\frac{1}{\delta_2}.
\end{cases}$$

Now, define the norm $\|\cdot\|_*$ on $E_1 \times E_2$ by

$$\|(x,y)\|_* \;=\; \|x\|_1 + \|y\|_2,\ \forall (x,y) \in E_1 \times E_2. \tag{6.17}$$

It is observed that $(E_1 \times E_2, \|\cdot\|_*)$ is a Banach space. Hence (6.14) implies that

$$\|(x_{n+1}, y_{n+1}) - (x_n, y_n)\|_* \;\leq \theta \|(x_n, y_n) - (x_{n-1}, y_{n-1})\|_*. \tag{6.18}$$

By condition (6.16), it follows that $\theta < 1$. Hence $\theta_n < 1$ for sufficiently large n. Therefore, (6.18) implies that $\{(x_n, y_n)\}$ is a Cauchy sequence in $E_1 \times E_2$. Let $(x_n, y_n) \to (x, y) \in E_1 \times E_2$ as $n \to \infty$. By μ_1-$\mathcal{H}$-Lipschitz continuity of A, we have

$$\begin{aligned} ||u_n - u_{n-1}||_1 &\leq (1 + (1+n)^{-1})\ \mathcal{H}_1(A(x_n), A(x_{n-1})) \\ &\leq (1 + (1+n)^{-1})\ \mu_1 ||x_n - x_{n-1}||_1. \end{aligned} \tag{6.19}$$

Since $\{x_n\}$ is a Cauchy sequence in E_1. Hence there exists $u \in E_1$ such that $\{u_n\} \to u$ as $n \to \infty$. Similarly, we can show that $\{v_n\} \in E_2, \{w_n\} \in E_1$ and $\{z_n\} \in E_2$ are Cauchy sequences and hence there exist $v \in E_2, w \in E_1$ and $z \in E_2$ such that $\{v_n\} \to v, \{w_n\} \to w$ and $\{z_n\} \to z$ as $n \to \infty$.

Next, we claim that $u \in A(x)$. Since $u_{n-1} \in A(x_{n-1})$, we have

$$\begin{aligned} d(u, A(x)) &\leq ||u - u_{n-1}||_1 + d(u_{n-1}, A(x)) \\ &\leq ||u_- u_{n-1}||_1 + \mathcal{H}_1(A(x_{n-1}), A(x)) \\ &\leq ||u - u_{n-1}||_1 + \mu_1\ ||x_{n-1} - x||_1 \to 0 \text{ as } n \to \infty. \end{aligned} \tag{6.20}$$

Since $A(x)$ is closed, we have $u \in A(x)$. Similarly, we can show that $v \in B(y), w \in C(x)$ and $z \in D(y)$. Furthermore, continuity of the mappings $g_1, g_2, A, B, C, D, K_1\ , K_2, M_1, M_2, N_1, N_2, J^{M_1(.,x)}_{P_1,\rho_1}, J^{M_2(.,y)}_{P_2,\rho_2}$ and Iterative Algorithm 5.1 gives that

$$g_1(x) = J^{M_1(.,x)}_{P_1,\rho_1}[P_1 \circ g_1(x) - \rho_1 N_1(u,v)], \tag{6.21}$$

$$g_2(x) = J^{M_2(.,y)}_{P_2,\rho_2}[P_2 \circ g_2(y) - \rho_2 N_2(w,z)]. \tag{6.22}$$

Finally, we define

$$w_1 = J^{M_1(.,x)}_{P_1,\rho_1}[P_1 \circ g_1(x) - \rho_1 N_1(u,v)], \tag{6.23}$$

$$w_2 = J^{M_2(.,y)}_{P_1,\rho_2}[P_2 \circ g_2(y) - \rho_2 N_2(w,z)]. \tag{6.24}$$

Using the similar arguments used to obtain (6.9) and (6.13) and using Lemma 2.5(b), we have the following estimates:

$$\begin{aligned} \|g_1(x_{n+1}) - w_1\|_1 \leq L_1 &\left(R_1 \left(\sqrt{\xi_1^2 - 2\rho_1\alpha_1 + 64c_1\rho_1^2\beta_1^2\mu_1^2\delta^2} \right) + \lambda_1 \right) \|x_n - x\|_1 \\ &+ \rho_1\gamma_1\eta_1 R_1 L_1 \delta^2 \|y_n - y\|_2^2, \end{aligned} \tag{6.25}$$

and

$$\|g_2(x_{n+1}) - w_2\|_2 \leq L_2 \left(R_2 \left(\sqrt{\xi_2^2 - 2\rho_2\alpha_2 + 64c_2\rho_2^2\gamma_2^2\eta_2^2\delta^2} \right) + \lambda_2 \right) \|y_n - y\|_2 + \rho_2\beta_2\mu_2 R_2 L_2 \delta^2 \|x_n - x\|_1^2. \tag{6.26}$$

Now, it follows from (6.17), (6.25) and (6.26) that

$$\begin{aligned} \|(g_1(x_{n+1}), g_2(y_{n+1})) - (w_1, w_2)\|_* &= \|g_1(x_{n+1}) - w_1\|_1 + \|g_2(y_{n+1}) - w_2\|_2 \\ &\leq L(\|x_n - x\|_1 + \|y_n - y\|_2), \end{aligned} \tag{6.27}$$

where

$$\begin{aligned} L &= \max\{l_1, l_2\}, \\ l_1 &:= L_1 \left(R_1 \sqrt{\xi_1^2 - 2\rho_1\alpha_1 + 64c_1\rho_1^2\beta_1^2\mu_1^2\delta^2} + \lambda_1 \right) + \rho_2\beta_2\mu_2 R_2 L_2 \delta^2 \\ l_2 &:= L_2 \left(R_2 \sqrt{\xi_2^2 - 2\rho_2\alpha_2 + 64c_2\rho_2^2\gamma_2^2\eta_2^2\delta^2} + \lambda_2 \right) + \rho_1\gamma_1\eta_1 R_1 L_1 \delta^2. \end{aligned}$$

Assume that l_1, l_2 are positive real numbers, then it follows from (6.27) that

$$\|(g_1(x_{n+1}), g_2(y_{n+1})) - (w_1, w_2)\|_* \to 0 \quad \text{as} \quad n \to \infty.$$

Thus,

$$g_1(x) = w_1 = J_{P_1,\rho_1}^{M_1(.,x)}[P_1 \circ g_1(x) - \rho_1 N_1(u, v)], \tag{6.28}$$

$$g_2(y) = w_2 = J_{P_2,\rho_2}^{M_2(.,y)}[P_2 \circ g_2(y) - \rho_1 N_2(w, z)]. \tag{6.29}$$

By Lemma 4.3, it follows that (x, y, u, v, w, z) is a solution of SMVI(4.1). This completes the proof. □

Remark 6.2.

(i) For $\lambda_1, \lambda_2, \rho_1, \rho_2 > 0$, it is clear that $\alpha_1 \leq \beta_1\mu_1$; $\alpha_2 \leq \gamma_2\eta_2$; $c_1 > \frac{2\rho_1\alpha_1 - \xi_1^2}{64\rho_1^2\beta_1^2\mu_1^2}$; $c_2 > \frac{2\rho_2\alpha_2 - \xi_2^2}{64\rho_2^2\gamma_2^2\eta_2^2}$; $2k_1 + 1 > 0$; $2k_2 + 1 > 0$. Further, $\theta < 1$ and condition (6.3) holds for some suitable set values of constants, for example,

- $\alpha_1 = .4$, $\beta_1 = .5$, $\gamma_1 = .4$, $\rho_1 = .1$, $\delta_1 = .2$, $\xi_1 = .1$, $k_1 = .1$, $\mu_1 = 1$, $\eta_1 = 1$.
- $\alpha_2 = .2$, $\beta_2 = .3$, $\gamma_2 = .2$, $\rho_2 = .2$, $\delta_2 = .3$, $\xi_2 = .2$, $k_2 = .2$, $\mu_2 = 1$, $\eta_2 = 1$.

(ii) Theorem 6.1 generalize and improve the corresponding theorems presented in [3–6, 10–12, 14, 15, 19, 20, 24, 25, 29] in the following sense:

- *These theorems are proved in more general space, particularly, in real uniformly smooth Banach space.*

- *Multi-valued mappings with closed and bounded values have been considered instead of multi-valued mappings with compact values.*
- *More general concepts of strongly accretivity, P-accretivity and mixed-Lipschitz continuity are considered.*

(iii) Using the method presented in this paper, one can extend the existence result for the system of n-generalized variational-like inclusions involving multi-valued mappings.

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