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$\omega\alpha$-Compactness and $\omega\alpha$-Connectedness in Topological Spaces

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Abstract : In this paper the concepts of $\omega\alpha$-compactness and $\omega\alpha$-connectedness are introduced and some of their properties are obtained using $\omega\alpha$-closed sets.

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1 Introduction

The notions of compactness and connectedness are very useful and fundamental notions of general topology also in the other advanced branches of mathematics. Many researchers [1–6] have investigated the basic properties of compactness and connectedness.

Recently, Benchalli et al. [7, 8] introduced and studied a new class of closed sets called $\omega\alpha$-closed sets and continuous maps in topological space. The aim of this paper is to introduce the concept of $\omega\alpha$-compactness and $\omega\alpha$-connectedness in topological spaces and is to give some characterization of $\omega\alpha$-compactness and $\omega\alpha$-connectedness. Further it is proved that $\omega\alpha$-connectedness is preserved under $\omega\alpha$- irresolute surjections.

2 Preliminaries

Throughout this paper (X,τ), (Y,σ) and (Z,η)(or simply X, Y and Z) represent topological spaces on which no separation axioms are assumed unless otherwise metioned. For a subset A of (X,τ), $cl(A)$, $Int(A)$, $\alpha cl(A)$ and A^c denote the closure of A, inerior of A, α-closure of A and the compliment of A in X respecively.

The following definitions are useful in the sequel.

Definition 2.1. Let (X,τ) be a topological space.Then,

1. A subset A of X is called $\omega\alpha$-closed set [7] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is ω-open in (X,τ).
2. A topological space (X,τ) is said to be GO-compact [1] (resp. αGO-compact [2]) if every g-open(resp. αg-open) cover of (X,τ) has a finite subcover.
3. A topological space (X,τ) is said to be GPR-compact [4] (resp. ω-compact [6]) if every GPR-open(resp. ω-open) cover of (X,τ) has a finite subcover.
4. A map $f:(X,\tau)\to(Y,\sigma)$ is called $\omega\alpha$-continuous [8] (resp. $\omega\alpha$-irresolute) if the inverse image of every closed (resp. $\omega\alpha$-closed) set in (Y,σ) is $\omega\alpha$-closed in (X,τ).
5. $f:(X,\tau)\to(Y,\sigma)$ is called strongly $\omega\alpha$-continuous [9] (resp. perfectly $\omega\alpha$-continuous) if the inverse image of every $\omega\alpha$-closed (resp. $\omega\alpha$-closed) set in (Y,σ) is closed (resp. clopen) in (X,τ).
6. $f:(X,\tau)\to(Y,\sigma)$ is called strongly g^*-continuous [5] if the inverse image of every strongly g closed set in (Y,σ) is closed in (X,τ)
7. A topological space (X,τ) is said to be $T_{\omega\alpha}$-space [7] (resp ${}_{\alpha g}T_{\omega\alpha}$, ${}_{\omega\alpha}T_{stg}$ space) if every $\omega\alpha$-closed (resp. αg-closed, $\omega\alpha$-closed) set is closed (resp. $\omega\alpha$-closed, strongly g-closed).

3 $\omega\alpha$-Compactness in Topological Spaces

In this section, we introduce the concept of $\omega\alpha$-compactness and studied some of their properties.

Definition 3.1. A collection $\{A_i : i \in I\}$ of $\omega\alpha$-open sets in a topological space (X,τ) is called a $\omega\alpha$- open cover of a subset A in (X,τ) if $A \subseteq \bigcup_{i\in I} A_i$.

Definition 3.2. A topological space(X,τ) is called $\omega\alpha$-compact if every $\omega\alpha$-open cover of (X,τ) has a finite subcover.

Definition 3.3. A subset A of a topological space (X,τ) is called $\omega\alpha$-compact relative to (X,τ) if for every collection $\{A_i : i\in I\}$ of $\omega\alpha$-open subsets of (X,τ) such that $A \subseteq \bigcup_{i\in I} A_i$, there exists a finite subset I_0 of I such that $A \subseteq \bigcup_{i\in I_0} A_i$.

Theorem 3.1. *A $\omega\alpha$-closed subset of $\omega\alpha$- compact space (X, τ) is $\omega\alpha$- compact relative to (X, τ).*

Proof. Let A be a $\omega\alpha$- closed subset of a topological space (X, τ). Then A^c is $\omega\alpha$-open in (X, τ). Let $S = \{A_i : i \in I\}$ be an $\omega\alpha$-open cover of A by $\omega\alpha$-open subsets in (X, τ). Then $S^* = S \cup A^c$ is a $\omega\alpha$- open cover of (X, τ).That is $X = (\bigcup_{i \in I} A_i) \bigcup A^c$. By hypothesis (X, τ) is $\omega\alpha$-compact and hence S^* is reducible to a finite subcover of (X, τ) say $X = A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_n} \cup A^c$, $A_{i_k} \in S^*$. But A and A^c are disjoint. Hence $A \subseteq A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_n} \in S$. Thus a $\omega\alpha$-open cover S of A contains a finite subcover. Hence A is $\omega\alpha$-compact relative to (X, τ). □

Theorem 3.2. *Every $\omega\alpha$-compact space is compact.*

Proof. Let (X, τ) be a $\omega\alpha$-compact space. Let $\{A_i : i \in I\}$ be an open cover of (X, τ). By [7], $\{A_i : i \in I\}$ is a $\omega\alpha$-open cover of (X, τ). Since (X, τ) is $\omega\alpha$-compact,$\omega\alpha$-open cover$\{A_i : i \in I\}$ of (X, τ) has a finite subcover say $\{A_i : i = 1, \dots, n\}$ for X. Hence (X, τ) is compact. □

Theorem 3.3. *If (X, τ) is compact and $T_{\omega\alpha}$ -space, then (X, τ) is $\omega\alpha$- compact.*

Theorem 3.4. *Every αGO-compact space is $\omega\alpha$-compact.*

Proof. Let (X, τ) be a αGO-compact space. Let $\{A_i : i \in I\}$ be an $\omega\alpha$-open cover of (X, τ) by $\omega\alpha$-open sets in (X, τ). From [7], $\{A_i : i \in I\}$ is αg-open cover of (X, τ) by αg-open sets. Since (X, τ) is αGO compact, the αg-open cover $\{A_i : i \in I\}$ of (X, τ) has a finite subcover say $\{A_i : i = 1, \dots, n\}$ of (X, τ).Hence (X, τ) is $\omega\alpha$-compact. □

Theorem 3.5. *If (X, τ) is $\omega\alpha$-compact and ${}_{\alpha g}T_{\omega\alpha}$, then (X, τ) is αGO-compact.*

Theorem 3.6. *Every GPR compact space is $\omega\alpha$-compact.*

Proof. Let (X, τ) be a GPR-copmact space. Let $\{A_i : i \in I\}$ be an $\omega\alpha$-open cover of (X, τ) by $\omega\alpha$-open sets. From [7], $\{A_i : i \in I\}$ is gpr-open cover of (X, τ), since (X, τ) is GPR-compact, the gpr-open cover $\{A_i : i \in I\}$ of (X, τ) has a finite sub cover say $\{A_i : i = 1, \dots, n\}$. Hence (X, τ) is $\omega\alpha$-compact. □

Theorem 3.7. *A $\omega\alpha$-closed subset of αGO-compact space (X, τ) is αGO-compact relative to (X, τ).*

Proof. From [7], every $\omega\alpha$-closed set is αg-closed and since αg-closed subset of a αGO-compact space is αGO-compact relative to (X, τ) [2], the result follows. □

Theorem 3.8. *The image of a $\omega\alpha$-compact space under $\omega\alpha$-continuous onto map is compact.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a $\omega\alpha$-continuoous map from $\omega\alpha$-compact space (X, τ) on to a topological space (Y, σ). Let $\{A_i : i \in I\}$ be an open cover of (Y, σ). Then $\{f^{-1}(A_i) : i \in I\}$ is a $\omega\alpha$ open cover of (X, τ), as f is $\omega\alpha$- continuous. Since (X, τ) is $\omega\alpha$-compact, the $\omega\alpha$-open cover of (X, τ),$\{f^{-1}(A_i) : i \in I\}$ has a finite subcover say $\{f^{-1}(A_i) : i = 1, \ldots, n\}$. Therefore $X = \bigcup_{i=1}^{n} f^{-1}(A_i)$ which implies $f(X) = \bigcup_{i=1}^{n}(A_i)$. Then $Y = \bigcup_{i=1}^{n}(A_i)$.That is $\{A_1, A_2, \ldots, A_n\}$ is a finite subcover of $\{A_i : i \in I\}$ for (Y, σ). Hence (Y, σ) is compact. □

Theorem 3.9. *If a map $f : (X, \tau) \to (Y, \sigma)$ is $\omega\alpha$-irresolute and a subset S of X is $\omega\alpha$-compact relative to (X, τ), then the image $f(S)$ is $\omega\alpha$-compact relative to (Y, σ).*

Proof. Let $\{A_i : i \in I\}$ be a collection of $\omega\alpha$-open sets in (Y, σ), such that $f(S) \subseteq \bigcup_{i \in I} A_i$. Then $S \subseteq \bigcup_{i=1}^{n} f^{-1}(A_i)$, where $\{f^{-1}(A_i : i \in I\}$ is $\omega\alpha$-open set in (X, τ). Since S is $\omega\alpha$-compact relative to (X, τ), there exists finite subcollection $\{A_1, A_2, \ldots, A_n\}$ such that $S \subseteq \bigcup_{i=1}^{n} f^{-1}(A_i)$. That is $f(S) \subseteq \bigcup_{i=1}^{n} A_i$. Hence $f(S)$ is $\omega\alpha$-compact relative to (Y, σ). □

Theorem 3.10. *If a map $f : (X, \tau) \to (Y, \sigma)$ is strongly $\omega\alpha$-continuous map from a compact space (X, τ) onto a topological space (Y, σ), then (Y, σ) is $\omega\alpha$-compact.*

Proof. Let $\{A_i : i \in I\}$ be an $\omega\alpha$-open cover of $(Y \sigma)$. Since f is strongly $\omega\alpha$-continuous, $\{f^{-1}(A_i : i \in I\}$ is an open cover of (X, τ). Again since (X, τ) is compact, the open cover $\{f^{-1}(A_i) : i \in I\}$ of (X, τ) has a finite subcover say $\{f^{-1}(A_i) : i = 1, \ldots, n\}$. Therefore $X = \bigcup_{i=1}^{n} f^{-1}(A_i)$, which implies $f(X) = \bigcup_{i=1}^{n} A_i$, so that $Y = \bigcup_{i=1}^{n} A_i$. That is $\{A_1, A_2, \ldots, A_n\}$ is a finite subcover of $\{A_i : i \in I\}$ for $(Y \sigma)$. Hence (Y, σ) is compact. □

Theorem 3.11. *If a map $f : (X, \tau) \to (Y, \sigma)$ is perfectly $\omega\alpha$-continuous map from a compact spce (X, τ) onto a topological space (Y, σ), then (Y, σ) is $\omega\alpha$-compact.*

Theorem 3.12. *A topological space (X, τ) is $\omega\alpha$-compact if and only if every family of $\omega\alpha$-closed sets of (X, τ) having finite intersection property has a non-empty intersection.*

Proof. Suppose (X, τ) is $\omega\alpha$-compact. Let $\{A_i : i \in I\}$ be a family of $\omega\alpha$ closed sets with finite intersection property.

Suppose $\bigcap_{i \in I} A_i = \phi$. Then $X - \bigcap_{i \in I} A_i = X$. This implies $\bigcup_{i \in I}(X - A_i) = X$. Thus the cover $\{X - A_i : i \in I\}$ is a $\omega\alpha$-open cover of (X, τ). Then, the $\omega\alpha$-open cover $\{X - A_i : i \in I\}$ has a finite subcover say $\{X - A_i : i = 1, \ldots, n\}$. This implies $X = \bigcup_{i \in I}(X - A_i)$ which implies $X = X - \bigcap_{i=1}^{n} A_i$, which implies $X - X = X - [X - \bigcap_{i=1}^{n} A_i]$ which implies $\phi = \bigcap_{i=1}^{n} A_i$. This contradicts the assumption. Hence $\bigcap_{i \in I} A_i \neq \phi$.

Conversely suppose (X, τ) is not $\omega\alpha$-compact. Then there exists an $\omega\alpha$-open cover of (X, τ) say $\{G_i : i \in I\}$ having no finite subcover. This implies for any finite subfamily $\{G_i : i = 1, \ldots, n\}$ of $\{G_i : i \in I\}$, we have $\bigcup_{i=1}^{n} G_i \neq X$, which implies $X - \bigcup_{i=1}^{n} G_i \neq X - X$, which implies $\bigcap_{i \in I}(X - G_i) \neq \phi$. Then the family $\{X - G_i : i \in I\}$ of $\omega\alpha$- closed sets has a finite intersection property. Also by assumption $\bigcap_{i\in I}(X - G_i) \neq \phi$ which implies $X - \bigcup_{i=1}^{n} G_i \neq \phi$, so that $\bigcup_{i=1}^{n} G_i \neq X$. This implies $\{G_i : i \in I\}$ is not a cover of (X, τ). This contradicts the fact that $\{G_i : i \in I\}$ is a cover for (X, τ). Therefore a $\omega\alpha$-open cover $\{G_i : i \in I\}$ of (X, τ) has a finite subcover $\{G_i : i = 1, \ldots, n\}$. Hence (X, τ) is $\omega\alpha$-compact. □

Theorem 3.13. *The image of a $\omega\alpha$-compact space under a strongly $\omega\alpha$-continuous map is $\omega\alpha$-compact.*

Theorem 3.14. *The image of a $\omega\alpha$-compact space under a $\omega\alpha$-irresolute map is $\omega\alpha$-compact.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ is $\omega\alpha$-irresolute map from a compact space (X, τ) onto a topological space (Y, σ). Let $\{A_i : i \in I\}$ be an $\omega\alpha$-open cover of (Y, σ). Then $\{f^{-1}(A_i) : i \in I\}$ is a $\omega\alpha$-open cover of (X, τ), since f is $\omega\alpha$-irresolute. As (X, τ) is $\omega\alpha$- compact, the $\omega\alpha$-open cover $\{f^{-1}(A_i) : i \in I\}$ of (X, τ) has a finite subcover say $\{f^{-1}(A_i) : i = 1, \ldots, n\}$. Therefore $X = \bigcup_{i=1}^{n} f^{-1}(A_i)$.Then $f(X) = \bigcup_{i=1}^{n} A_i$, that is $Y = \bigcup_{i=1}^{n} A_i$. Thus $\{A_1, A_2, \ldots, A_n\}$ is a finite subcover of $\{A_i : i \in I\}$ for (Y, σ). Hence (Y, σ) is $\omega\alpha$- compact. □

4 Countably $\omega\alpha$-Compactness in Topological Spaces

In this section, we study the concept of Countably $\omega\alpha$-compactness and their properties.

Definition 4.1. A topological space (X, τ) is said to be countably $\omega\alpha$-compact if every countable $\omega\alpha$-open cover of (X, τ) has a finite subcover.

Theorem 4.1. *If (X, τ) is a countably $\omega\alpha$-compact space, then (X, τ) is countably compact*

Theorem 4.2. *If (X, τ) is a countably compact and $T_{\omega\alpha}$-space, then (X, τ) is countably $\omega\alpha$-compact.*

Theorem 4.3. *Every $\omega\alpha$-compact space is countably $\omega\alpha$-compact.*

Theorem 4.4. *If $f : (X, \tau) \to (Y, \sigma)$ is $\omega\alpha$-continuous map form a countably $\omega\alpha$-compact space (X, τ) onto a topological space (Y, σ), then (Y, σ) is countably compact.*

Theorem 4.5. *Let $f : (X, \tau) \to (Y, \sigma)$ is strongly $\omega\alpha$-continuous map form a countably compact space (X, τ) onto a topological space (Y, σ), then $(Y, \sigma))$ is countably $\omega\alpha$-compact.*

Theorem 4.6. *Let $f : (X, \tau) \to (Y, \sigma)$ is strongly g^*-continuous map form a countably compact space (X, τ) onto a topological space (Y, σ) and if (Y, σ) is ${}_{\omega\alpha}T_{stg}$ -space, then (Y, σ) countably $\omega\alpha$-compact.*

Proof. Let $\{A_i : i \in I\}$ be an countably $\omega\alpha$-open cover of (Y, σ) by $\omega\alpha$-open sets. Since (Y, σ) is ${}_{\omega\alpha}T_{stg}$-space ,$\{A_i : i \in I\}$ is countably strongly g-open cover of (Y, σ). Then $\{f^{-1}(A_i) : i \in I\}$ is a countable open cover of (X, τ), since f is strongly g^*-continuous map. As (X, τ) is countably compact, the countable open cover $\{f^{-1}(A_i) : i \in I\}$ of (X, τ) has a finite subcove say $\{f^{-1}(A_i) : i = 1, \ldots, n\}$. Therefore $X = \bigcup_{i=1}^{n} f^{-1}(A_i)$.Then $f(X) = \bigcup_{i=1}^{n} A_i$, that is $Y = \bigcup_{i=1}^{n} A_i$. Thus $\{A_1, A_2, \ldots, A_n\}$ is a finite subcover of $\{A_i : i \in I\}$ for (Y, σ). Hence (Y, σ) is $\omega\alpha$-compact. □

Theorem 4.7. *If a map $f : (X, \tau) \to (Y, \sigma))$ is perfectly $\omega\alpha$-continuous map form a countably compact space (X, τ) onto a topological space (Y, σ), then (Y, σ) is countably $\omega\alpha$-compact.*

Theorem 4.8. *The image of a countably $\omega\alpha$-compact space under $\omega\alpha$-irresolute map is countably $\omega\alpha$-compact.*

Theorem 4.9. *A space (X, τ) is countably $\omega\alpha$-compact if and only if every countable family of $\omega\alpha$-closed sets of (X, τ) having finite intersection property has a non-empty intersection.*

Theorem 4.10. *A $\omega\alpha$-closed subset of a countably $\omega\alpha$-compact space is countably $\omega\alpha$-compact.*

Definition 4.2. A topological space (X, τ) is said to be $\omega\alpha$-Lindelöf space if every $\omega\alpha$-open cover of (X, τ) has a countable subcover.

Theorem 4.11. *Every $\omega\alpha$-Lindelöf space is Lindelöf space.*

Theorem 4.12. *If (X, τ) is Lindelöf and $T_{\omega\alpha}$- space, then (X, τ) is $\omega\alpha$-Lindelöf space.*

Theorem 4.13. *Every $\omega\alpha$-compact space is $\omega\alpha$-Lindelöf space.*

Proof. Let (X, τ) is $\omega\alpha$-compact space. Let $\{A_i : i \in I\}$ be an $\omega\alpha$-open cover of (X, τ). Then $\{A_i : i \in I\}$ has a finite subcover say $\{A_i : i = 1, \ldots, n\}$, since (X, τ) is $\omega\alpha$-compact. Since every finite subcover is always a countable subcover and therefore, $\{A_i : i = 1, \ldots, n\}$ is countable subcover of $\{A_i : i \in I\}$ for (X, τ). Hence (X, τ) is $\omega\alpha$-Lindelöf space. □

Theorem 4.14. *If a map $f : (X, \tau) \to (Y, \sigma)$ is $\omega\alpha$-continuous map form a $\omega\alpha$-Lindelöf space (X, τ) onto a topological space (Y, σ), then (Y, σ) is Lindelöf space.*

Theorem 4.15. *The image of a $\omega\alpha$-Lindelöf space under $\omega\alpha$-irresolute map is $\omega\alpha$-Lindelöf.*

Proof. $f : (X, \tau) \to (Y, \sigma)$ is $\omega\alpha$-irresolute map form a $\omega\alpha$-Lindelöf space (X, τ) onto a topological space (Y, σ). Let $\{A_i : i \in I\}$ be an $\omega\alpha$-open cover of (Y, σ), then $\{f^{-1}(A_i) : i \in I\}$ is an $\omega\alpha$-open cover of (X, τ) as f is $\omega\alpha$-irresolute. Since (X, τ) is $\omega\alpha$- Lindelöf, the $\omega\alpha$-open cover$\{f^{-1}(A_i) : i \in I\}$ of (X, τ) has a countable subcover say $\{f^{-1}(A_i) : i = 1, \ldots, n\}$. Therefore $X = \bigcup_{i=1}^{n} f^{-1}(A_i)$ which implies $f(X) = Y = \bigcup_{i=1}^{n} A_i$ that is $\{A_1, A_2, \ldots, A_n\}$ is a countable subfamily of $\{A_i : i \in I\}$ for (Y, σ). Hence (Y, σ) is Lindelöf space. □

Theorem 4.16. *If (X, τ) is $\omega\alpha$-Lindelöf and countably $\omega\alpha$-compact space then (X, τ) is $\omega\alpha$-compact.*

Proof. Suppose (X, τ) is $\omega\alpha$-Lindelöf and countably $\omega\alpha$-compact space. Let $\{A_i : i \in I\}$ be an $\omega\alpha$- open cover of (X, τ). Since (X, τ) is $\omega\alpha$-Lindelöf, $\{A_i : i \in I\}$ has a countable sucover say $\{A_{i_n} : n \in N\}$. Therefore, $\{A_{i_n} : n \in N\}$ is a countable subcover of (X, τ) and $\{A_{i_n} : n \in N\}$ is subfamily of $\{A_i : i \in I\}$ and so $\{A_{i_n} : n \in N\}$ is a countably $\omega\alpha$-open cover of (X, τ). Again since (X, τ) is countably $\omega\alpha$-compact, $\{A_{i_n} : n \in N\}$ has a finite subcover say $\{A_{i_k} : k = 1, \ldots, n\}$. Therefore $\{A_{i_k} : k = 1, \ldots, n\} \subseteq \{A_{i_n} : n \in N\}$ and $\{A_{i_n} : n \in N\} \subseteq \{A_i : i \in I\}$. Therefore $\{A_{i_k} : k = 1, \ldots, n\}$ is a finite subcover of $\{A_i : i \in I\}$ for (X, τ). Hence (X, τ) is $\omega\alpha$-compact space. □

Theorem 4.17. *A $\omega\alpha$-closed subspace of a $\omega\alpha$-Lindelöf space is $\omega\alpha$-Lindelöf.*

Proof. Let (X, τ) be a $\omega\alpha$-Lindelöf space. Let (Y, τ_y) be a $\omega\alpha$-closed subspace of (X, τ). Let $G = \{G_i : i \in I\}$ be an $\omega\alpha$-open cover of (Y, τ_y). Now G_i is open in (Y, τ_y) for all $i \in I$. Now G_i can be expressed as $Y \cap H_i$,that is $G_i = Y \bigcap H_i$ for all $i \in I$ where H_i is $\omega\alpha$-open in (X, τ). Then $\{H_i : i \in I\} \cup (X - Y)$ is an $\omega\alpha$-open cover of (X, τ). Since (X, τ) is $\omega\alpha$-Lindelöf space, there is an $\omega\alpha$-open cover of (X, τ) which has a countable subcover say $\{H_{i_n} : n \in N\} \cup (X - Y)$. Let $u = \{Y \cap H_{i_n} : n \in N\}$. But $Y \cap H_{i_n} = G_{i_n}$, for all $i \in I$. Therefore $u = \{G_{i_n} : n \in N\} \subseteq \{G_i : n \in I\}$, u is a countable subcover of G for $(Y \tau_y)$. Therefore every $\omega\alpha$-open cover of (Y, τ_y) has a countable subcover u. Hence $(Y \tau_y)$ is $\omega\alpha$-Lindelöf space. □

5 $\omega\alpha$-Connectedness in Topological Spaces

Definition 5.1. A topological space (X, τ) is said to be $\omega\alpha$-connected if X cannot be written as a disjoint union of two non empty $\omega\alpha$-open sets.

A subset of (X, τ) is $\omega\alpha$-connected if it is $\omega\alpha$-connected as a subspace.

Theorem 5.1. *For a topological space (X, τ) the following are equivalent:*

1. *(X, τ) is $\omega\alpha$-connected*
2. *The only subsets of (X, τ) which are both $\omega\alpha$-open and $\omega\alpha$-closed are the empty set ϕ and X.*

3. *Each $\omega\alpha$-continuous map of (X,τ) into a discrete space (Y,σ) with at least two points is a constant map.*

Proof. $1 \Rightarrow 2$: Let G be a $\omega\alpha$-open and $\omega\alpha$-closed subset of (X,τ). Then $X-G$ is also both $\omega\alpha$-open and $\omega\alpha$-closed. Then $X = G \cup (X-G)$ a disjoint union of two non-empty $\omega\alpha$-open sets which contradicts the fact that (X,τ) is $\omega\alpha$-connected. Hence $G = \phi$ or X.

$2 \Rightarrow 1$: Suppose that $X = A \cup B$ where A and B are disjoint non-empty $\omega\alpha$-open subsets of (X,τ). Since $A = X - B$, then A is both $\omega\alpha$-open and $\omega\alpha$-closed. By assumption $A = \phi$ or X, which is a contradiction. Hence (X,τ) is $\omega\alpha$-connected.

$2 \Rightarrow 3$: Let $f : (X,\tau) \to (Y,\sigma)$ be a $\omega\alpha$-continuous map, where (Y,σ) is discrete space with at least two points. Then $f^{-1}(\{y\})$ is $\omega\alpha$- closed and $\omega\alpha$-open for each $y \in Y$. That is (X,τ) is covered by $\omega\alpha$ -closed and $\omega\alpha$-open covering $\{f^{-1}(\{y\}) : y \in Y\}$. By assumption, $\{f^{-1}(\{y\}) = \phi$ or X for each $y \in Y$. If $\{f^{-1}(\{y\}) = \phi$ for each $y \in Y$, then f fails to be a map. Therefore there exist at least one point say $f^{-1}(\{y_1\}) \neq \phi$, $y_1 \in Y$ such that $f^{-1}(\{y_1\}) = X$. This shows that f is a constant map.

$3 \Rightarrow 2$: Let G be both $\omega\alpha$-open and $\omega\alpha$-closed in (X,τ). Suppose $G \neq \phi$. Let $f : (X,\tau) \to (Y,\sigma)$ be a $\omega\alpha$-continuous map defined by $f(G) = \{a\}$ and $f(X-G) = \{b\}$ where $a \neq b$ and $a, b \in Y$. By assumption, f is constant so $G = X$. □

Theorem 5.2. *Every $\omega\alpha$-connected space is connected but converse need not true in general.*

Example 5.3. *Let $X = \{a,b,c\}$ and $\tau = \{\phi, X\}$. Then (X,τ) is connected but not an $\omega\alpha$-connected space because $X = \{a\} \cup \{b,c\}$ wherer $\{a\}$ and $\{b,c\}$ are $\omega\alpha$-open sets in (X,τ).*

Theorem 5.4. *Let $f : (X,\tau) \to (Y,\sigma)$ be a $\omega\alpha$-continuous surjection and (X,τ) is $\omega\alpha$-connected, then (Y,σ) is connected*

Proof. Suppose that (Y,σ) is not connected. Let $Y = A \cup B$ where A and B are disjoint non-empty open subsets in $(Y,\sigma))$. Since f is $\omega\alpha$-continuous, $X = f^{-1}(A) \cup f^{-1}(B)$, where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint non-empty $\omega\alpha$-open subsets in (X,τ). This contradicts the fact that (X,τ) is $\omega\alpha$-connected. Hence (Y,σ) is connected. □

Theorem 5.5. *Suppose that (X,τ) is $T_{\omega\alpha}$- space, then (X,τ) is connected if and only if (X,τ) is $\omega\alpha$-connected.*

Theorem 5.6. *Let $f : (X,\tau) \to (Y,\sigma)$ be a $\omega\alpha$-irresolute surjection and (X,τ) is $\omega\alpha$-connected, then (Y,σ) is $\omega\alpha$-connected.*

Theorem 5.7. *The image of a connected space under strongly $\omega\alpha$-continuous map is $\omega\alpha$-connected.*

References

[1] K. Balachandran, P. Sundaram, H. Maki, On generalized continuous maps in topological spaces, Mem. Fac. Kochi Univ. Ser. A, Math. 12 (1991) 5–13.

[2] R. Devi, Studies on Generalizations of Closed Maps and Homeomorphisms in Topological Spaces, Ph.D. Thesies, Bharathiyar University, Coimbatore, 1994.

[3] G. Di Maio, T. Noiri, On s-Closed Spaces, Indian J. Pure Appl. Math. 18 (1987) 226–233.

[4] Y. Gnanambal, K. Balachandran, On gpr-continuous functions in topological spaces, Indian J. Pure Appl. Math. 30 (6) (1999) 581–593.

[5] A. Pushpalatha, Studies on Generalizations of Mappings in Topological Spaces, Ph.D. Thesies, Bharathiyar University, Coimbatore, 2000.

[6] M. Sheik John, A Study on Generalizations of Closed Sets and Continuous Maps in Topological and Bitopological Spaces, Ph.D. Thesies, Bharathiyar University, Coimbatore, 2002.

[7] S.S. Benchalli, P.G. Patil, T.D. Rayanagoudar, $\omega\alpha$-Closed sets in topological spaces, The Global J. Appl. Maths Math. Sciences 2 (1-2) (2009) 53–63.

[8] S.S. Benchalli, P.G. Patil, Some new continuous maps in topological spaces, J. Advanced Studies in Topology 1 (2) (2010) 16–21.

[9] P.G. Patili, Some Studies in Topological Spaces, Ph.D. Thesies, Karnatak University Dharwad, Karnataka, 2007.

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