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A Fundamental Theorem of Co-Homomorphisms for Semirings

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Abstract : The quotient structure of a semiring with non-zero identity modulo a Q-strong co-ideal has been introduced and studied in [1]. In this paper, we will introduce the notions of co-homomorphisms and Maximal co-homomorphisms for semirings. Using these notions, the fundamental theorem of co-homomorphisms will be generalized to include a large class of semirings.

Keywords : semiring; co-ideal; strong co-ideal; partitioning strong co-ideal; subtractive co-ideal; co-homomorphism; maximal co-homomorphism.
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1 Introduction

P. J. Allen [2] introduced the notion of a Q-ideal and a construction process was presented by which one can build the quotient structure of a semiring modulo a Q-ideal. Maximal homomorphisms were defined and examples of such homomorphisms were given. Using these notions, the fundamental theorem of homomorphisms for rings was generalized to include a large class of semirings. The present authors [3] have presented the notion of a Q-strong co-ideal I in the semiring R and constructed the quotient semiring R/I. In this paper, we extend the definition and results given by Allen to a more general Q-strong co-ideal case. In this paper, we introduce the notion of co-homomorphism and maximal co-homomorphism. We show if I is a Q-strong co-ideal of semiring R and $\phi : R \to R/I$ with $\phi(a) = qI$,

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where q is the unique element of Q such that $a \in qI$, then ϕ is a maximal co-homomorphism. Also, it is shown if ϕ is a co-homomorphism from the semiring R onto R' that is maximal, then $R/co - Ker(\phi) \cong R'$.

For the sake of completeness, we state some definitions and notations used throughout. A commutative semiring R is defined as an algebraic system $(R, +, .)$ such that $(R, +)$ and $(R, .)$ are commutative semigroups, connected by $a(b+c) = ab + ac$ for all $a, b, c \in R$, and there exists $0, 1 \in R$ such that $r + 0 = r$ and $r0 = 0r = 0$ and $r1 = 1r = r$ for each $r \in R$. In this paper all semirings considered will be assumed to be commutative semirings with non-zero identity. In this paper, $\mathcal{B}$ denotes the boolean semiring $\{0, 1\}$, which $1 + 1 = 1$.

Definition 1.1. *Let R be a semiring.*

(1) A non-empty subset I of R is called co-ideal, denoted by $I \trianglelefteq^c R$, if it is closed under multiplication and satisfies the condition $r + a \in I$ for all $a \in I$ and $r \in R$ (clearly, $0 \in I$ if and only if $I = R$) ([3], [4]). A co-ideal I is called strong co-ideal if $1 \in I$ [1].

(2) A co-ideal I of R is called subtractive if for each $x, y \in R$ with $x, xy \in I$, then $y \in I$ [4].

(3) A proper co-ideal I of R is said to be maximal if J is a co-ideal of R with $I \subsetneqq J$, then $J = R$. It is known that maximal co-ideals are strong co-ideal [5].

(4) A mapping φ from the semiring R into the semiring R' will be called a homomorphism if $\varphi(a+b) = \varphi(a) + \varphi(b)$ and $\varphi(a.b) = \varphi(a).\varphi(b)$ for each $a, b \in R$. An isomorphism is a one-to-one homomorphism. The semirings R and R' will be called isomorphic (denoted by $R \cong R'$) if there exists an isomorphism from R onto R' [2].

Definition 1.2. *(See [1]) A strong co-ideal I of a semiring R is called a partitioning co-ideal (= Q-strong co-ideal) if there exists a subset Q of R such that*

(1) $R = \bigcup\{qI : q \in Q\}$, where $qI = \{qt : t \in I\}$.

(2) If $q_1, q_2 \in Q$, then $(q_1 I) \cap (q_2 I) \neq \emptyset$ if and only if $q_1 = q_2$.

Lemma 1.3. *(See [1]) Let I be a Q-strong co-ideal of the semiring R. If $x \in R$, then there exists a unique $q \in Q$ such that $xI \subseteq qI$. In particular, $x = qa$ for some $a \in I$.*

Let I be a Q-strong co-ideal of a semiring R and let $R/I = \{qI : q \in Q\}$. Then R/I forms a semiring under the binary operations $\oplus$ and $\odot$ defined as follows:

(1) $(q_1 I) \oplus (q_2 I) = q_3 I$, where q_3 is the unique element in Q such that $(q_1 I + q_2 I) \subseteq q_3 I$; and

(2) $(q_1 I) \odot (q_2 I) = q_3 I$, where q_3 is the unique element in Q such that $(q_1 q_2) I \subseteq q_3 I$ (see [1]).

Proposition 1.4. *(See [1]) Every Q-strong co-ideal I of a semiring R is subtractive.*

Lemma 1.5. *(See [5]) If D is a maximal co-ideal of a semiring R, then $R - D$ is an ideal.*

2 Co-Homomorphism of semirings

We begin with the key definition of this paper.

Definition 2.1. *Let R and R' be two semirings. The map $\phi : R \to R'$ is called co-homomorphism if satisfies the following conditions:*
(1) $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in R$.
(2) $\phi(a + b) = \phi(a) + \phi(b)$ for all $a, b \in R$.
(3) $\phi(0) = 0$.
(4) $\phi(1) = 1$.
(5) If $\phi(r) = 1$ for some $r \in R$, then $\phi(a + r) = 1$ for all $a \in R$.

One can easily see that every co-homomorphism is a semiring homomorphism. The following example shows that a homomorphism need not be a co-homomorphism.

Example 2.2. *Let $\mathbb{Z}^+ \cup \{0\}$ be the semiring of positive integers with the usual addition and multiplication and i be the identity homomorphism of semiring $\mathbb{Z}^+ \cup \{0\}$. It is clear that $i(1) = 1$ and $i(r + 1) \neq 1$ for each $r \in \mathbb{Z}^+ \cup \{0\}$. So i is not a co-homomorphism.*

Proposition 2.3. *Let D be a co-ideal of a semiring R such that $R - D$ is an ideal of R. Then D is a subtractive strong co-ideal of R.*

Proof. Let $xy \in D$ and $x \in D$ for some $x, y \in R$. If $y \notin D$, then $y \in R - D$. By hypothesis, $R - D$ is an ideal of R, therefore $xy \in R - D$, a contradiction. Thus D is a subtractive co-ideal of R. Clearly, $1 \in D$ since D is a subtractive co-ideal. □

The converse of Proposition 2.3 is not true, as the following example shows.

Example 2.4. *Let $X = \{a, b, c\}$. Then $R = (P(X), \cup, \cap)$ is a semiring, where $P(X)$ is the set of all subsets of X. An inspection will show that $I = \{X, \{a, b\}\}$ is a Q-strong co-ideal of R, where $Q = \{\{c\}, \{a, c\}, \{b, c\}, X\}$. Thus I is a subtractive co-ideal of R by Proposition 1.4. It can be seen $R - I$ is not an ideal of R, because $\{a\}, \{b\} \in R - I$ and $\{a\} \cup \{b\} = \{a, b\} \notin R - I$.*

Proposition 2.5. *If D is a maximal co-ideal of R, then D is subtractive.*

Proof. Apply Lemma 1.5 and Proposition 2.3. □

The following example shows that the converse of Proposition 2.5 is not true.

Example 2.6. *Let R be the set of all non-negative integers. Define $a + b = \gcd(a, b)$ and $a \times b = \operatorname{lcm}(a, b)$,(take $0 + 0 = 0$ and $0 \times 0 = 0$). Then $(R, +, \times)$ is easily checked to be a commutative semiring. Let I be the set of all non-negative odd integers, then I is a co-ideal of R. An inspection shows that $R - I$ is an ideal of R. It can be seen I is not a maximal co-ideal of R, because $I \subsetneqq R - \{0\}$ and $R - \{0\}$ is a maximal co-ideal of R.*

Theorem 2.7. *Let D be a co-ideal of R such that $R - D$ is an ideal of R. Then there exists a co-homomorphism from R onto $\mathcal{B}$.*

Proof. Let $\phi : R \to \mathcal{B}$ with

$$\phi(x) = \begin{cases} 0 & \text{if } x \notin D, \\ 1 & \text{if } x \in D \end{cases}$$

We will show that ϕ is a co-homomorphism.

(1) $\phi(a+b) = \phi(a)+\phi(b)$ for all $a, b \in R$. We consider the various possibilities for a, b.
Case 1: $a, b \in D$. Since D is a co-ideal, $a + b \in D$. So $\phi(a + b) = 1$. Also $\phi(a) + \phi(b) = 1 + 1 = 1$. Thus $\phi(a + b) = \phi(a) + \phi(b)$.
Case 2: $a \notin D$ and $b \notin D$. Since $I = R-D$ is an ideal of R and $a, b \in I$, $a+b \in I$ and so $a + b \notin D$. It is clear that $\phi(a + b) = \phi(a) + \phi(b) = 0$.
Case 3: $(a \in D,\ b \notin D)$ or $(a \notin D,\ b \in D)$. In these two, we have $a + b \in D$. So $1 = \phi(a + b) = \phi(a) + \phi(b) = 1 + 0 = 1$.

(2) $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in R$. We consider the various possibilities for a, b.
Case 1: $a, b \in D$. Since D is a co-ideal, $ab \in D$, and so $\phi(ab) = 1$. Since $a, b \in D$, $\phi(a) = 1$ and $\phi(b) = 1$. Therefore $\phi(ab) = \phi(a)\phi(b)$.
Case 2: $a \notin D$ and $b \notin D$. Since $I = R - D$ is an ideal of R, $ab \in I$ and so $ab \notin D$. Thus $\phi(ab) = 0$. Therefore $0 = \phi(ab) = \phi(a)\phi(b)$.
Case 3: $(a \in D,\ b \notin D)$ or $(a \notin D,\ b \in D)$. Since $R - D$ is an ideal, D is a subtractive co-ideal by Proposition 2.3. Therefore $ab \notin D$. So $0 = \phi(ab) = \phi(a)\phi(b)$.

(3) $\phi(1) = 1$ is clear, since $1 \in D$.

(4) $\phi(0) = 0$ is clear, since $0 \notin D$.

(5) If $\phi(r) = 1$, then $r \in D$. Hence $a+r \in D$ for each $a \in R$. Thus $\phi(a+r) = 1$.

It is clear that ϕ is onto. □

Definition 2.8. *Let R and R' be two semirings and $\phi : R \to R'$ be a co-homomorphism. Set $co - Ker(\phi) = \{r \in R : \phi(r) = 1\}$.*

Remark 2.9. *It is clear that $co-Ker(\phi)$ is a strong co-ideal of R and in Theorem 2.7, $co - Ker(\phi) = \{x \in R : \ \phi(x) = 1\} = D$.*

Definition 2.10. *A co-homomorphism ϕ with $co - Ker(\phi) = K$ from a semiring R onto the semiring R' is said to be maximal if for each $a \in R'$ there exists $q_a \in \phi^{-1}(\{a\})$ such that $xK \subseteq q_aK$ for each $x \in \phi^{-1}(\{a\})$.*

Example 2.11. *Let $R = \mathbb{Z}^+ \cup \{0\}$ be the semiring of positive integers and $\phi : R \to \mathcal{B}$ with*

$$\phi(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x \in R - \{0\}. \end{cases}$$

It can be checked that ϕ is a co-homomorphism. Put $q_1 = 1$ and $q_0 = 0$. Then for each $x \in R - \{0\}$, we have $x(co - Ker(\phi)) \subseteq q_1(co - Ker(\phi))$ and for $x = 0$, $x(co - Ker(\phi)) \subseteq q_0(co - Ker(\phi))$. Therefore ϕ is a maximal co-homomorphism.

Proposition 2.12. *Let R be a semiring and I be a Q-strong co-ideal of R. If $\phi : R \to R/I$ with $\phi(a) = qI$, where q is the unique element of Q such that $a \in qI$, then ϕ is a maximal co-homomorphism.*

Proof. We prove the proposition in six steps.

(1) $\phi(ab) = \phi(a) \odot \phi(b)$ for all $a, b \in R$. Let q_1, q_2, q be elements of Q such that $ab \in qI$, $a \in q_1 I$ and $b \in q_2 I$. Hence $\phi(a) = q_1 I$, $\phi(b) = q_2 I$ and $\phi(ab) = qI$. Let $q' \in Q$ such that $q_1 q_2 I \subseteq q' I$ then $\phi(a) \odot \phi(b) = q_1 I \odot q_2 I = q' I$. We will show that $q = q'$. Since $ab \in q_1 I q_2 I \subseteq q_1 q_2 I \subseteq q' I$, $ab \in (q' I) \cap (qI)$ and so $q = q'$. Therefore $\phi(ab) = \phi(a) \odot \phi(b)$.

(2) $\phi(a + b) = \phi(a) \oplus \phi(b)$, for all $a, b \in R$. Let $q \in I$ such that $a + b \in qI$, then $\phi(a + b) = qI$. Let $q_1 \in Q$ and $q_2 \in Q$ such that $a \in q_1 I$ and $b \in q_2 I$, then $\phi(a) = q_1 I$ and $\phi(b) = q_2 I$. Let $q' \in Q$ such that $q_1 I + q_2 I \subseteq q' I$, then $\phi(a) \oplus \phi(b) = q_1 I \oplus q_2 I = q' I$. Since $a + b \in q_1 I + q_2 I$, $a + b \in q' I$ and hence $a + b \in (q' I) \cap (qI)$. Therefore $q = q'$ and so $\phi(a + b) = \phi(a) \oplus \phi(b)$.

(3) $\phi(0) = 0$. Let $q_0 \in Q$ be unique element such that $0 \in q_0 I$. Therefore $\phi(0) = q_0 I$ where $q_0 I$ is zero element of R/I.

(4) $\phi(1) = 1$ is clear.

(5) Let $\phi(r) = q_e I = I$ where $q_e I$ is the identity element of R/I, then by definition of ϕ, $r \in I$. Thus for each $a \in R$, $a + r \in I$ (since I is a co-ideal), and hence $\phi(a + r) = I$ as desired.

(6) It is clear that $co - Ker(\phi) = I$. Since I is a Q-strong co-ideal, for each $qI \in R/I$ and $x \in \phi^{-1}(qI)$, $xI \subseteq qI$. Thus ϕ is a maximal co-homomorphism. □

Lemma 2.13. *Let ϕ be a co-homomorphism from the semiring R onto semiring R'. If ϕ is maximal, then $co - Ker(\phi) = K$ is a Q-strong co-ideal of R.*

Proof. As ϕ is a maximal co-homomorphism, for each $a \in R'$ there exists $q_a \in \phi^{-1}(\{a\})$ such that $xK \subseteq q_a K$ for each $x \in \phi^{-1}(\{a\})$. First, we show that $R = \cup\{q_a K : a \in R'\}$. Let $r \in R$, then $\phi(r) \in R'$. Let $\phi(r) = b$. Then $r \in \phi^{-1}(\{b\})$. Since ϕ is maximal, there exists $q_b \in \phi^{-1}(\{b\})$ such that $rK \subseteq q_b K$. As $1 \in K$, we have $r \in q_b K$. Thus $R \subseteq \cup\{q_a K : a \in R'\}$. The other side is clear. Next, let $a, b \in R'$ and $x \in q_a K \cap q_b K$, so $x \in \phi^{-1}(\{b\}) \cap \phi^{-1}(\{a\})$. Hence $\phi(x) = a = b$. As $q_a \in \phi^{-1}(\{a\})$, $\phi(q_a) = a$. Since $a = b$ and for each $x \in \phi^{-1}(\{b\})$, $xK \subseteq q_b K$, we have $q_a K \subseteq q_b K$. Similarly $q_b K \subseteq q_a K$. Hence $q_a K = q_b K$. Also, if $q_a = q_b$, then $q_a = q_b \in q_a K \cap q_b K$(because $1 \in K$). Hence K is a Q-strong co-ideal. □

Lemma 2.14. *Let R, R', ϕ and Q be as stated in Lemma 2.13 and let q_a, q_b and q_c be elements in Q and $K = co - Ker(\phi)$.*
(1) If $(q_aK + q_bK) \subseteq q_cK$, then $a + b = c$.
(2) If $q_aq_bK \subseteq q_cK$, then $ab = c$.

Proof. (1) Since $q_a + q_b \in (q_aK + q_bK) \subseteq q_cK$, there exists $k \in K$ such that $q_a + q_b = q_ck$. Thus $a + b = \phi(q_a) + \phi(q_b) = \phi(q_a + q_b) = \phi(q_ck) = \phi(q_c)\phi(k) = c$.
(2) It can be proved by a similar way as in (1). □

Theorem 2.15. *If ϕ is a co-homomorphism from the semiring R onto R' that is maximal, then $R/co - Ker(\phi) \cong R'$.*

Proof. Let $co - Ker(\phi) = K$. By Lemma 2.13, K is a Q-strong co-ideal and $R = \cup\{q_aK : a \in R'\}$. Let $\bar{\phi} : R/K \to R'$ with $\bar{\phi}(q_aK) = a$ (for each $x \in \phi^{-1}(\{a\}), xK \subseteq q_aK$). Let $q_aK = q_bK$. Since K is a Q-strong co-ideal, $q_a = q_b$. So $a = \phi(q_a) = \phi(q_b) = b$. Thus $\bar{\phi}$ is well-defined. Now we show $\bar{\phi}$ is a isomorphism.

(1)$\bar{\phi}(q_aK \odot q_bK) = \bar{\phi}(q_aK)\bar{\phi}(q_bK)$. Let $q_c \in Q$ such that $q_aq_bK \subseteq q_cK$. Then $q_aK \odot q_bK = q_cK$. Thus by Lemma 2.14, $ab = c$ and so $\bar{\phi}(q_aK \odot q_bK) = \bar{\phi}(q_aK)\bar{\phi}(q_bK)$.

(2) $\bar{\phi}(q_aK \oplus q_bK) = \bar{\phi}(q_aK) + \bar{\phi}(q_bK)$. Let $q_c \in Q$ such that $q_aK + q_bK \subseteq q_cK$, then $q_aK \oplus q_bK = q_cK$. By Lemma 2.14, $a + b = c$. Thus $\bar{\phi}(q_aK \oplus q_bK) = \bar{\phi}(q_aK) + \bar{\phi}(q_bK)$.

(3) $\bar{\phi}$ is monomorphism. Let $\bar{\phi}(q_aK) = \bar{\phi}(q_bK)$. Hence $a = b$. Since for each $x \in \phi^{-1}(\{b\})$, $xK \subseteq q_bK$, we have $q_aK \subseteq q_bK$. Similarly $q_bK \subseteq q_aK$. Hence $q_aK = q_bK$.

(4) $\bar{\phi}$ is epimorphism. Let $a \in R'$. Since ϕ is epic, $\phi^{-1}(\{a\}) \neq \emptyset$. Since ϕ is maximal, there exists $q_a \in Q$ such that $q_a \in \phi^{-1}(\{a\})$ and for each $x \in \phi^{-1}(\{a\})$, $xK \subseteq q_aK$. Thus $\bar{\phi}(q_aK) = a$. Thus $\bar{\phi}$ is epic. □

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