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# Existence and Convergence Theorems of Fixed Points of a Lipschitz Pseudo-contraction by an Iterative Shrinking Projection Technique in Hilbert Spaces

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**Abstract :** The aim of this paper is to provide some existence theorems of a Lipschitz pseudo-contraction by the way of a hybrid shrinking projection method involving some necessary and sufficient conditions. The method allows us to obtain a strong convergence iteration for finding some fixed points of a Lipschitz pseudo-contraction in the framework of real Hilbert spaces. In addition, we also provide certain applications of the main theorems to confirm the existence of the zeros of a Lipschitz monotone operator along with its convergent results.

**Keywords :** Lipschitz pseudo-contraction; iterative shrinking projection technique; Hilbert space; fixed point.
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## 1 Introduction

There are several attempts to establish an iteration method to find a fixed point of some well-known nonlinear mappings, for instant, nonexpansive mapping. We note that Mann's iterations [1] have only weak convergence even in a Hilbert space (see e.g., [2]). Nakajo and Takahashi [3] modified the Mann iteration method so that strong convergence is guaranteed, later well known as a hybrid projection method. Since then, the hybrid method has received rapid developments. For the details, the readers are referred to papers [4–24] and the references cited therein. In 2008, Takahashi, Takeuchi and Kobota [19] introduced an alternative projection method, subsequently well known as the shrinking projection method, and they showed several strong convergence theorems for a family of nonexpansive mappings; see also [25]. In 2009, Aoyama, Kohsaka and Takahashi [26] applied the hybrid shrinking projection method along with creating some necessary and sufficient conditions to confirm the existence of a fixed point of firmly nonexpansive mapping.

Let $H$ be a real Hilbert space, a mapping $T$ with domain $D(T)$ and range $R(T)$ in $H$ is called firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle, \quad \forall x, y \in D(T),$$

nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in D(T),$$

Throughout this paper, $I$ stands for an identity mapping. The mapping $T$ is said to be a strict pseudo-contraction if there exists a constant $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in D(T). \tag{1.1}$$

In this case, $T$ may be called as $k$-strict pseudo-contraction. In the even that $k = 1$, $T$ is said to be a pseudo-contraction, i.e.,

$$\|Tx - Ty\|^2 \leqslant \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in D(T). \tag{1.2}$$

It is easy to see that (1.2) is equivalent to

$$\langle x - y, (I - T)x - (I - T)y\rangle \geqslant 0, \quad \forall x, y \in D(T).$$

We use $F(T)$ to denote the set of fixed point of $T$ (i.e. $F(T) = \{x \in D(T) : Tx = x\}$). The class of pseudo-contractions extendclass of the class of strict pseudo-contractions, the class of nonexpansive mappings and firmly nonexpansive mappings. That is

$$\boxed{\text{firmly nonexpansive}} \Rightarrow \boxed{\text{nonexpansive}} \Rightarrow \boxed{\text{strict pseudo-contraction}} \Rightarrow \boxed{\text{pseudo-contraction}}.$$

However, the following examples show that the converse is not true.

**Example 1.1** (Chidume, Mutangadura [27]). *Take* $H = \mathbb{R}^2$, $B = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$, $B_1 = \{x \in B : \|x\| \leq \frac{1}{2}\}$, $B_2 = \{x \in B : \frac{1}{2} \leq \|x\| \leq 1\}$. *If* $x = (a, b) \in H$ *we define* $x^{\perp}$ *to be* $(b, -a) \in H$. *Define* $T : B \to B$ *by*

$$Tx = \begin{cases} x + x^{\perp}, & x \in B_1, \\ \dfrac{x}{\|x\|} - x + x^{\perp}, & x \in B_2. \end{cases}$$

**Example 1.2** (Zhou [28]). *Take $H = \mathbb{R}$ and define $T : H \to H$ by*

$$Tx = \begin{cases} 1, & x \in (-\infty, -1)\,, \\ \sqrt{1-(1+x)^2}, & x \in [-1, 0)\,, \\ -\sqrt{1-(x-1)^2}, & x \in [0, 1]\,, \\ 1, & x \in (1, \infty)\,. \end{cases}$$

Then, $T$ is a Lipschitz and pseudo-contraction but not a strict pseudo-contraction.

**Example 1.3.** *Let $H$ be a real Hilbert space and $\alpha \in (1, \infty)$. Define $T_\alpha : H \to H$ by*

$$T_\alpha x = -\alpha x, \quad \forall x \in H.$$

*Then, $T_\alpha$ is a strict pseudo-contraction but not a nonexpansive mapping.*

Indeed, it is clear that $T_\alpha$ is not nonexpansive. On the other hand, let us consider

$$\begin{aligned}\|T_\alpha x - T_\alpha y\|^2 = \|(-\alpha x) - (-\alpha y)\|^2 = \alpha^2\|x-y\|^2 &= \left(1 + \frac{\alpha^2-1}{(1+\alpha)^2}(1+\alpha)^2\right)\|x-y\|^2 \\ &= \|x-y\|^2 + \frac{\alpha^2-1}{(1+\alpha)^2}\|(1-(-\alpha))x - (1-(-\alpha))y\|^2 \\ &= \|x-y\|^2 + \frac{\alpha-1}{\alpha+1}\|(I-T_\alpha)x - (I-T_\alpha)y\|^2 \\ &\le \|x-y\|^2 + \kappa\|(I-T_\alpha)x - (I-T_\alpha)y\|^2\end{aligned}$$

for all $\kappa \in \left[\frac{\alpha-1}{\alpha+1}, 1\right)$. Thus $T_\alpha$ is a strict pseudo-contraction.

**Example 1.4.** *Take $H \neq \{0\}$ and let $T = -I$, it is not hard to verify that $T$ is nonexpansive but not firmly nonexpansive.*

From a practical point of view, strict pseudo-contractions have more powerful applications than nonexpansive mappings do in solving inverse problems (see Scherzer [29]). Therefore, it is important to develop theory of iterative methods for strict pseudo-contractions. Within the past several decades, many authors have been devoted to the studies on the existence and convergence of fixed points for strict pseudocontractions. In 1967, Browder and Petryshyn [30] introduced a convex combination method to study strict pseudo-contractions in Hilbert spaces. On the other hand, Marino and Xu [12] and Zhou [31] developed some iterative scheme for finding a fixed point of a strict pseudocontraction mapping.

In 2009, Yao, Liou, Marino [32] introduced the hybrid iterative algorithm for pseudo-contractive mapping in Hilbert spaces as follows:

Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $T : C \to C$ be a pseudo-contraction. Let $\{\alpha_n\}$ be a sequence in $(0, 1)$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define a sequence $\{x_n\}$ of $C$ as follows:

$$\begin{cases} y_n &= (1-\alpha_n)x_n + \alpha_n T x_n, \\ C_{n+1} &= \left\{ v \in C_n : \|\alpha_n (I-T) y_n\|^2 \leq 2\alpha_n \langle x_n - v, (I-T)y_n \rangle \right\}, \\ x_{n+1} &= P_{C_{n+1}}(x_0). \end{cases} \tag{1.3}$$

**Theorem 1.5** (Yao, Liou, Marino [32]). *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $T : C \to C$ be a L-Lipschitz pseudo-contraction such that $F(T) \neq \varnothing$. Assume the sequence $\{\alpha_n\} \subset [a, b]$ for some $a, b \in \left(0, \frac{1}{L+1}\right)$. Then the sequence $\{x_n\}$ generated by (1.3) converges strongly to $P_{F(T)}(x_0)$.*

In 2009, Aoyama, Kohsaka and Takahashi [26] provided the useful and interesting lemma to confirm that the sequence generated by the shrinking projection method is well defined even if the firmly nonexpansive mapping $T$ has no fixed points:

**Lemma 1.6** (Aoyama, Kohsaka, Takahashi [26, Lemma 4.2]). *Let $H$ be a Hilbert space, $C$ a nonempty closed convex subset of $H$, $T : C \to C$ a firmly nonexpansive mapping and $x_0 \in H$. Let $\{x_n\}$ be a sequence in $C$ and $\{C_n\}$ a sequence of closed convex subsets of $H$ generated by $C_1 = C$ and*

$$\begin{cases} x_n = P_{C_n}(x_0), \\ C_{n+1} = \{ z \in C_n : \langle T x_n - z, x_n - T x_n \rangle \geq 0 \}, \end{cases}$$

*for all $n \in \mathbb{N}$. Then $C_n$ is nonempty for every $n \in \mathbb{N}$, and consequently, $\{x_n\}$ is well defined.*

By using the lemma mentioned above, they proved the following theorem:

**Theorem 1.7** (Aoyama, Kohsaka, Takahashi [26, Theorem 4.3]). *Let $H$ be a Hilbert space, $C$ a nonempty closed convex subset of $H$, $T : C \to C$ a firmly nonexpansive mapping and $x_0 \in H$. Let $\{x_n\}$ be a sequence in $C$ and $\{C_n\}$ a sequence of closed convex subsets of $H$ generated by $C_1 = C$ and*

$$\begin{cases} x_n = P_{C_n}(x_0), \\ C_{n+1} = \{ z \in C_n : \langle T x_n - z, x_n - T x_n \rangle \geq 0 \}, \end{cases}$$

*for all $n \in \mathbb{N}$. Then the following are equivalent:*

1. *$\bigcap_{n=1}^{\infty} C_n$ is nonempty;*
2. *$\{x_n\}$ is bounded;*
3. *$F(T)$ is nonempty.*

Motivated and inspired by the results mentioned above, in this paper, we provide some existence theorems of a Lipschitz pseudo-contraction by the way of the shrinking projection method involving some necessary and sufficient conditions. Then we prove a strong convergence theorem and present its applications to confirm the existence of the zeros of a Lipschitz monotone operator along with its convergent results.

Throughout the paper, we will use the notation:

1. $\to$ for strong convergence and $\rightharpoonup$ for weak convergence,
2. $\omega_w(x_n) = \{x : \exists x_{n_i} \rightharpoonup x\}$ denotes the weak $\omega$-limit set of $\{x_n\}$.

## 2 Preliminaries

In this section, some definitions are provided and some relevant lemmas which are useful to prove in the next section are collected. Most of them are known others are not hard to find and understand the proof.

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and let $C$ be a closed convex subset of $H$. For every point $x \in H$ there exists a unique nearest point in $C$, denoted by $P_C(x)$, such that

$$\|x - P_C x\| \leqslant \|x - y\|, \quad \forall y \in C.$$

The mapping $P_C$ is called the *metric projection* of $H$ onto $C$. It is well known that $P_C$ is a firmly nonexpansive mapping of $H$ onto $C$, that is

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle, \quad \forall x, y \in H.$$

Furthermore, for any $x \in H$ and $z \in C$,

$$z = P_C x \Leftrightarrow \langle x - z, z - y \rangle \geq 0, \quad \forall y \in C.$$

Moreover, $P_C x$ is characterized by the following:

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \quad \forall y \in C.$$

It is obvious that the following equality holds for all $x, y \in H$:

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2 \langle x - y, y \rangle, \quad \forall x, y \in H. \tag{2.1}$$

**Lemma 2.1** ([33, Problem 1.2(2)]). *Let $\{a_n\}$ be a sequence of real numbers. Then, $\lim_{n\to\infty} a_n = 0$ if and only if for any subsequence $\{a_{n_i}\}$ of $\{a_n\}$, there exists a subsequence $\left\{a_{n_{i_j}}\right\}$ of $\{a_{n_i}\}$ such that $\lim_{j\to\infty} a_{n_{i_j}} = 0$.*

**Lemma 2.2** ([34]). *Let $H$ be a real Hilbert space, $C$ a closed convex subset of $H$ and $T : C \to C$ a continuous pseudo-contractive mapping, then*

1. *$F(T)$ is closed convex subset of $C$.*

2. *$I-T$ is demiclosed at zero, i.e., if $\{x_n\}$ is a sequence in $C$ such that $x_n \rightharpoonup z$ and $(I-T)x_n \to 0$, then $(I-T)z = 0$.*

**Lemma 2.3** ([35, Theorem 7.1.8])**.** *Let $K$ be a bounded closed convex subset of a Hilbert space $H$ and $A : K \to H$ a continuous monotone mapping. Then there exists an element $u_0 \in K$ such that $\langle v - u_0, Au_0\rangle \geq 0$ for all $v \in K$.*

# 3 Main Results

In this section, motivated by Aoyama, Kohsaka and Takahashi [26] (see also, Matsushita and Takahashi [36]), we discuss the existence of fixed point of a Lipschitz pseudo-contraction by using the shrinking projection technique playing as the tool to guarantee the existence of fixed point of a Lipschitz pseudo-contraction.

Every iteration process generated by the shrinking projection method for a Lipschitz pseudo-contraction $T$ is well defined even if $T$ is fixed point free.

**Lemma 3.1.** *Let $H$ be a Hilbert space, $C$ a nonempty closed convex subset of $H$, $T : C \to C$ a L-Lipschitz pseudo-contraction and $x_0 \in H$. Let $\{x_n\}$ be a sequence in $C$ and $\{C_n\}$ a sequence of closed convex subsets of $H$ generated by $C_1 = C$ and*

$$\begin{cases} x_1 = P_{C_1}(x_0), \\ y_n = (1-\alpha_n)x_n + \alpha_n T x_n; \quad 0 \leq \alpha_n \leq \dfrac{1}{L+1}, \\ C_{n+1} = \left\{ z \in C_n : \|\alpha_n (I-T)y_n\|^2 \leq 2\alpha_n \langle x_n - z, (I-T)y_n\rangle \right\}, \\ x_{n+1} = P_{C_{n+1}}(x_0), \end{cases} \tag{3.1}$$

*for all $n \in \mathbb{N}$. Then $C_n$ is nonempty for every $n \in \mathbb{N}$, and consequently, $\{x_n\}$ is well defined.*

**Proof** Clearly, $C_1$ is nonempty. Suppose that $C_n$ is nonempty for some $n \in \mathbb{N}$. Since $C_n \subset C_{n-1} \subset \ldots \subset C_1$, we have $C_1, C_2, \ldots, C_n$ are nonempty and hence $\{x_1, x_2, \ldots, x_n\}$ is well define. Put $r = \max\{\|y_i\| : i = 1, 2, \ldots, n\}$ and $B_r = \{z \in H : \|z\| \leq r\}$. Obviously $C \cap B_r$ is a nonempty bounded closed convex subset of $H$. Let $I$ denote the identity mapping on $C$. Since $I - T$ is continuous and monotone, it follows from Lemma 2.3 that there exists $u \in C \cap B_r$ such that

$$\langle y - u, (I-T)u\rangle \geq 0 \quad \forall y \in C \cap B_r.$$

In particular, we have

$$\langle y_i - u, (I-T)u\rangle \geq 0 \tag{3.2}$$

for every $i = 1, 2 \dots, n$. On the other hand, by employing the identity (2.1) and then adding and subtracting the terms $y_i$ and $(I - T)u$, we obtain

$$\begin{aligned}
&\|x_i - u - \alpha_i(I-T)y_i\|^2 \\
&=\|x_i - u\|^2 - \|\alpha_i(I-T)y_i\|^2 - 2\alpha_i \langle x_i - u - \alpha_i(I-T)y_i, (I-T)y_i\rangle \\
&=\|x_i - u\|^2 - \|\alpha_i(I-T)y_i\|^2 - 2\alpha_i \langle x_i - y_i - \alpha_i(I-T)y_i, (I-T)y_i\rangle \\
&\quad - 2\alpha_i \langle y_i - u, (I-T)y_i\rangle \\
&=\|x_i - u\|^2 - \Big(\|\alpha_i(I-T)y_i\|^2 + 2\langle x_i - y_i - \alpha_i(I-T)y_i, \alpha_i(I-T)y_i\rangle\Big) \\
&\quad - 2\alpha_i \langle y_i - u, (I-T)y_i - (I-T)u\rangle - 2\alpha_i \langle y_i - u, (I-T)u\rangle .
\end{aligned} \tag{3.3}$$

By using the identity (2.1) again, it follows that

$$\begin{aligned}
&\|\alpha_i(I-T)y_i\|^2 + 2\langle x_i - y_i - \alpha_i(I-T)y_i, \alpha_i(I-T)y_i\rangle \\
&\quad = \|\alpha_i(I-T)y_i\|^2 + \|x_i - y_i\|^2 - \|\alpha_i(I-T)y_i\|^2 - \|x_i - y_i - \alpha_i(I-T)y_i\|^2 \\
&\quad = \|x_i - y_i\|^2 - \|x_i - y_i - \alpha_i(I-T)y_i\|^2 .
\end{aligned} \tag{3.4}$$

Substituting (3.4) in (3.3), and by $\alpha_n \geq 0$ for all $n \in \mathbb{N}$, the monotonicity of $(I - T)$ and (3.2), we have

$$\begin{aligned}
&\|x_i - u - \alpha_i(I-T)y_i\|^2 \\
&=\|x_i - u\|^2 - \|x_i - y_i\|^2 + \|x_i - y_i - \alpha_i(I-T)y_i\|^2 \\
&\quad - 2\alpha_i \langle y_i - u, (I-T)y_i - (I-T)u\rangle - 2\alpha_i \langle y_i - u, (I-T)u\rangle \\
&=\|x_i - u\|^2 - \|x_i - y_i\|^2 - \|x_i - y_i - \alpha_i(I-T)y_i\|^2 + 2\|x_i - y_i - \alpha_i(I-T)y_i\|^2 \\
&\quad - 2\alpha_i \langle y_i - u, (I-T)y_i - (I-T)u\rangle - 2\alpha_i \langle y_i - u, (I-T)u\rangle \\
&\leq\|x_i - u\|^2 - \|x_i - y_i\|^2 - \|x_i - y_i - \alpha_i(I-T)y_i\|^2 + 2\|x_i - y_i - \alpha_i(I-T)y_i\|^2 .
\end{aligned} \tag{3.5}$$

We observe that

$$\begin{aligned}
&\|x_i - y_i - \alpha_i(I-T)y_i\|^2 \\
&\qquad = \langle x_i - y_i - \alpha_i(I-T)y_i, x_i - y_i - \alpha_i(I-T)y_i\rangle \\
&\qquad = \alpha_i \langle (I-T)x_i - (I-T)y_i, x_i - y_i - \alpha_i(I-T)y_i\rangle \\
&\qquad \leq \alpha_i \|(I-T)x_i - (I-T)y_i\| \, \|x_i - y_i - \alpha_i(I-T)y_i\| \\
&\qquad \leq \alpha_i (L+1) \|x_i - y_i\| \, \|x_i - y_i - \alpha_i(I-T)y_i\| \\
&\qquad \leq \frac{\alpha_i (L+1)}{2} \Big(\|x_i - y_i\|^2 + \|x_i - y_i - \alpha_i(I-T)y_i\|^2\Big) .
\end{aligned} \tag{3.6}$$

Joining (3.5) for the term $\|x_i - y_i - \alpha_i(I-T)y_i\|^2$ with (3.6) and by $0 \leq \alpha_n \leq$

$\frac{1}{L+1}$, we have

$$
\begin{aligned}
&\|x_i - u - \alpha_i(I-T)y_i\|^2 \\
&\leq \|x_i - u\|^2 - \|x_i - y_i\|^2 - \|x_i - y_i - \alpha_i(I-T)y_i\|^2 + 2\|x_i - y_i - \alpha_i(I-T)y_i\|^2 \\
&\leq \|x_i - u\|^2 - \|x_i - y_i\|^2 - \|x_i - y_i - \alpha_i(I-T)y_i\|^2 \\
&\quad + \alpha_i (L+1) \left( \|x_i - y_i\|^2 + \|x_i - y_i - \alpha_i(I-T)y_i\|^2 \right) \qquad (3.7) \\
&= \|x_i - u\|^2 + (\alpha_i (L+1) - 1) \left( \|x_i - y_i\|^2 + \|x_i - y_i - \alpha_i(I-T)y_i\|^2 \right) \\
&\leq \|x_i - u\|^2.
\end{aligned}
$$

Notice that

$$\|x_i - u - \alpha_i(I-T)y_i\|^2 = \|x_i - u\|^2 - 2\alpha_i \langle x_i - u, (I-T)y_i \rangle + \|\alpha_i(I-T)y_i\|^2. \tag{3.8}$$

Combining (3.7) and (3.8), we have

$$\|\alpha_i(I-T)y_i\|^2 \leq 2\alpha_i \langle x_i - u, (I-T)y_i \rangle . \tag{3.9}$$

for every $i = 1, 2, \ldots, n$. This shows that $u \in C_{n+1}$. By induction on $n$, we obtain the desired result.

The following theorem provides some necessary and sufficient conditions to confirm the existence of a fixed point of a Lipschitz pseudo-contraction in Hilbert spaces.

**Theorem 3.2.** *Let all the assumptions be as in Lemma 3.1 and* $0 < a \leq \alpha_n \leq b < \frac{1}{L+1}$ *for all* $n \in \mathbb{N}$. *Then the following are equivalent:*

1. $\bigcap_{n=1}^{\infty} C_n$ *is nonempty;*
2. $\{x_n\}$ *is bounded;*
3. $F(T)$ *is nonempty.*

**Proof** [(i)⇒(ii)] Suppose that $\bigcap_{n=1}^{\infty} C_n \neq \varnothing$. Let $u \in \bigcap_{n=1}^{\infty} C_n$, it follows from the nonexpansiveness of $P_{C_n}$ that

$$\|x_n - u\| \leq \|P_{C_n}x_0 - P_{C_n}u\| \leq \|x_0 - u\|, \quad \forall n \in \mathbb{N}.$$

This shows that $\{x_n\}$ is bounded.

[(ii)⇒(iii)] Suppose that $\{x_n\}$ is bounded, we observe that

$$
\begin{aligned}
0 \leq \|x_{n+1} - x_n\|^2 &= \|x_{n+1} - P_{C_n}x_0\|^2 \\
&\leq \|x_{n+1} - x_0\|^2 - \|P_{C_n}x_0 - x_0\|^2 \qquad (3.10) \\
&= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2.
\end{aligned}
$$

This shows that $\{\|x_n - x_0\|\}$ is non-decreasing and then with the boundedness of $\{x_n\}$, we have $\lim_{n\to\infty} \|x_n - x_0\|$ exists. By using (3.10), we obtain

$$\|x_{n+1} - x_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

Since $x_{n+1} \in C_{n+1}$ and $0 < a \leq \alpha_n \leq b < \frac{1}{L+1}$, so we have

$$\|\alpha_n(I-T)y_n\|^2 \leq 2\alpha_n \langle x_n - x_{n+1}, (I-T)y_n\rangle \to 0 \quad \text{as} \quad n \to \infty,$$

and then

$$\|y_n - Ty_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

Furthermore, it follows from the Lipschitzian of $T$ that

$$\begin{aligned}\|x_n - Tx_n\| &\leq \|x_n - y_n\| + \|y_n - Ty_n\| + \|Ty_n - Tx_n\| \\ &\leq \|x_n - y_n\| + \|y_n - Ty_n\| + L\,\|y_n - x_n\| \\ &= \alpha_n(L+1)\,\|x_n - Tx_n\| + \|y_n - Ty_n\|.\end{aligned}$$

By simple calculation, we have

$$\begin{aligned}\|x_n - Tx_n\| &\leq \left(\frac{1}{1-\alpha_n(L+1)}\right)\|y_n - Ty_n\| \\ &\leq \left(\frac{1}{1-b\,(L+1)}\right)\|y_n - Ty_n\| \to 0 \quad \text{as} \quad n \to \infty.\end{aligned} \tag{3.11}$$

Since $\{x_n\}$ is bounded, the reflexivity of $H$ allows a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup p \in C$ as $i \to \infty$. By using (3.11) and Lemma 2.2 (ii) the demicloseness of $(I-T)$, we obtain $p - Tp = 0$ that is $p \in F(T) \neq \varnothing$.

[(iii)⇒(i)] Suppose that $F(T) \neq \varnothing$. We will show that $F(T) \subset C_n$ for every $n \in \mathbb{N}$. Let $p \in F(T)$, then we have $(I-T)p = 0$. Let us replace $u$ in the proof of Lemma 3.1 with $p$, it is not difficult to see that all equalities and inequalities are satisfied until (3.9). This implies that $p \in C_n$ for all $n \in \mathbb{N}$. Therefore $F(T) \subset \bigcap_{n=1}^{\infty} C_n \neq \varnothing$.

**Theorem 3.3.** *Let all the assumptions be as in Lemma 3.2. Then, if* $\bigcap_{n=1}^{\infty} C_n \neq \varnothing$ *(⇔* $\{x_n\}$ *is bounded ⇔* $F(T) \neq \varnothing$*), then the sequence* $\{x_n\}$ *generated by* (3.1) *converges strongly to some points of* $C$ *and its strong limit point is a member of* $F(T)$*, that is* $\lim_{n\to\infty} x_n = P_{F(T)}x_0 \in F(T)$.

**Proof** We will show that $\lim_{n\to\infty} x_n$ exists. Furthermore, the limit point $p = \lim_{n\to\infty} x_n = P_{F(T)}x_0 \in F(T)$.

Suppose that $\bigcap_{n=1}^{\infty} C_n \neq \varnothing$, then by Theorem 3.2 ensures that the sequence $\{x_n\}$ is bounded. Let $\{x_{n_i}\}$ be any subsequence of $\{x_n\}$, then the reflexivity of

$H$ allows a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ such that $x_{n_{i_j}} \rightharpoonup p \in C$ as $j \to \infty$ and then applying Lemma 2.1 we can conclude that $x_n \rightharpoonup p \in C$ as $n \to \infty$. By using (3.11) and the demicloseness of $(I - T)$, we obtain $p - Tp = 0$ that is $p \in F(T)$. Since $P_{F(T)}x_0 \in F(T) \subset C_n$, we observe that

$$\|x_n - x_0\| \leq \|P_{C_n}x_0 - x_0\| \leq \left\|P_{F(T)}x_0 - x_0\right\| \tag{3.12}$$

for every $n \in \mathbb{N}$. Since $\|\cdot\|^2$ is weakly lower semicontinuous and $\{\|x_n - x_0\|\}$ is convergent, it follows from (3.12) that

$$\|p - x_0\|^2 \leq \liminf_{n\to\infty} \|x_n - x_0\|^2 = \lim_{n\to\infty} \|x_n - x_0\|^2 \leq \left\|P_{F(T)}x_0 - x_0\right\|^2.$$

Taking into account $p \in F(T)$, we obtain $p = P_{F(T)}x_0$. This shows that $x_n \rightharpoonup P_{F(T)}x_0$ and $\|x_n - x_0\| \to \|P_{F(T)}x_0 - x_0\|$. Consequently, from (2.1), we obtain

$$\begin{aligned}\left\|x_n - P_{F(T)}x_0\right\|^2 &= \left\|x_n - x_0 - \left(P_{F(T)}x_0 - x_0\right)\right\|^2 \\ &= \|x_n - x_0\|^2 - \left\|P_{F(T)}x_0 - x_0\right\|^2 \\ &\quad - 2\left\langle x_n - P_{F(T)}x_0, P_{F(T)}x_0 - x_0\right\rangle \to 0.\end{aligned}$$

This completes the proof.

# 4 Deduced Theorems and Applications

In this section, some deduced theorems and applications of the main results are provided in order to guarantee the existence of fixed points of a nonexpansive mapping, the existence of the zeros of a Lipschitz monotone operator. Moreover, we also have the methods that can be used to find fixed points and zero points mentioned above.

If $T : C \to C$ is nonexpansive ($\Leftrightarrow T$ is 1-Lipschitz pseudo-contraction), then we have the following corollaries.

Every iteration process generated by the shrinking projection method for a nonexpansive mapping $T$ is well defined even if $T$ is fixed point free.

**Corollary 4.1.** *Let $H$ be a Hilbert space, $C$ a nonempty closed convex subset of $H$, $T : C \to C$ a nonexpansive mapping and $x_0 \in H$. Let $\{x_n\}$ be a sequence in $C$ and $\{C_n\}$ a sequence of closed convex subsets of $H$ generated by $C_1 = C$ and*

$$\begin{cases} x_1 = P_{C_1}x_0, \\ y_n = (1 - \alpha_n)x_n + \alpha_n Tx_n; 0 \leq \alpha_n \leq \dfrac{1}{2}, \\ C_{n+1} = \left\{z \in C_n : \|\alpha_n(I - T)y_n\|^2 \leq 2\alpha_n \langle x_n - z, (I - T)y_n\rangle\right\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \end{cases} \tag{4.1}$$

*for all $n \in \mathbb{N}$. Then $C_n$ is nonempty for every $n \in \mathbb{N}$, and consequently, $\{x_n\}$ is well defined.*

**Corollary 4.2.** *Let all the assumptions be as in Lemma 4.1 and* $0 < a \leq \alpha_n \leq b < \frac{1}{2}$*. Then the following are equivalent:*

1. $\bigcap_{n=1}^{\infty} C_n$ *is nonempty;*
2. $\{x_n\}$ *is bounded;*
3. $F(T)$ *is nonempty.*

**Corollary 4.3.** *Let all the assumptions be as in Lemma 4.2. Then, if* $\bigcap_{n=1}^{\infty} C_n \neq \varnothing$ *(*$\Leftrightarrow \{x_n\}$ *is bounded* $\Leftrightarrow F(T) \neq \varnothing$*), then* $\{x_n\}$ *converges strongly to some points of* $C$ *and its strong limit point is a member of* $F(T)$*, that is* $\lim_{n\to\infty} x_n = P_{F(T)}(x_0) \in F(T)$*.*

Recall that a mapping $A$ is said to be monotone, if $\langle x - y, Ax - Ay\rangle \geqslant 0$ for all $x, y \in H$ and inverse strongly monotone if there exists a real number $\gamma > 0$ such that $\langle x - y, Ax - Ay\rangle \geqslant \gamma\|Ax - Ay\|^2$ for all $x, y \in H$. For the second case, $A$ is said to be $\gamma$-inverse strongly monotone. It follows immediately that if $A$ is $\gamma$-inverse strongly monotone, then $A$ is monotone and Lipschitz continuous, that is, $\|Ax - Ay\| \leqslant \frac{1}{\gamma}\|x - y\|$. It is well known (see, e.g., [37]) that if $A$ is monotone, then the solutions of the equation $Ax = 0$ correspond to the equilibrium points of some evolution systems. Therefore, it is important to focus on finding the zero points of monotone mappings. The pseudo-contractive mapping and strictly pseudo-contractive mapping are strongly related to the monotone mapping and inverse strongly monotone mapping, respectively. It is well known that

1. $A$ is monotone $\Leftrightarrow T := (I - A)$ is pseudo-contractive.
2. $A$ is inverse strongly monotone $\Leftrightarrow T := (I - A)$ is strictly pseudo-contractive.

Indeed, for (ii), we notice that the following equality always holds in a real Hilbert space

$$\|(I - A)x - (I - A)y\|^2 = \|x - y\|^2 + \|Ax - Ay\|^2 - 2\langle x - y, Ax - Ay\rangle \quad \forall x, y \in H, \tag{4.2}$$

with out loss of generality we can assume that $\gamma \in (0, \frac{1}{2}]$ and then it yields

$$\begin{aligned}
\langle x - y, Ax - Ay\rangle \geqslant \gamma\|Ax - Ay\|^2 &\Leftrightarrow -2\langle x - y, Ax - Ay\rangle \leqslant -2\gamma\|Ax - Ay\|^2 \\
&\Leftrightarrow \|(I - A)x - (I - A)y\|^2 \leqslant \|x - y\|^2 + (1 - 2\gamma)\|Ax - Ay\|^2 \\
&\qquad \text{(via (4.2))} \\
&\Leftrightarrow \|Tx - Ty\|^2 \leqslant \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2 \\
&\qquad \text{(where } T := (I - A) \text{ and } k := 1 - 2\gamma).
\end{aligned}$$

Every iteration process generated by the shrinking projection method for a $L_A$-Lipschitz monotone operator $A$ is well defined even if $A$ has no zeros.

**Corollary 4.4.** *Let $H$ be a Hilbert space and $A : H \to H$ be a $L_A$-Lipschitz monotone operator. Let $x_0 \in H$, $C_1 = C$ and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 = P_{C_1}x_0, \\ y_n = (I - \alpha_n A)\, x_n; 0 \leq \alpha_n \leq \dfrac{1}{L_A + 2}, \\ C_{n+1} = \left\{ z \in C_n : \|\alpha_n A y_n\|^2 \leq 2\alpha_n \langle x_n - z, Ay_n \rangle \right\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \end{cases} \tag{4.3}$$

*for all $n \in \mathbb{N}$. Then $C_n$ is nonempty for every $n \in \mathbb{N}$, and consequently, $\{x_n\}$ is well defined.*

**Proof** Let $T := (I - A)$. Then $T$ is $(L_A + 1)$-Lipschitz pseudo-contraction,

$$y_n = (I - \alpha_n A)\, x_n = (I - \alpha_n (I - (I - A)))\, x_n = (1 - \alpha_n)x_n + \alpha_n T x_n.$$

Hence, applying Theorem 3.1, we have the desired result.

The following theorem provides some necessary and sufficient conditions to confirm the existence of a zeros of a $L_A$-Lipschitz monotone operator in Hilbert spaces.

**Corollary 4.5.** *Let all the assumptions be as in Corollary 4.4 and $0 < a \leq \alpha_n \leq b < \frac{1}{L_A+2}$ for all $n \in \mathbb{N}$. Then the following are equivalent:*

1. $\bigcap\limits_{n=1}^{\infty} C_n$ *is nonempty;*
2. $\{x_n\}$ *is bounded;*
3. $A^{-1}(0)$ *is nonempty.*

**Proof** Let $T := (I - A)$. Then $T$ is $(L_A + 1)$-Lipschitz pseudo-contraction, it is not difficult to show that $F(T) = A^{-1}(0)$. Hence, applying Theorem 3.2, we have the desired result.

**Corollary 4.6.** *Let all the assumptions be as in Corollary 4.5. Then, if $\bigcap\limits_{n=1}^{\infty} C_n \neq \varnothing$ ($\Leftrightarrow \{x_n\}$ is bounded $\Leftrightarrow A^{-1}(0) \neq \varnothing$), then the sequence $\{x_n\}$ generated by (4.3) converges strongly to some points of $H$ and its strong limit point is a member of $A^{-1}(0)$, that is $\lim\limits_{n\to\infty} x_n = P_{A^{-1}(0)}x_0 \in A^{-1}(0)$.*

**Proof** Let $T := (I - A)$ and by applying Theorem 3.3, we have the desired result.

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