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On Pure Ideals in Ordered Ternary Semigroups

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Abstract : In this paper, we introduce the concepts of pure ideal, weakly pure ideal and purely prime ideal in an ordered ternary semigroup. We obtain some characterizations of pure ideals and prove that the set of all purely prime ideals is topologized.

Keywords : semigroup; ordered ternary semigroup; weakly regular; pure ideal; weakly pure ideal; purely prime ideal; topology.
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1 Introduction and Preliminaries

In [1], Ahsan and Takahashi introduced the notions of pure ideal and purely prime ideal in a semigroup. Recently, Bashir and Shabir [2] defined the concepts of pure ideal, weakly pure ideal and purely prime ideal in a ternary semigroup without order. The authors gave some characterizations of pure ideals and showed that the set of all purely prime ideals of a ternary semigroup is topologized. In this paper, we do in the line of Bashir and Shabir. We introduce the concepts of pure ideal, weakly pure ideal and purely prime ideal in an ordered ternary semigroup. We characterize pure ideals and prove that the set of all purely prime ideals of an ordered ternary semigroup is topologized. Note that the results on ternary semigroups without order become then special cases.

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For the rest of this section, definitions and results used throughout the paper will be recalled.

Let T be a nonempty set. Then T is called a *ternary semigroup* ([3, 4, 5]) if there exists a ternary operation $T\times T\times T\to T$, written as $(x_1,x_2,x_3)\mapsto[x_1x_2x_3]$, such that

$$[[x_1x_2x_3]x_4x_5]=[x_1[x_2x_3x_4]x_5]=[x_1x_2[x_3x_4x_5]]$$

for all $x_1,x_2,x_3,x_4,x_5\in T$.

Let $(T,[\ ])$ be a ternary semigroup. For nonempty subsets A_1,A_2 and A_3 of T, let

$$[A_1A_2A_3]=\{[x_1x_2x_3]\mid x_1\in A_1,x_2\in A_2,x_3\in A_3\}.$$

For $x\in T$, let $[xA_1A_2]=[\{x\}A_1A_2]$. The other cases can be defined analogously.

The notion of ternary semigroup was first introduced by Banach who showed by an example that a ternary semigroup does not necessary reduce to an ordinary semigroup [6]. Sioson [5] studied ideals and radicals on ternary semigroups. The authors in [3] investigated some properties of quasi-ideals and bi-ideals in ternary semigroups. Recently, the authors in [2] introduced and studied pure ideals in ternary semigroups; the authors in [4] studied regular ternary semigroups; and the author in [7] and [8] characterized (ordered) ternary semigroup to be left and right simple.

A ternary semigroup $(T,[\ ])$ is called an *ordered ternary semigroup* [9] if there is a partial order $\leq$ on T such that

$$x\leq y\Rightarrow[xx_1x_2]\leq[yx_1x_2],[x_1xx_2]\leq[x_1yx_2],[x_1x_2x]\leq[x_1x_2y]$$

for all $x,y,x_1,x_2\in T$.

Let $(T,[\ ],\leq)$ be an ordered ternary semigroup. For $A\subseteq T$, let

$$(A]=\{x\in T\mid x\leq a\text{ for some }a\in A\}.$$

For $A,B,C\subseteq T$, we have

(1) $A\subseteq(A]$,

(2) $A\subseteq B$ implies $(A]\subseteq(B]$,

(3) $((A]]=(A]$,

(4) $[(A](B](C]]\subseteq([ABC]]$.

In [9], the notion of ideal extensions of ordered ternary semigroups have been introduced and studied. In [10], the authors described fuzzy ideals and fuzzy filters of ordered ternary semigroups.

Let $(T,[\ ],\leq)$ be an ordered ternary semigroup. A nonempty subset A of T is called a *left (respectively, right, lateral) ideal* of T if

(i) $[TTA]\subseteq A$ (respectively, $[ATT]\subseteq A$, $[TAT]\subseteq A$).

(ii) If $x \in A$ and $y \in T$ such that $y \leq x$, then $y \in A$.

The second condition means that $A = (A]$. A nonempty subset A of T is called a *two-sided ideal* of T if A is both a left and a right ideal of T. Lateral ideals are also known as *middle ideals*. If A is a left, right and lateral ideal of T, then it is called an *ideal* of T. If A, B and C are two-sided ideals of T, then $([ABC]]$ is a two-sided ideal of T. Intersection of ideals of T is an ideal of T if it is nonempty. Union of ideals of T is an ideal of T. Finite intersection of ideals of T is an ideal of T.

Let $(T, [\ ], \leq)$ be an ordered ternary semigroup and A a nonempty subset of T. In [9], the intersection of all left ideals of T containing A, denoted by $(A)_l$, is the smallest left ideal of T containing A and $(A)_l = (A \cup [TTA]]$. Similarly,

$$\begin{aligned}(A)_r &= (A \cup [ATT]] \\ (A)_m &= (A \cup [TAT] \cup [T[TAT]T]] \\ (A)_t &= (A \cup [TTA] \cup [ATT] \cup [T[TAT]T]] \\ (A) &= (A \cup [TTA] \cup [ATT] \cup [TAT] \cup [T[TAT]T]]\end{aligned}$$

are the right, lateral, two-sided, and ideal of T generated by A, respectively.

Let $(T, [\ ], \leq)$ be an ordered ternary semigroup. An element a of T is said to be *regular* if there exists $x \in T$ such hat $a \leq [axa]$, and T is called *regular* if every element of T is regular. Note that T is regular if and only if $a \in ([aTa]]$ for all $a \in T$.

2 Pure ideals in ordered ternary semigroups

In [2], Bashir and Shabir studied pure ideals in ternary semigroup. In this section we define pure ideals in ordered ternary semigroups.

Definition 2.1. Let $(T, [\ ], \leq)$ be an ordered ternary semigroup. A two-sided ideal A of T is called a *left (respectively, right) pure two-sided ideal* if for each $x \in A$ there exist $y, z \in A$ such that $x \leq [yzx]$ (respectively, $x \leq [xyz]$). An ideal A of T is called *left (respectively, right) pure ideal* if for each $x \in A$ there exist $y, z \in A$ such that $x \leq [yzx]$ (respectively $x \leq [xyz]$). Similarly, we define one-sided left and right pure ideals.

Theorem 2.2. *Let $(T, [\ ], \leq)$ be an ordered ternary semigroup and A a two-sided ideal of T. Then A is right pure two-sided ideal if and only if $B \cap A = ([BAA]]$ for all right ideals B of T.*

Proof. Assume that A is right pure two-sided ideal. Let B be a right ideal of T. We have $[BAA] \subseteq [BTT] \subseteq B$. Then $([BAA]] \subseteq (B] = B$. Since $[BAA] \subseteq [TTA] \subseteq A$, so $([BAA]] \subseteq (A] = A$. Hence $([BAA]] \subseteq B \cap A$. To prove the reverse inclusion, let $x \in B \cap A$. By assumption, there exist $y, z \in A$ such that $x \leq [xyz]$. Since $[xyz] \in [BAA]$, we obtain $x \in ([BAA]]$. Thus $B \cap A \subseteq ([BAA]]$.

Conversely, suppose that $B \cap A = ([BAA]]$ for all right ideals B of T. Let $x \in A$. Since $(\{x\} \cup [xTT]]$ is a right ideal of T and $[TTA] \subseteq A$, we have

$$(\{x\} \cup [xTT]] \cap A = ([(\{x\} \cup [xTT]]AA]] \subseteq ([xAA] \cup [[xTT]AA]] \subseteq ([xAA]].$$

Since $x \in (\{x\} \cup [xTT]] \cap A$, $x \in ([xAA]]$. Hence A is a right pure two-sided ideal of T. □

Definition 2.3. An ordered ternary semigroup $(T, [\ ], \leq)$ is said to be *right weakly regular* if for any $x \in T$, $x \in ([[xTT][xTT][xTT]]]$.

Note that every regular ordered ternary semigroup is right weakly regular.

Theorem 2.4. *Let $(T, [\ ], \leq)$ be an ordered ternary semigroup. The following are equivalent.*

(i) *T is right weakly regular.*

(ii) *$([AAA]] = A$ for all right ideals A of T.*

(iii) *$B \cap A = ([BAA]]$ for all right ideals B and all two-sided ideals A of T.*

(iv) *$B \cap A = ([BAA]]$ for all right ideals B and all ideals A of T.*

Proof. (i) ⇒ (ii). Assume that T is right weakly regular. Let A be a right ideal of T. Since $[AAA] \subseteq [ATT] \subseteq A$, we have $([AAA]] \subseteq A$. Let $x \in A$. By assumption, $x \in ([[xTT][xTT][xTT]]] \subseteq ([AAA]]$. Then $A \subseteq ([AAA]]$, whence $([AAA]] = A$.

(ii) ⇒ (i). Assume that $([AAA]] = A$ for all right ideals A of T. Let $x \in T$. Since $(\{x\} \cup [xTT]]$ is a right ideal of T, we have

$$\begin{aligned}(\{x\} \cup [xTT]] &= ([(\{x\} \cup [xTT]](\{x\} \cup [xTT]](\{x\} \cup [xTT]]]\\ &\subseteq (\{[xxx]\} \cup [[xxx]TT] \cup [[xxT]Tx] \cup [[xxT]T[xTT]] \cup [[xTT]xx]\\ &\quad \cup [[xTT]x[xTT]] \cup [[xTT][xTT]x] \cup [[xTT][xTT][xTT]]].\end{aligned}$$

Since $x \in (\{x\} \cup [xTT]]$, we obtain (by calculations) $x \in ([[xTT][xTT][xTT]]]$. Hence T is right weakly regular.

(i) ⇒ (iii). Assume that T is right weakly regular. Let B and A be a right ideal and a two-sided ideal of T, respectively. Since $[BAA] \subseteq [BTT] \subseteq B$, $([BAA]] \subseteq B$. Similarly, $([BAA]] \subseteq A$. Then $([BAA]] \subseteq B \cap A$. Let $x \in B \cap A$. We have $([[xTT][xTT][xTT]]] \subseteq ([BAA]]$. By assumption, we get $x \in ([[xTT][xTT][xTT]]]$, hence $x \in ([BAA]]$. Thus $B \cap A \subseteq ([BAA]]$, whence $B \cap A = ([BAA]]$.

That (iii) ⇒(iv) is clear.

(iv) ⇒ (i). Assume that $B \cap A = ([BAA]]$ for all right ideals B and all ideals A of T. To prove that T is right weakly regular, let $x \in T$. We have

$$(\{x\} \cup [xTT]] \text{ and } (\{x\} \cup [xTT] \cup [TTx] \cup [TxT] \cup [T[TxT]T]]$$

are right ideal and ideal of T, respectively. Then

$$\begin{aligned}&(\{x\} \cup [xTT]] \cap (\{x\} \cup [xTT] \cup [TTx] \cup [TxT] \cup [T[TxT]T]]\\ &\quad = ((\{x\} \cup [xTT]](\{x\} \cup [xTT] \cup [TTx] \cup [TxT]\\ &\qquad \cup [T[TxT]T]](\{x\} \cup [xTT] \cup [TTx] \cup [TxT] \cup [T[TxT]T]]]\end{aligned}$$

$$\begin{aligned}&\subseteq (\{[xxx]\} \cup [[xxx]TT] \cup [x[xTT]x] \cup [x[xTx]T] \cup [x[xTT][xTT]] \cup [x[TTx]x] \\ &\quad \cup [[xTT][xxT]T] \cup [[xTT][xTT]x] \cup [[xTT][xTx]T] \cup [[xTT][xTT][xTT]] \\ &\quad \cup [[xTx]Tx] \cup [[xTx][TxT]T] \cup [[xTx][TTx]T]] \\ &\subseteq ([[xTT][xTT][xTT]]].\end{aligned}$$

Thus $x \in ([[xTT][xTT][xTT]]]$. Hence T is right weakly regular ordered ternary semigroup. □

Theorem 2.5. *Let $(T, [\ ], \leq)$ be an ordered ternary semigroup. The following are equivalent.*

(i) T is right weakly regular.

(ii) Every two-sided ideal A of T is right pure.

(iii) Every ideal A of T is right pure.

Proof. This follows from Theorem 2.2 and Theorem 2.4. □

Definition 2.6. Let $(T, [\ ], \leq)$ be an ordered ternary semigroup. An element 0 of T is called a zero of T if

(i) $[0x_1x_2] = [x_1 0 x_2] = [x_1x_2 0] = 0$ for all $x_1, x_2 \in T$.

(ii) $0 \leq x$ for all $x \in T$.

Theorem 2.7. *Let $(T, [\ ], \leq)$ be an ordered ternary semigroup with zero 0.*

(i) *$\{0\}$ is a right pure ideal of T.*

(ii) *Union of any right pure two-sided ideals (respectively, ideal) of T is a right pure two-sided ideal (respectively, ideals) of T.*

(iii) *Finite intersection of right pure two-sided ideals (respectively, ideal) of T is a right pure two-sided ideal (respectively, ideals) of T.*

Proof. (i) This is obvious.

(ii) Let A_i, $i \in I$ be right pure two-sided ideals of T. We have $\cup_{i\in I} A_i$ is a two-sided ideal of T. Let $x \in \cup_{i\in I} A_i$. Then $x \in A_j$ for some $j \in I$. Since A_j is right pure two-sided ideal, there exist $y, z \in A_j$ such that $x \leq [xyz]$. Since $y, z \in A_j \subseteq \cup_{i\in I} A_i$, we have $\cup_{i\in I} A_i$ is right pure.

(iii) Let $A_1, A_2, \ldots, A_n$ be right pure two-sided ideals of T. Then $\cap_{i=1}^n A_i$ is a two-sided ideal of T. Let $x \in \cap_{i=1}^n A_i$. For $k \in \{1, 2, \ldots, n\}$, there exist $y_k, z_k \in A_k$ such that $x \leq [xy_kz_k]$. We have

$$x \leq [[xy_nz_n] \cdots [y_2z_2y_1]z_1].$$

Since $[[y_nz_ny_{n-1}] \cdots [y_2z_2y_1]z_1] \in \cap_{i=1}^n A_i$, we have $\cap_{i=1}^n A_i$ is a right pure two-sided ideal of T. □

Theorem 2.8. *Let T be an ordered ternary semigroup with zero 0 and A a two-sided ideal of T. Then A contains the largest right pure two-sided ideal of T, denoted by $\mathcal{S}(A)$. $\mathcal{S}(A)$ is called the pure part of A.*

Proof. Since $\{0\}$ is a right pure two-sided ideal of T contained in A, it follows that the union of all right pure two-sided ideals of T contained in A exists, and hence it is the largest right pure two-sided ideal of T contained in A. □

Similarly, we have the following.

Theorem 2.9. *Let T be an ordered ternary semigroup with zero 0 and A an ideal of T. Then A contains the largest right pure ideal of T.*

Theorem 2.10. *Let $(T, [\ ], \leq)$ be an ordered ternary semigroup with zero 0. Let A, B and A_i, $i \in I$ be two-sided ideals of T.*

(i) $\mathcal{S}(A \cap B) = \mathcal{S}(A) \cap \mathcal{S}(B)$.

(ii) $\bigcup_{i\in I} \mathcal{S}(A_i) \subseteq \mathcal{S}(\bigcup_{i\in I} A_i)$.

Proof. (i) Since $\mathcal{S}(A) \subseteq A$ and $\mathcal{S}(B) \subseteq B$, we have $\mathcal{S}(A) \cap \mathcal{S}(B) \subseteq A \cap B$. Hence $\mathcal{S}(A) \cap \mathcal{S}(B) \subseteq \mathcal{S}(A \cap B)$. Since $\mathcal{S}(A \cap B) \subseteq A \cap B \subseteq A$, we get $\mathcal{S}(A \cap B) \subseteq \mathcal{S}(A)$. Similarly, $\mathcal{S}(A \cap B) \subseteq \mathcal{S}(B)$. Then $\mathcal{S}(A \cap B) \subseteq \mathcal{S}(A) \cap \mathcal{S}(B)$, whence $\mathcal{S}(A \cap B) = \mathcal{S}(A) \cap \mathcal{S}(B)$.

(ii) Since $\mathcal{S}(A_i) \subseteq A_i$ for all $i \in I$, we have $\bigcup_{i\in I} \mathcal{S}(A_i) \subseteq \bigcup_{i\in I} A_i$. Then $\bigcup_{i\in I} \mathcal{S}(A_i) \subseteq \mathcal{S}(\bigcup_{i\in I} A_i)$. □

Definition 2.11. A right pure two-sided ideal A of an ordered ternary semigroup $(T, [\ ], \leq)$ is said to be *purely maximal* if for any proper right pure two-sided ideal B of T, $A \subseteq B$ implies $A = B$.

Definition 2.12. A proper right pure two-sided ideal A of an ordered ternary semigroup $(T, [\ ], \leq)$ is said to be *purely prime* if for any right pure two-sided ideals B_1, B_2 of T, $B_1 \cap B_2 \subseteq A$ implies $B_1 \subseteq A$ or $B_2 \subseteq A$.

Theorem 2.13. *Every purely maximal two-sided ideal of an ordered ternary semigroup $(T, [\ ], \leq)$ is purely prime.*

Proof. Let A be a purely maximal two-sided ideal of T. Let B and C be right pure two-sided ideals of T such that $B \cap C \subseteq A$ and $B \nsubseteq A$. Since $B \cup A$ is a right pure two-sided ideal such that $A \subset B \cup A$, so $T = B \cup A$. We have

$$C = C \cap T = C \cap (B \cup A) = (C \cap B) \cup (C \cap A) \subseteq A.$$

Then A is purely prime. □

Theorem 2.14. *The pure part of any maximal two-sided ideal of an ordered ternary semigroup $(T, [\ ], \leq)$ with zero is purely prime.*

Proof. Let A be a maximal two-side ideal of T. To show that $\mathcal{S}(A)$ is purely prime, let B, C be right pure two-side ideals of T such that $B \cap C \subseteq \mathcal{S}(A)$. If $B \subseteq A$, then $B \subseteq \mathcal{S}(A)$. Suppose that $B \nsubseteq A$. We have $B \cup A$ is a two-side ideal of T. By maximality of A, $T = B \cup A$, and hence $C \subseteq A$. Thus $C \subseteq \mathcal{S}(A)$. □

Theorem 2.15. *Let $(T, [\ ], \leq)$ be an ordered ternary semigroup and A a right pure two-sided ideal of T. If $x \in T \setminus A$, then there exists a purely prime two-sided ideal B of T such that $A \subseteq B$ and $x \notin B$.*

Proof. Let $P = \{B \mid B$ is a right pure two-sided ideal of T, $A \subseteq B$ and $x \notin B\}$. Since $A \in P$, $P \neq \emptyset$. Under the usual inclusion, P is a partially ordered set. Let $B_k, k \in K$ be a totally ordered subset of P. By Theorem 2.7, $\bigcup_{k\in K} B_k$ is a right pure two-sided ideal of T. Since $A \subseteq \bigcup_{k\in K} B_k$ and $x \notin \bigcup_{k\in K} B_k$, we have $\bigcup_{k\in K} B_k \in P$. By Zorn's Lemma, P has a maximal element, say M. Then M is a right pure two-sided ideal, $A \subseteq M$ and $x \notin M$. We shall show that M is purely prime. Let A_1 and A_2 be right pure two-sided ideals of T such that $A_1 \nsubseteq M$ and $A_2 \nsubseteq M$. Since A_1, A_2 and M are right pure two-sided ideals of T, we obtain $A_1 \cup M$ and $A_2 \cup M$ are right pure two-sided ideals of T such that $M \subset A_1 \cup M$ and $M \subset A_2 \cup M$. Thus $x \in A_i \cup M$ $(k = 1, 2)$. Since $x \notin M$, $x \in A_1 \cap A_2$. Hence $A_1 \cap A_2 \nsubseteq M$. This shows that M is purely prime. □

Theorem 2.16. *Any proper right pure two-sided ideal A of an ordered ternary semigroup $(T, [\ ], \leq)$ is the intersection of all the purely prime two-sided ideals of T containing A.*

Proof. By Theorem 2.15, there exists purely prime ideals containing A. Let $\{B_i : i \in I\}$ be the set of all purely prime two-sided ideals of T containing A. We have $A \subseteq \bigcap_{i\in I} B_i$. To show that $\bigcap_{k\in K} B_k \subseteq A$. Let $x \notin A$. By Theorem 2.15, there exists purely prime ideal B_j such that $A \subseteq B_j$ and $x \notin B_j$. Hence $x \notin \bigcap_{i\in I} B_i$. □

3 Weakly pure ideals in ordered ternary semigroups

In this section, we introduce the concept of weakly pure ideal in ordered ternary semigroups.

Definition 3.1. Let $(T, [\ ], \leq)$ be an ordered ternary semigroup. A two-sided ideal A of T is called *left (respectively, right) weakly pure* if $A \cap B = ([AAB]]$ (respectively, $A \cap B = ([BAA]]$) for all two-sided ideals B of S.

In an ordered ternary semigroup, every left (right) pure two-sided ideals is left (right) weakly pure.

Theorem 3.2. *Let* $(T, [\ ], \leq)$ *be an ordered ternary semigroup with zero* 0. *If* A *and* B *are two-sided ideals of* T*, then*

$$BA^{-1} = \{t \in T \mid \forall x, y \in A, [xyt] \in B\}$$
$$A_{-1}B = \{t \in T \mid \forall x, y \in A, [txy] \in B\}$$

are two-sided ideals of T.

Proof. We shall show that BA^{-1} is a two-sided ideal of T. That $A_{-1}B$ is a two-sided ideal of T can be proved similarly. Clearly, $0 \in BA^{-1}$. Let $u, v \in T$ and $t \in BA^{-1}$. To show that $[uvt] \in BA^{-1}$, let $x, y \in A$. Since $[yuv] \in A$, we have $[xy[uvt]] = [x[yuv]t] \in B$. Thus $[uvt] \in BA^{-1}$.

Let $x \in BA^{-1}$ and $y \in T$ be such that $y \leq x$. Let $z, w \in A$. Since $[zwy] \leq [zwx]$ and $[zwx] \in B$, we have $[zwy] \in B$. Hence $y \in BA^{-1}$. Therefore, BA^{-1} is a two-sided ideal of T. □

Theorem 3.3. *Let* $(T, [\ ], \leq)$ *be an ordered ternary semigroup and* A *a two-sided ideal of* T. *Then* A *is left (right) weakly pure two-sided ideal if and only if* $(BA^{-1}) \cap A = A \cap B$ $((BA_{-1}) \cap A = A \cap B)$ *for all ideals* B *of* T.

Proof. Suppose that A is left weakly pure two-sided ideal. Let B be a ideal of T. By Theorem 3.2, BA^{-1} is an two-side ideal of T, and thus $A \cap BA^{-1} = ([AA(BA^{-1})]]$. Since $[AA(BA^{-1})] \subseteq [ATT] \subseteq A$, we have $([AA(BA^{-1}])] \subseteq (A] = A$. Let $t \in ([AA(BA^{-1})]]$ be such that $t \leq [xyz]$ for some $x, y \in A, z \in BA^{-1}$. By definition of BA^{-1}, $[xyz] \in B$. Thus $t \in B$. This proves that $A \cap BA^{-1} \subseteq A \cap B$. For the reverse inclusion, let $a \in A \cap B$. Since $[xya] \in B$ for any $x, y \in A$, we have $a \in BA^{-1}$. We get $a \in BA^{-1} \cap A$, and then $A \cap B \subseteq BA^{-1} \cap A$.

Conversely, assume that $(BA^{-1}) \cap A = A \cap B$ for all ideal B of T. To show that A is left weakly pure two-sided ideal, let C be any ideal of T. To show that $A \cap C = ([AAC]]$. By assumption, $A \cap C = CA^{-1} \cap A$. Since $[AAC] \subseteq [ATT] \subseteq A$, $([AAC]] \subseteq A$. Let $t \in ([AAC]]$ such that $t \leq [xyz]$ for some $x, y \in A, z \in C$ and let $a, b \in A$. Since $[a[bxy]z] = ab[xyz] \in C$, we obtain $[xyz] \in CA^{-1}$, and so $t \in CA^{-1}$. Then $([AAC]] \subseteq CA^{-1}$. This proves that $([AAC]] \subseteq A \cap C$. For the reverse inclusion, we have $C \subseteq ([AAC]]A^{-1}$ because $c \in C, a, b \in A$ implies $[abc] \in [AAC] \subseteq ([AAC]]$. Then $A \cap C \subseteq ([AAC]]A^{-1} \cap A = A \cap ([AAC]] \subseteq ([AAC]]$. □

Theorem 3.4. *Let* $(T, [\ ], \leq)$ *be an ordered ternary semigroup. The following are equivalent.*

(i) *Every two-sided ideal is left weakly pure two-sided ideal.*

(ii) *For every two-sided ideal* A *of* T, $[AAA] = A$.

(iii) *Every two-sided ideal is right weakly pure two-sided ideal.*

Proof. This can be proved similarly as Proposition 4.4 in [2]. □

4 Pure spectrum of an ordered ternary semigroup

Let $(T, [\,], \leq)$ be an ordered ternary semigroup with zero such that $[TTT] = T$. The set of all right pure ideals of T and the set of all proper purely prime ideals of T will be denoted by $P(T)$ and $P'(T)$, respectively. For $A \in P(T)$, let

$$I_A = \{J \in P'(T) \mid A \nsubseteq J\} \text{ and } \tau(T) = \{I_A \mid A \in P(T)\}.$$

Theorem 4.1. *$\tau(T)$ forms a topology on $P'(T)$.*

Proof. Since $\{0\}$ is a right pure ideal of T and $I_{\{0\}} = \emptyset$, we have $\emptyset \in \tau(T)$. Since T is a right pure ideal of T such that $I_T = P'(T)$, we get $P'(T) \in \tau(T)$.

Let $\{I_{A_\alpha} \mid \alpha \in \Lambda\} \subseteq \tau(T)$. We have $\bigcup_{\alpha\in\Lambda} I_{A_\alpha} = I_{\bigcup_{\alpha\in\Lambda} A_\alpha}$, whence $\bigcup_{\alpha\in\Lambda} I_{A_\alpha} \in \tau(T)$.

Let $I_{A_1}, I_{A_2} \in \tau(T)$. We shall show that $I_{A_1} \cap I_{A_2} = I_{A_1\cap A_2}$, therefore let $J \in I_{A_1} \cap I_{A_2}$. We have $J \in P'(T)$, $A_1 \nsubseteq J$ and $A_2 \nsubseteq J$. Suppose that $A_1 \cap A_2 \subseteq J$. Since J is purely prime, $A_1 \subseteq J$ or $A_2 \subseteq J$. A contradiction. Then $J \in I_{A_1\cap A_2}$, hence $I_{A_1} \cap I_{A_2} \subseteq I_{A_1\cap A_2}$. For the reverse inclusion, let $J \in I_{A_1\cap A_2}$. Since $A_1 \cap A_2 \nsubseteq J$, $A_1 \nsubseteq J$ and $A_2 \nsubseteq J$. This implies that $J \in I_{A_1} \cap I_{A_2}$, thus $I_{A_1\cap A_2} \subseteq I_{A_1} \cap I_{A_2}$.

Then $\tau(T)$ forms a topology on $P'(T)$. □

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