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On the Otimes Operator Related to Nonlinear Heat Equation

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Abstract : In this paper, we are finding nonlinear heat equation in n-dimensional. By method of Fourier transform in sense of Distribution theory we obtain the solution in the convolution form. On the suitable we obtain the solution nonlinear triharmonic heat equation.

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1 Introduction

It is well known that for the heat equation

$$\frac{\partial}{\partial t}u(x,t) = c^2\triangle u(x,t) \tag{1.1}$$

with the initial condition

$$u(x,0) = f(x)$$

where $\triangle = \sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}$ is the Laplace operator and $(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times (0,\infty)$, we obtain

$$u(x,t) = \frac{1}{(4c^2\pi t)^{n/2}}\int_{\mathbb{R}^n}\exp\left(-\frac{|x-y|^2}{4c^2t}\right)f(y)dy \tag{1.2}$$

as the solution of (1.1). Now, (1.2) can be written $u(x,t) = E(x,t) * f(x)$ where

$$E(x,t) = \frac{1}{(4c^2\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4c^2 t}\right). \tag{1.3}$$

$E(x,t)$ is called *the heat kernel*, where $|x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$ and $t > 0$, [1, pp. 208–209].

Next, Nonlaopon and Kananthai [2] study the equation

$$\frac{\partial}{\partial t} u(x,t) = c^2 \Box u(x,t),$$

and $\Box^k$ is the ultra-hyperbolic operator iterated $k-$ times and is defined by

$$\Box^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}\right)^k. \tag{1.4}$$

They obtained the ultra-hyperbolic heat kernel

$$E(x,t) = \frac{i^q}{(4c^2\pi t)^{n/2}} \exp\left(-\frac{\sum_{i=1}^{p} x_i^2 - \sum_{j=p+1}^{p+q} x_j^2}{4c^2 t}\right), \tag{1.5}$$

where $p + q = n$ and n is the dimension of the Euclidean space R^n and $i = \sqrt{-1}$ for finding the kernel $E(x,t)$.

In 1996, Kananthai [3] first introduced the operator $\diamondsuit^k$ and is named Diamond operator and is defined by

$$\diamondsuit^k = \left(\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2\right)^k. \tag{1.6}$$

The operator $\diamondsuit^k$ can be written as the product of the operators in the form

$$\diamondsuit^k = \triangle^k \Box^k = \Box^k \triangle^k, \tag{1.7}$$

where $\triangle^k$ is the Laplacian operator iterated $k-$ times and is defined by

$$\triangle^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right)^k. \tag{1.8}$$

The Fourier transform of the Diamond operator also has been studied and the elementary solution of such operator, [4]. In 2009, Satsanit [5] has first introduced the operator $\otimes^k$, where $\otimes^k$ is defined by

$$\begin{aligned}
\otimes^k &= \left(\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^3 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^3 \right)^k \\
&= \left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^k \\
&\quad \times \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^2 + \left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right) \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right) + \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right]^k \\
&= (\Box)^k \left(\triangle^2 - \frac{1}{4}(\triangle + \Box)(\triangle - \Box) \right)^k \\
&= \left(\frac{3}{4} \Diamond \triangle + \frac{1}{4} \Box^3 \right)^k, \qquad (1.9)
\end{aligned}$$

where $\triangle, \Box$ and $\Diamond$ are defined by (1.8), (1.4) and (1.6) with $k = 1$ respectively.

Now, the purpose of this work is to study the equation

$$\frac{\partial}{\partial t} u(x,t) - c^2(-\otimes)^k u(x,t) = f(x,t,u(x,t)) \qquad (1.10)$$

for $(x,t) \in \mathbb{R}^n \times (0,\infty)$. We consider the equation (1.10) which is in the form of nonlinear heat equation with the following conditions on u and f as follows

(1) $u(x,t) \in C^{(6k)}(\mathbb{R}^n)$ for any $t > 0$ where $C^{(6k)}(\mathbb{R}^n)$ is the space of continuous function with $6k$-derivatives.

(2) f satisfies the Lipchitz condition,

$$|f(x,t,u) - f(x,t,w)| \leq A|u - w|$$

where A is constant with $0 < A < 1$.

(3) $\int_0^\infty \int_{\mathbb{R}^n} \left| t^{-n/6k} f(x,t,u(x,t)) \right| dxdt < \infty$ for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $0 < t < \infty$ and $u(x,t)$ is continuous function on $\mathbb{R}^n \times (0,\infty)$.

Under such conditions of f and u and for the spectrum of $E(x,t)$, we obtain the convolution

$$u(x,t) = E(x,t) * f(x,t,u(x,t))$$

as a unique solution of (1.10) where $E(x,t)$ is defined by (2.9).

2 Preliminaries

Definition 2.1. Let $f(x) \in \mathrm{L}_1(\mathbb{R}^n)$-the space of integrable function in $\mathbb{R}^n$. The Fourier transform of $f(x)$ is defined by

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(\xi,x)} f(x)\, dx, \tag{2.1}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $(\xi, x) = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$ and $dx = dx_1\, dx_2 \ldots dx_n$.

Also, the inverse of Fourier transform is defined by

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi,x)} \widehat{f}(\xi)\, d\xi. \tag{2.2}$$

If f is a distribution with compact supports then Eq.(2.1) can be written as

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \left\langle f(x), e^{-i\xi . x} \right\rangle. \tag{2.3}$$

[6, Theorem 7.4-3, p.187].

Lemma 2.2. *Given the function*

$$f(x) = \exp\left(\left(\sum_{i=1}^{p} x_i^2\right)^3 - \left(\sum_{j=p+1}^{p+q} x_j^2\right)^3\right)^k$$

where

$$(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, \quad p+q = n, \left(\sum_{i=1}^{p} x_i^2\right)^3 < \left(\sum_{j=p+1}^{p+q} x_j^2\right)^3$$

and k is the positive odd number. Then

$$\left| \int_{\mathbb{R}^n} f(x) dx \right| \leq \frac{\pi^{n/2}}{9k^2} \cdot \frac{\Gamma(\frac{p}{6k})\Gamma(\frac{q}{6k})}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})},$$

where $\frac{p+q}{2} = \frac{n}{2}$ and Γ denoted the gamma function. That is $\int_{\mathbb{R}^n} f(x)dx$ is bounded.

Proof.

$$\int_{\mathbb{R}^n} f(x)dx = \int_{\mathbb{R}^n} \exp\left(\left(\sum_{i=1}^{p} x_i^2\right)^3 - \left(\sum_{j=p+1}^{p+q} x_j^2\right)^3\right)^k dx$$

$$= \int_{\mathbb{R}^n} \exp\left(-\left(\left(\sum_{j=p+1}^{p+q} x_j^2\right)^3 - \left(\sum_{i=1}^{p} x_i^2\right)^3\right)^k\right) dx$$

for k is a positive odd number. By changing the coordinate, we put

$$x_1 = iy_1, \quad x_2 = iy_2, \ldots, \quad x_p = iy_p$$

$$dx_1 = idy_1, \quad dx_2 = idy_2, \ldots, \quad dx_p = idy_p$$

and

$$x_{p+1} = y_{p+1}, \quad x_{p+2} = y_{p+2}, \ldots, \quad x_{p+q} = y_{p+q}$$

$$dx_{p+1} = dy_{p+1}, \quad dx_{p+2} = dy_{p+2}, \ldots, \quad dx_{p+q} = dy_{p+q},$$

where $i = \sqrt{-1}$. Then we obtain

$$\int_{\mathbb{R}^n} f(x)dx = i^p \int_{\mathbb{R}^n} \exp\left(-\left(\left(\sum_{j=p+1}^{p+q} y_j^2\right)^3 + \left(\sum_{i=1}^{p} y_i^2\right)^3\right)\right)^k dy \tag{2.4}$$

Let us transform to bipolar coordinates defined by

$$y_1 = rw_1, y_2 = rw_2, \ldots, \quad y_p = rw_p$$

$$and \quad y_{p+1} = sw_{p+1}, y_{p+2} = sw_{p+2}, \ldots, y_{p+q} = sw_{p+q}, \quad p+q = n$$

where $w_1^2 + w_2^2 + \cdots + w_p^2 = 1$ and $w_{p+1}^2 + w_{p+2}^2 + \cdots + w_{p+q}^2 = 1$, Thus

$$\int_{\mathbb{R}^n} f(x)dx = i^p \int_{\mathbb{R}^n} \exp\left(-\left(r^6 + s^6\right)^k\right) r^{p-1} s^{q-1} dr ds d\Omega_p d\Omega_q \tag{2.5}$$

where

$$dy = r^{p-1} s^{q-1} dr ds d\Omega_p d\Omega_q, \tag{2.6}$$

$d\Omega_p$ and $d\Omega_q$ are the elements of surface area on the unit sphere in R^p and R^q respectively. By computing directly, we obtain

$$\int_{\mathbb{R}^n} f(x)dx = i^p \Omega_p \Omega_q \int_0^\infty \int_0^\infty \exp\left(-\left(r^6 + s^6\right)^k\right) r^{p-1} s^{q-1} dr ds \tag{2.7}$$

where $\Omega_p = \dfrac{2\pi^{p/2}}{\Gamma(p/2)}$ and $\Omega_q = \dfrac{2\pi^{q/2}}{\Gamma(q/2)}$. Since $(r^6 + s^6)^k \geq r^{6k} + s^{6k}$, we have

$$\exp(-\left(r^6 + s^6\right)^k\right) \leq \exp(-\left(r^{6k} + s^{6k}\right)$$

Thus

$$|\int_{\mathbb{R}^n} f(x)dx| \leq \Omega_p \Omega_q \int_0^\infty \int_0^\infty \exp\left(-r^{6k} - s^{6k}\right) r^{p-1} s^{q-1} dr ds$$
$$= \Omega_p \Omega_q \int_0^\infty \exp\left(-r^{6k}\right) r^{p-1} dr \int_0^\infty \exp\left(-s^{6k}\right) s^{q-1} ds.$$

Put $u = r^{6k}, dr = \frac{1}{6k}u^{\frac{1}{6k}-1}du$ and $v = s^{6k}, ds = \frac{1}{6k}v^{\frac{1}{6k}-1}dv$ in the above equation, we obtain

$$\begin{aligned}\left|\int_{\mathbb{R}^n} f(x)dx\right| &\leq \frac{\Omega_p\Omega_q}{(6k)^2}\int_0^\infty e^{-u}u^{\frac{1}{6k}-1}du\int_0^\infty e^{-v}v^{\frac{1}{6k}-1}dv \\ &= \frac{\Omega_p\Omega_q}{(6k)^2}\Gamma\left(\frac{p}{6k}\right)\Gamma\left(\frac{q}{6k}\right) \\ &= \frac{2\pi^{p/2}2\pi^{q/2}}{(6k)^2}\frac{\Gamma(\frac{p}{6k})\Gamma(\frac{q}{6k})}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})} \\ &= \frac{\pi^{n/2}}{9k^2}\frac{\Gamma(\frac{p}{6k})\Gamma(\frac{q}{6k})}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})},\end{aligned}$$

where $\frac{p+q}{2} = \frac{n}{2}$. That is $\int_{\mathbb{R}^n} f(x)dx$ is bounded. □

Lemma 2.3. *(The Fourier transform of $\otimes^k\delta$)*

$$\mathcal{F}\otimes^k\delta = \frac{(-1)^{3k}}{(2\pi)^{n/2}}\left[\left(\xi_1^2+\xi_2^2+\cdots+\xi_p^2\right)^3 - \left(\xi_{p+1}^2+\xi_{p+2}^2+\cdots+\xi_{p+q}^2\right)^3\right]^k,$$

where $\mathcal{F}$ is the Fourier transform defined by (2.1) and if the norm of ξ is given by $\|\xi\| = \left(\xi_1^2+\xi_2^2+\cdots+\xi_n^2\right)^{1/2}$ then

$$\mathcal{F}\otimes^k\delta \leq \frac{M}{(2\pi)^{n/2}}\|\xi\|^{6k}.$$

Since M is positive constant thus $\mathcal{F}\otimes^k\delta$ is bounded and continuous on the space $\mathcal{S}'$ of the tempered distribution. Moreover, by Eq.(2.9)

$$\otimes^k\delta = \mathcal{F}^{-1}\frac{(-1)^{3k}}{(2\pi)^{n/2}}\left[\left(\xi_1^2+\xi_2^2+\cdots+\xi_p^2\right)^3 - \left(\xi_{p+1}^2+\xi_{p+2}^2+\cdots+\xi_{p+q}^2\right)^3\right]^k$$

Proof. See [5]. □

Lemma 2.4. *Given the operator*

$$L = \frac{\partial}{\partial t} - c^2(-\otimes)^k, \tag{2.8}$$

where $(-\otimes)^k$ is the operator defined by (1.5), k is a positive odd integer, $u(x,t)$ is an unknown function for $(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times (0,\infty)$, and c is a positive constant. Then we obtain

$$E(x,t) = \frac{1}{(2\pi)^n}\int_{R^n}\exp\left[-c^2\left[\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^3 - \left(\sum_{i=1}^{p}\xi_i^2\right)^3\right]^k t + i(\xi,x)\right]d\xi, \tag{2.9}$$

where $\sum_{j=p+1}^{p+q}\xi_j^2 > \sum_{i=1}^{p}\xi_i^2$, as an elementary solution for the operator L defined by (2.8).

Proof. We have to find function $E(x,t)$ from the equation

$$L\left(E(x,t)\right)=\delta(x,t),$$

where $\delta(x,t)$ is dirac delta function for $(x,t)\in\mathbb{R}^n\times(0,\infty)$.We can write

$$\frac{\partial}{\partial t}E(x,t)-c^2(-\otimes)^kE(x,t)=\delta(x)\cdot\delta(t). \tag{2.10}$$

By taking the Fourier transform defined by (2.1) to both sides of (2.10), we obtain

$$\frac{\partial}{\partial t}\widehat{E}(\xi,t)-c^2\left[\left(\sum_{i=1}^{p}\xi_i^2\right)^3-\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^3\right]^k\widehat{E}(\xi,t)=\frac{1}{(2\pi)^{n/2}}\delta(t),$$

which has solution

$$\widehat{E}(\xi,t)=\frac{H(t)}{(2\pi)^{n/2}}\exp\left[c^2t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^3-\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^3\right)^k\right] \tag{2.11}$$

where $H(t)$ is a heaviside function and $H(t)=1$ for $t>0$. Since k is a positive odd number and $\sum_{j=p+1}^{p+q}\xi_j^2>\sum_{i=1}^{p}\xi_i^2$, thus $\widehat{E}(\xi,t)$ is bounded and can be written by

$$\widehat{E}(\xi,t)=\frac{H(t)}{(2\pi)}\exp\left[-c^2t\left(\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^3-\left(\sum_{i=1}^{p}\xi_i^2\right)^3\right)^k\right]. \tag{2.12}$$

Now, the inverse Fourier transform

$$E(x,t)=\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}e^{i(\xi,x)}\widehat{E}(\xi,t)d\xi.$$

Thus

$$\begin{aligned}E(x,t)&=\frac{H(t)}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}e^{i(\xi,x)}\exp\left[-c^2t\left(\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^3-\left(\sum_{i=1}^{p}\xi_i^2\right)^3\right)^k\right]\\&=\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}e^{i(\xi,x)}\exp\left[-c^2t\left(\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^3-\left(\sum_{i=1}^{p}\xi_i^2\right)^3\right)^k\right]\end{aligned}$$

or

$$E(x,t)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\exp\left[-c^2t\left[\left(\sum_{j=p=1}^{p+q}\xi_j^2\right)^3-\left(\sum_{i=1}^{p}\xi_i^2\right)^3\right]^k+i(\xi,x)\right]d\xi. \tag{2.13}$$

as an elementary solution of (2.10) and bounded if $\sum_{j=p+1}^{p+q}\xi_j^2>\sum_{i=1}^{p}\xi_i^2$and k is positive odd number. □

3 Main Results

Theorem 3.1. *The kernel $E(x,t)$ defined by (2.13) has the following properties:*

(1) $E(x,t)\in\mathcal{C}^\infty$-the space of continuous function for $x\in\mathbb{R}^n$, $t>0$ with infinitely differentiable.

(2) $\left(\frac{\partial}{\partial t}-c^2(-\otimes)^k\right)E(x,t)=0$ for $t>0$and k is positive odd number.

(3) $E(x,t)>0$ for $t>0$.

(4) $|E(x,t)|\leq\frac{M(t)}{9.2^n\pi^{n/2}k^2(c^2t)^{\frac{n}{6k}}}\frac{\Gamma(\frac{p}{6k})\Gamma(\frac{q}{6k})}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})}$, for $t>0$, where $M(t)$ is a positive constant. Thus $E(x,t)$ is bounded for any fixed $t>0$.

(5) $\lim_{t\to0}E(x,t)=\delta$.

Proof. (1) From (2.13), since

$$\frac{\partial^n}{\partial x^n}E(x,t)=\frac{1}{(2\pi)^n}\int_{R^n}\frac{\partial^n}{\partial x^n}\exp\left[c^2\left[\left(\sum_{j=p+1}^{p+q}\xi_i^2\right)^3-\left(\sum_{i=1}^{p}\xi_j^2\right)^3\right]^k t+i(\xi,x)\right]d\xi.$$

Thus $E(x,t)\in\mathcal{C}^\infty$ for $x\in\mathbb{R}^n$, $t>0$.

(2) By computing directly, we obtain

$$\left(\frac{\partial}{\partial t}-c^2(-\otimes)^k\right)E(x,t)=0.$$

for $t>0$ where $E(x,t)$ is defined by (2.13).

(3) $E(x,t)>0$ for $t>0$ is obvious by (2.13).

(4) We have

$$E(x,t)=\frac{1}{(2\pi)^n}\int_{R^n}\exp\left[-c^2t\left[\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^3-\left(\sum_{i=1}^{p}\xi_i^2\right)^3\right]^k+i(\xi,x)\right]d\xi.$$

put

$$\xi_1 = iy_1, \quad \xi_2 = iy_2, \ldots, \quad \xi_p = iy_p$$
$$d\xi_1 = idy_1, \quad d\xi_2 = idy_2, \ldots, \quad d\xi_p = idy_p$$

and

$$\xi_{p+1} = y_{p+1}, \quad \xi_{p+2} = y_{p+2}, \ldots, \quad \xi_{p+q} = y_{p+q}$$
$$d\xi_{p+1} = dy_{p+1}, \quad d\xi_{p+2} = dy_{p+2}, \ldots, \quad d\xi_{p+q} = dy_{p+q},$$

where $i = \sqrt{-1}$. Thus, we obtain

$$E(x,t) = \frac{i^p}{(2\pi)^n} \int_{R^n} \exp\left[-c^2 t \left[\left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^3 - \left(\sum_{i=1}^{p} \xi_i^2\right)^3\right]^k - \sum_{r=1}^{p} x_r y_r + i \sum_{j=p+1}^{p+q} x_j y_j\right] dy.$$

$$|E(x,t)| \leq \frac{M}{(2\pi)^n} \int_{R^n} \exp\left[-c^2 t \left(\left(\sum_{j=p+1}^{p+q} y_j^2\right)^3 + \left(\sum_{r=1}^{p} y_r^2\right)^3\right)\right]^k dy, \qquad (3.1)$$

where M is a positive constant. The same process as Lemma 2.1 then (3.1) becomes

$$|E(x,t)| \leq \frac{M(t)}{9.2^n \pi^{n/2} k^2 (c^2 t)^{\frac{n}{6k}}} \frac{\Gamma(\frac{p}{6k})\Gamma(\frac{q}{6k})}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})}$$

(5) We have

$$E(x,t) = \frac{1}{(2\pi)^n} \int_{R^n} \exp\left[-c^2 t \left[\left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^3 - \left(\sum_{i=1}^{p} \xi_i^2\right)^3\right]^k + i(\xi, x)\right] d\xi.$$

Since $E(x,t)$ exists, we have

$$\begin{aligned} \lim_{t\to 0} E(x,t) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(\xi,x)}\, d\xi \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(\xi,x)}\, d\xi \\ &= \delta(x), \qquad \text{for } x \in \mathbb{R}^n \end{aligned} \qquad (3.2)$$

[7, p.396, Eq.(10.2.19b)]. □

Theorem 3.2. *Given the nonlinear equation*

$$\frac{\partial}{\partial t} u(x,t) - c^2(-\otimes)^k u(x,t) = f(x,t,u(x,t)) \qquad (3.3)$$

for $(x,t) \in \mathbb{R}^n \times (0,\infty)$. *We consider the equation (3.3) which is in the form of nonlinear heat equation with the following conditions on* u *and* f *as follows*

(1) $u(x,t) \in C^{(6k)}(\mathbb{R}^n)$ *for any* $t>0$ *where* $C^{(6k)}(\mathbb{R}^n)$ *is the space of continuous function with* $6k$*-derivatives.*

(2) f *satisfies the Lipchitz condition,*

$$|f(x,t,u) - f(x,t,w)| \leq A|u-w|$$

where A *is constant with* $0 < A < 1$.

(3) $\int_0^\infty \int_{\mathbb{R}^n} \left| t^{-n/6k} f(x,t,u(x,t)) \right| dxdt < \infty$ *for* $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $0 < t < \infty$ *and* $u(x,t)$ *is continuous function on* $\mathbb{R}^n \times (0,\infty)$.

Under such conditions of f *and* u, *we obtain the convolution*

$$u(x,t) = E(x,t) * f(x,t,u(x,t))$$

as a unique solution of (3.3) where $E(x,t)$ *is an elementary solution defined by (2.9). In particular, if we put* $k=1$ *and* $p=0$ *in (3.3) then (3.3) reduces to the equation*

$$\frac{\partial}{\partial t} u(x,t) - c^2 \triangle^3 u(x,t) = f(x,t,u(x,t)),$$

where is related to the nonlinear triharmonic heat equation.

Proof. Convolving both sides of (3.1) with $E(x,t)$, that is

$$E(x,t) * \left[\frac{\partial}{\partial t} u(x,t) - c^2 (-\otimes)^k u(x,t) \right] = E(x,t) * f(x,t,u(x,t))$$

or

$$\left[\frac{\partial}{\partial t} E(x,t) - c^2 (-\otimes)^k E(x,t) \right] * u(x,t) = E(x,t) * f(x,t,u(x,t)),$$

so

$$\delta(x,t) * u(x,t) = E(x,t) * f(x,t,u(x,t)).$$

Thus

$$\begin{aligned} u(x,t) &= E(x,t) * f(x,t,u(x,t)) \\ &= \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} E(r,s) f(x-r, t-s, u(x-r,t-s)) drds \end{aligned}$$

where $E(r,s)$ is given by definition (2.4). We next show that $u(x,t)$ is bounded on $\mathbb{R}^n \times (0,\infty)$. We have

$$\begin{aligned} |u(x,t)| &\leq \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} |E(r,s)| \cdot f(x-r,t-s,u(x-r,t-s)) drds. \\ &\leq \frac{M}{9.2^n \pi^{n/2} k^2 c^{n/3k}} \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} s^{-n/6k} |f(x-r,t-s,u(x-r,t-s))| drds \\ &\leq \frac{MN}{9.2^n \pi^{n/2} k^2 c^{n/3k}} \end{aligned} \tag{3.4}$$

where $N = \int_{-\infty}^{\infty}\int_{\mathbb{R}^n} |f(x-r, t-s, u(x-r, t-s))|drds$. Thus $u(x,t)$ is bounded on $\mathbb{R}^n \times (0,\infty)$. To show that $u(x,t)$ is unique. Now, we next to show that $u(x,t)$ is unique. Let $w(x,t)$ be another solution of (3.1), then

$$w(x,t) = E(x,t) * f(x,t,w(x,t))$$

for $(x,t) \in \Omega_0 \times (0,T]$ the compact subset of $\mathbb{R}^n \times [0,\infty)$ and $E(x,t)$ is defined by (2.6).

Now, define $\|u(x,t)\| = \sup_{\substack{x\in\Omega_0 \\ 0<t\leq T}} |u(x,t)|$.

Now,

$$\begin{aligned} |u(x,t) - w(x,t)| &= |E(x,t) * f(x,t,u(x,t)) - E(x,t) * f(x,t,w(x,t))| \\ &\leq \int_{-\infty}^{\infty}\int_{\mathbb{R}^n} |E(r,s)| \cdot |f(x-r, t-s, u(x-r, t-s)) \\ &\quad - f(x-r, t-s, w(x-r, t-s))|drds \\ &\leq A|E(r,s)| \int_{-\infty}^{\infty}\int_{\mathbb{R}^n} |u(x-r,t-s) - w(x-r,t-s)|drds \end{aligned}$$

by (2.9) and the condition (2) of the theorem. Now, for $(x,t) \in \Omega_0 \times (0,T]$ we have

$$\begin{aligned} |u-w| &\leq A|E(r,s)|\|u-w\| \int_0^T ds \int_{\Omega_0} dr \\ &= A|E(r,s)|TV(\Omega_0)\|u-w\| \end{aligned} \tag{3.5}$$

where $V(\Omega_0)$ is the volume of the surface on Ω_0.

Choose $A|E(r,s)|TV(\Omega_0) \leq 1$ or $A \leq \dfrac{1}{|E(r,s)|TV(\Omega_0)}$. Thus from (3.5),

$$\|u-w\| \leq \alpha\|u-w\| \quad \text{where} \quad \alpha = A|E(r,s)|TV(\Omega_0) \leq 1.$$

It follows that $\|u-w\| = 0$, thus $u = w$. That is the solution u of (3.3) is unique. In particular, if we put $k = 1$ and $p = 0$ in (3.3), then (3.3) reduces to the nonlinear heat equation

$$\frac{\partial}{\partial t}u(x,t) - c^2\triangle^3 u(x,t) = f(x,t,u(x,t))$$

which has solution

$$u(x,t) = E(x,t) * f(x,t,u(x,t))$$

where $E(x,t)$ is defined by (2.9) with $k = 1$ and $p = 0$. That is complete of proof. □

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