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Fixed Points, Common Fixed Points and Invariant Approximation[1]

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Abstract : Some new fixed point and common fixed point theorems for nonexpansive and generalized nonexpansive mapppings have been proved in the framework of metric spaces. Invaraint approximation results have been obtained for these types of mappings as applications. Our results generalize and extend several known results available in the literature.

Keywords : Banach operator pair; best approximation; convex metric space; starshaped set; commuting and compatible maps.
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1 Introduction and Preliminaries

Fixed point theory is one of the famous and traditional theory in mathematics and has a lot of applications. In fixed point theory the importance of various contractive inequalities cannot be over emphasized. Fixed point and common

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fixed point theorems for different types of mappings have been investigated extensively by various researchers (see, e.g., [1–25] and references cited therein). In this paper, we prove some fixed point and common fixed point results for nonexpansive and generalized nonexpansive mappings. Invariant approximation results are also obtained for these types of mappings as applications. Our results extend and generalize some of the known results of Al-Thagafi [1], Al-Thagafi and Shahzad [2, 3], Beg et al. [4], Chandok [11], Chandok and Narang [6, 8], Chen and Li [13], Dotson [14, 15], Habiniak [17], Hicks and Humphires [18], Narang and Chandok [19, 20], O'Regan and Shahzad [22], and of Shahzad [24].

First, we recall some basic definitions and notations.

For a metric space (X, d), a continuous mapping $W : X \times X \times [0, 1] \to X$ is said to be a *convex structure* on X if for all $x, y \in X$ and $\lambda \in [0, 1]$,

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y)$$

holds for all $u \in X$. The metric space (X, d) together with a convex structure is called a *convex metric space* [25].

A nonempty subset K of a convex metric space (X, d) with a convex structure W is said to be a *convex set* [25] if $W(x, y, \lambda) \in K$ for all $x, y \in K$ and $\lambda \in [0, 1]$. The set K is said to be *p-starshaped* [16] if there exists a $p \in K$ such that $W(x, p, \lambda) \in K$ for all $x \in K$ and $\lambda \in [0, 1]$ that is the segment $[p, x] = \{W(x, p, \lambda) : \lambda \in [0, 1]\}$ joining p to x is contained in K for all $x \in K$.

Clearly, each convex set is a starshaped set but converse is not true.

A convex metric space (X, d) is said to satisfy *Property (I)* [16] if for all $x, y, q \in X$ and $\lambda \in [0, 1]$,

$$d(W(x, q, \lambda), W(y, q, \lambda)) \leq \lambda d(x, y).$$

A normed linear space and each of its convex subsets are simple examples of convex metric spaces with a convex structure W given by $W(x, y, \lambda) = \lambda x + (1-\lambda)y$ for $x, y \in X$ and $\lambda \in [0, 1]$. There are many convex metric spaces which are not normed linear spaces (see [16, 25]). Property (I) is always satisfied in a normed linear space.

For a nonempty subset M of a metric space (X, d) and $p \in X$, an element $y \in M$ is said to be a *best approximant* to p or a *best M-approximant* to p if $d(p, y) = dist(p, M) \equiv \inf\{d(p, y) : y \in M\}$. The set of all best approximants to p in M is denoted by $P_M(p)$. Also we define sets $C_M^I(p) = \{x \in M : Ix \in P_M(p)\}$, and $D_M^I(p) = P_M(p) \cap C_M^I(p) = \{x \in P_M(p) : Ix \in P_M(p)\}$, where I is a self mapping of X.

For a convex subset M of a convex metric space (X, d), a mapping $g : M \to X$ is said to be *affine* if for all $x, y \in M$, $g(W(x, y, \lambda)) = W(gx, gy, \lambda)$ for all $\lambda \in [0, 1]$. A mapping g is said to be *affine with respect to* $p \in M$ if $g(W(x, p, \lambda)) = W(gx, gp, \lambda)$ for all $x \in M$ and $\lambda \in [0, 1]$.

Suppose (X, d) is a metric space, M a nonempty subset of X, and S, T are self mappings of M. A mapping T is said to be

(i) *S-contraction* if there exists $k \in [0, 1)$ such that $d(Tx, Ty) \leq kd(Sx, Sy)$ for all $x, y \in M$;

(ii) *S-nonexpansive* if $d(Tx, Ty) \leq d(Sx, Sy)$ for all $x, y \in M$.

A mapping $S : M \to M$ is called *T-selector* if $Sx \in Tx$ for each $x \in M$. A point $x \in M$ is a *common fixed (coincidence) point* of S and T if $x = Sx = Tx(Sx = Tx)$. The set of all fixed points (respectively, coincidence points) of S and T is denoted by $F(S, T)$ (respectively, $C(S, T)$). The pair (S, T) is said to be *commuting* on M if $STx = TSx$ for all $x \in M$.

The ordered pair (T, I) of two self maps of a metric space (X, d) is called a *Banach operator pair* [13] if $F(I)$, the set of fixed points of I, is T-invariant, i.e. $T(F(I)) \subseteq F(I)$. Obviously, a commuting pair (T, I) is a Banach operator pair but not conversely (see [13]). If (T, I) is a Banach operator pair then (I, T) need not be a Banach operator pair (see [13]). If the self maps T and I of X satisfy $d(ITx, Tx) \leq kd(Ix, x)$, for all $x \in X$ and for some $k \geq 0$, $ITx = TIx$ whenever $x \in F(I)$, that is $Tx \in F(I)$, then (T, I) is a Banach operator pair. This class of non-commuting mappings is different from the known classes of non-commuting mappings viz. R-weakly commuting, R-subweakly commuting, compatible, weakly compatible and C_q-commuting etc. existing in the literature.

Let C be a nonempty subset of a metric space (X, d) and $\mathfrak{F} = \{f_\alpha : \alpha \in C\}$ a family of functions from $[0, 1]$ into C, having the property $f_\alpha(1) = \alpha$, for each $\alpha \in C$. Such a family $\mathfrak{F}$ is said to be

i) *contractive* if there exists a function $\Phi : (0, 1) \to (0, 1)$ such that for all $\alpha, \beta \in C$ and for all $t \in (0, 1)$, we have
$$d(f_\alpha(t), f_\beta(t)) \leq \Phi(t) d(\alpha, \beta).$$

ii) *jointly continuous* if $t \to t_\circ$ in $[0, 1]$ and $\alpha \to \alpha_\circ$ in C imply $f_\alpha(t) \to f_{\alpha_\circ}(t_\circ)$ in C.

In normed linear spaces, these notions were discussed by Dotson [15].

Example 1.1. *Any subspace, a convex set with* 0 *and a starshaped subset with center* 0 *of a normed linear space have a contractive jointly continuous family of functions.*

If C is a starshaped subset (of a normed linear space) with star-center q then the family $\mathfrak{F} = \{f_\alpha : \alpha \in C\}$ defined by $f_\alpha(t) = (1 - t)q + t\alpha$ is contractive if we take $\Phi(t) = t$ for $0 < t < 1$, and is jointly continuous. The same is true for starshaped subsets of convex metric spaces with Property (I), if we define the family $\mathfrak{F}$ as $f_x(\alpha) = W(x, q, \alpha)$, then
$$\begin{aligned} d(f_x(\alpha), f_y(\alpha)) &= d(W(x, q, \alpha), W(y, q, \alpha)) \\ &\leq \alpha\ d(x, y), \end{aligned}$$
so taking $\Phi(\alpha) = \alpha$, $0 < \alpha < 1$, the family is a contractive jointly continuous family and so the class of subsets of X with the property of contractiveness and joint continuity contains the class of starshaped sets which in turn contains the class of convex sets.

2 Main Results

We begin the section with the following theorem which extends and generalizes the corresponding results of Al-Thagafi and Shahzad [2, 3], Beg et al. [4], Dotson [14, 15], Habiniak [17] and of Narang and Chandok [19].

Theorem 2.1. *Let D be a nonempty closed subset of a metric space (X, d) and T be a self mapping of D. Suppose that D has a contractive jointly continuous family $\mathfrak{F} = \{f_\alpha : \alpha \in D\}$, cl $(T(D))$ is compact and T is nonexpansive on D. Then $F(T)$ is nonempty.*

Proof. Define $T_n : D \to D$ as $T_n x = f_{Tx}(k_n)$ where $k_n = \frac{n}{n+1}$, $n = 1, 2, 3, \ldots$, $f_{Tx} \in \mathfrak{F}$. Since $T(D) \subseteq cl\ (T(D)) \subseteq D$ and $0 < k_n < 1$, each T_n is well defined and maps D into D. Further, for all $x, y \in D$ and for each n

$$\begin{aligned} d(T_n x, T_n y) &= d(f_{Tx}(k_n), f_{Ty}(k_n)) \\ &\leq \Phi(k_n) d(Tx, Ty) \\ &\leq \Phi(k_n) d(x, y). \end{aligned}$$

Therefore, each T_n is a contraction mapping on D. Now as $T(D) \subseteq cl\ (T(D)) \subseteq D$, T is nonexpansive and $cl\ (T(D))$ is compact, $cl\ (T_n(D))$ is compact for each n and hence is complete and so by Banach contraction principle, each T_n has a unique fixed point $x_n \in D$. Since $cl\ (T_n(D))$ is compact, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to x \in D$. Since $T_{n_i} x_{n_i} = x_{n_i}$, we have $T_{n_i} x_{n_i} \to x$. Now T being nonexpansive, is continuous and so $Tx_{n_i} \to Tx$. Consider,

$$T_{n_i} x_{n_i} = f_{Tx_{n_i}}(k_{n_i}) \to f_{Tx}(1),$$

that is, $T_{n_i} x_{n_i} \to Tx$. Therefore $Tx = x$ that is $x \in D$ is a fixed point of T. □

Since starshaped subsets of convex metric spaces with Property (I) have contractive jointly continuous family, we have the following results.

Corollary 2.2. *Let D be a subset of a convex metric space (X, d) with Property (I) and T a self mapping of D. Suppose that D is q-starshaped, cl $(T(D))$ is a subset of D, cl $(T(D))$ is compact and T is nonexpansive on D then $F(T)$ is nonempty.*

Corollary 2.3 ([4, Theorem 3])**.** *Let (X, d) be a convex metric space satisfying property (I) and D a closed q-starshaped subset of X. If T is a nonexpansive self mapping of D and cl $(T(D))$ is compact, then T has a fixed point.*

Since $cl\ (T(D))$ is compact if T/D is compact, we have the following results.

Corollary 2.4. *Let T be a mapping on a metric space (X, d), D a T-invariant subset of X such that T/D is compact, x a T-invariant point and cl $(T(D)) \subseteq D$. If $P_D(x)$ is a nonempty closed set for which there exists a contractive jointly continuous family $\mathfrak{F}$ of functions and T is nonexpansive on $P_D(x) \cup \{x\}$ then $P_D(x)$ contains a T-invariant point.*

Corollary 2.5 ([19]). *Let T be a mapping on a metric space (X, d), D a closed T-invariant subset of X such that T/D is compact and x a T-invariant point. If $P_D(x)$ is a nonempty set for which there exists a contractive jointly continuous family $\mathfrak{F}$ of functions and T is non-expansive on $P_D(x) \cup \{x\}$ then $P_D(x)$ contains a T-invariant point.*

Corollary 2.6 ([4, Theorem 10]). *Let (X, d) be a convex metric space satisfying Property (I) and T a nonexpansive mapping on X. Let D be a T-invariant subset of X, T/D compact and x a T-invariant point. If the set of best D-approximant to x is a nonempty, convex or starshaped set then it contains a T-invariant point.*

Remark 2.7.

a. *By comparing Lemma 2.3 of Al-Thagafi and Shahzad [3] with Theorem 2.1, starshapedness of D has been relaxed with contractive jointly continuous family and the spaces undertaken are metric spaces.*

b. *Theorem 1 of Dotson [14] and [15] are special cases of Theorem 2.1.*

c. *Habiniak [17, Theorem 4] proved Corollary 2.3 in normed linear spaces.*

d. *Thoerem 8 of Habiniak [17] is a special case of Corollary 2.4.*

The following theorem extends and generalizes corresponding results of Al-Thagafi [1], Al-Thagafi and Shahzad [3], Chandok and Narang [6], Chen and Li [13] and of Habiniak [17].

Theorem 2.8. *Let D be a subset of a metric space (X, d) and T, I be self mappings of D. Suppose that $F(I)$ has a contractive jointly continuous family, cl $(T(F(I)))$ is a subset of $F(I)$, cl $(T(D))$ is compact and T is I-nonexpansive on D, then $F(I, T)$ is nonempty.*

Proof. Since T is I-nonexpansive on D, T is nonexpansive on $F(I)$. It follows from Theorem 2.1 that $F(I, T)$ is nonempty. □

Since starshaped subsets of convex metric spaces with Property (I) have contractive jointly continuous family, we have the following results.

Corollary 2.9. *Let D be a subset of a convex metric space (X, d) with Property (I) and T, g be self mappings of D. Suppose that $F(g)$ is q-starshaped, cl $(T(F(g)))$ is a subset of $F(g)$, cl $(T(D))$ is compact and T is g-nonexpansive on D, then $F(g, T)$ is nonempty.*

Corollary 2.10. *Let (X, d) be a metric space and T, I be self mappings of X. If $p \in X$, $D \subseteq P_M(p)$, $G = D \cap F(I)$ has a contractive jointly continuous family, cl $(T(G)) \subseteq G$, cl $(T(D))$ is compact and T is I-nonexpansive on D, then $P_M(p) \cap F(I, T)$ is nonempty.*

Corollary 2.11. *Let (X, d) be a convex metric space with Property (I) and T, I be self mappings of X. If $p \in X$, $D \subseteq P_M(p)$, $G = D \cap F(I)$ is q-starshaped, cl $(T(G)) \subseteq G$, cl $(T(D))$ is compact and T is I-nonexpansive on D, then $P_M(p) \cap F(I, T)$ is nonempty.*

Corollary 2.12 ([3, Corollary 2.5])**.** *Let X be a normed linear space and T, I be self mappings of X. If $p \in X$, $D \subseteq P_M(p)$, $G = D \cap F(I)$ is q-starshaped, cl $(T(G)) \subseteq G$, cl $(T(D))$ is compact and T is I-nonexpansive on D, then $P_M(p) \cap F(I, T)$ is nonempty.*

Remark 2.13.

a. *By comparing Theorem 2 of Chandok and Narang [6] with Theorem 2.8, their assumptions D is closed, (T, I) is a Banach operator pair, I is continuous on D and D has a contractive jointly continuous family with $If_x(k) = f_{Ix}(k)$ are replaced with cl $(T(F(I))) \subseteq F(I)$ and $F(I)$ has a contractive jointly continuous family and the spaces undertaken are metric spaces.*

b. *By comparing Theorem 3.3 of Chen and Li [13] with Theorem 2.8, their assumptions $q \in F(I)$, D is closed and q-starshaped, (T, I) is a Banach operator pair, and I is continuous on D are replaced with cl $(T(F(I))) \subseteq F(I)$ and starshapedness of $F(I)$ has been relaxed to contractive jointly continuous family and the spaces undertaken are metric spaces.*

c. *By comparing Lemma 2.2 of Shahzad [24] with Theorem 2.8, his assumptions $q \in F(I)$, D is closed and q-starshaped, $T(D) \subseteq I(D)$, (I, T) is R-subweakly commuting, T is continuous on D and I is linear are replaced with cl $(T(F(I))) \subseteq F(I)$ and $F(I)$ has a contractive jointly continuous family and the spaces undertaken are metric spaces.*

d. *By comparing Theorem 2.4 of Al-Thagafi and Shahzad [3] with Theorem 2.8, starshapedness of $F(I)$ has been replaced with contractive and jointly continuous family and the spaces undertaken are metric spaces.*

e. *Theorem 4 of Habiniak [17] is a special case of Theorem 2.8, by taking I as the identity mapping.*

f. *By comparing Theorem 4 of Chandok and Narang [6] with Corollary 2.10, their assumption $p \in F(I, T)$, $D^I_M(p) \cap F(I)$ has a contractive jointly continuous family with $If_x(k) = f_{Ix}(k)$, cl $(T(D^I_M(p)))$ is compact, (T, I) is a Banach operator pair on $D^I_M(p)$, T is I-nonexpansive on $D^I_M(p) \cup \{p\}$ and I is continuous on cl $T(D^I_M(p))$ are replaced with $p \in X$, $D \subseteq P_M(p)$, $G = D \cap F(I)$ has a contractive jointly continuous family, cl $(T(G)) \subseteq G$, T is I-nonexpansive on D and cl $(T(D))$ is compact and the spaces undertaken are metric spaces.*

g. *By comparing Theorem 4.2 of Chen and Li [13] with Corollary 2.10, their assumptions $p \in F(I, T)$, $D^I_M(p) \cap F(I)$ is q-starshaped, cl $(T(D^I_M(p)))$ is compact, (T, I) is a Banach operator pair on $D^I_M(p)$, T is I-nonexpansive* *on $D_M^I(p) \cup \{p\}$ and I is continuous on cl $(T(D_M^I(p)))$ are replaced with $p \in X$, $D \subseteq P_M(p)$, $G = D \cap F(I)$ has a contractive jointly continuous family, cl $(T(G)) \subseteq G$, T is I-nonexpansive on D and cl $(T(D))$ is compact and the spaces undertaken are metric spaces.*

Let $G_\circ$ denotes the class of closed convex subsets containing a point $x_\circ$ of a convex metric space (X, d) with Property (I). For $M \in G_\circ$ and $p \in X$, let $M_p = \{x \in M : d(x, x_\circ) \le 2d(p, x_\circ)\}$. Then $P_M(p) \subset M_p \in G_\circ$ as $x \in P_M(p) \Rightarrow d(p, x) = dist(p, M) \Rightarrow d(x, x_\circ) \le d(x, p) + d(p, x_\circ) \le 2d(p, x_\circ) \Rightarrow x \in M_p$.

Theorem 2.14. *Let (X, d) be a convex metric space with Property (I) and T, g be self mappings of X. If $p \in X$ and $M \in G_\circ$ such that $T(M_p) \subseteq M$, cl $(T(M_p))$ is compact, and $d(Tx, p) \le d(x, p)$ for all $x \in M_p$, then $P_M(p)$ is nonempty, closed and convex with $T(P_M(p)) \subseteq P_M(p)$. If, in addition, D is a subset of $P_M(p)$, $G = D \cap F(g)$ is q-starshaped, cl $(T(G)) \subseteq G$, and T is g-nonexpansive on D, then $P_M(p) \cap F(g, T)$ is nonempty.*

Proof. If $p \in M$ then the results are obvious. So assume that $p \notin M$. If $x \in M \backslash M_p$ then $d(x, x_\circ) > 2d(p, x_\circ)$ and so $d(p, x) \ge d(x, x_\circ) - d(p, x_\circ) > d(p, x_\circ) \ge dist(p, M)$. Thus $\alpha = dist(p, M) \le d(p, x_\circ)$. Since *cl* $(T(M_p))$ is compact, and the distance function is continuous, there exists $z \in cl\ (T(M_p))$ such that $\beta = dist(p, cl\ (T(M_p)) = d(p, z)$. Hence

$$\begin{aligned}\alpha = dist(p, M) &\le dist(p, cl\ (T(M_p)))\\ &= \beta\\ &= dist(p, T(M_p))\\ &\le d(p, Tx)\\ &\le d(p, x)\end{aligned}$$

for all $x \in M_p$. Therefore, $\alpha = \beta = dist(p, M)$, i.e., $dist(p, M) = dist(p, cl\ (T(M_p))$ $= d(p, z)$ that is $z \in P_M(p)$ and so $P_M(p)$ is nonempty. The closedness and convexity of $P_M(p)$ follow from that of M. Now to prove $T(P_M(p)) \subseteq P_M(p)$, let $y \in T(P_M(p))$. Then $y = Tx$ for $x \in P_M(p)$. Consider

$$d(p, y) = d(p, Tx) \le d(p, x) = dist(p, M)$$

and so $y \in P_M(p)$ as $P_M(p) \subset M_p \Rightarrow T(P_M(p)) \subset M$ that is $y \in M$.

Since *cl* $(T(P_M(p)))$ is compact, the result follows from Corollary 2.11. □

Corollary 2.15 ([3, Theorem 2.6])**.** *Let X be a normed linear space and T, g are self mappings of X. If $p \in X$ and $M \in G_\circ$ such that $T(M_p) \subseteq M$, cl $(T(M_p))$ is compact, and $\|Tx - p\| \le \|x - p\|$ for all $x \in M_p$, then $P_M(p)$ is nonempty, closed and convex with $T(P_M(p)) \subseteq P_M(p)$. If, in addition, D is a subset of $P_M(p)$, $G = D \cap F(g)$ is q-starshaped, cl $(T(G)) \subseteq G$, and T is g-nonexpansive on D, then $P_M(p) \cap F(g, T)$ is nonempty.*

Remark 2.16. *Theorem 2.14 extends and generalizes corresponding results of Al-Thagafi [1], Al-Thagafi and Shahzad [2], Habiniak [17], Hicks and Humphires [18], Narang and Chandok [20], Shahzad [24].*

We shall now use the following lemma of Al-Thagafi and Shahzad [3] for our next theorem.

Lemma 2.17. *Let C be a nonempty subset of a metric space (X, d), $T, g : C \to C$ and cl $(T(F(g))) \subseteq F(g)$. Suppose that cl $(T(C))$ is complete and T, g satisfy for all $x, y \in C$ and $0 \leq h < 1$*

$$d(Tx, Ty) \leq h \max\{d(gx, gy), d(Tx, gx), d(Ty, gy), d(Tx, gy), d(Ty, gx)\}.$$

If $F(g)$ is nonempty, then there is a common fixed point of T and g.

Theorem 2.18. *Let C be a nonempty subset of a convex metric space (X, d) with Property (I), $T, g : C \to C$. If $F(g)$ is q-starshaped, cl $(T(F(g))) \subseteq F(g)$, cl $(T(C))$ is compact, T is continuous on C and T, g satisfy for all $x, y \in C$*

$$\begin{aligned} d(Tx, Ty) \leq \max\{&d(gx, gy), dist(gx, W(Tx, q, k)), dist(gy, W(Ty, q, k)), \\ &dist(gx, W(Ty, q, k)), dist(gy, W(Tx, q, k))\}, \end{aligned}$$

then there is a common fixed point of T and g.

Proof. For each $n \in \mathbb{N}$, define $T_n : C \to C$ by $T_n(x) = W(Tx, q, k_n)$, for each $x \in C$ where $\{k_n\}$ is a sequence in $(0, 1)$ such that $k_n \to 1$. Then each T_n is a self mapping of C. Since *cl* $(T(F(g))) \subseteq F(g)$ and $F(g)$ is q-starshaped, so *cl* $(T_n(F(g))) \subseteq F(g)$ for each n. Consider

$$\begin{aligned} d(T_n x, T_n y) &= d(W(Tx, q, k_n), W(Ty, q, k_n)) \\ &\leq k_n d(Tx, Ty) \\ &\leq k_n \max\{d(gx, gy), dist(gx, W(Tx, q, k)), dist(gy, W(Ty, q, k)), \\ &\qquad dist(gx, W(Ty, q, k)), dist(gy, W(Tx, q, k))\} \end{aligned}$$

which implies for each $k_n \in (0, 1)$, we have

$$\begin{aligned} d(T_n x, T_n y) &\leq k_n \max\{d(gx, gy), dist(gx, W(Tx, q, k_n)), dist(gy, W(Ty, q, k_n)), \\ &\qquad dist(gx, W(Ty, q, k_n)), dist(gy, W(Tx, q, k_n))\} \\ &\leq k_n \max\{d(gx, gy), d(gx, T_n x), d(gy, T_n y), d(gx, T_n y), d(gy, T_n x)\} \end{aligned}$$

for all $x, y \in C$. As *cl* $(T(C))$ is compact, *cl* $(T_n(C))$ is compact for each n and hence complete. Now by Lemma 2.17, there exists $x_n \in C$ such that x_n is common fixed point of g and T_n for each n. The compactness of *cl* $(T(C))$ implies there exists a subsequence $\{Tx_{n_i}\}$ of $\{Tx_n\}$ such that $Tx_{n_i} \to y \in$ *cl* $(T(C))$. Since $\{Tx_n\}$ is a sequence in $T(F(g))$, $y \in$ *cl* $(T(F(g))) \subseteq F(g)$. Now, as $k_{n_i} \to 1$, we have

$$x_{n_i} = T_{n_i} x_{n_i} = W(Tx_{n_i}, q, k_{n_i}) \to y.$$

By the continuity of T, we have $Ty = y$ and hence $F(T) \cap F(g) \neq \emptyset$. $\square$

If $F(g)$ is closed and (T, g) is a Banach operator pair, we have $cl\ (T(F(g))) \subseteq F(g)$ and so above theorem gives the following results.

Corollary 2.19. *Let C be a nonempty subset of a convex metric space (X, d) with Property (I), $T, g : C \to C$. If $F(g)$ is q-starshaped and closed, (T, g) is Banach operator pair, cl $(T(C))$ is compact, T is continuous on C and T, g satisfy for all $x, y \in C$*

$$d(Tx, Ty) \leq \max\{d(gx, gy), dist(gx, W(Tx, q, k)), dist(gy, W(Ty, q, k)), \\ dist(gx, W(Ty, q, k)), dist(gy, W(Tx, q, k))\},$$

then there is a common fixed point of T and g.

Corollary 2.20. *Let (X, d) be a convex metric space with Property (I), $T, g : X \to X$. If $p \in X$, $D \subseteq P_M(p)$, $G = D \cap F(g)$ is q-starshaped, cl $(T(G))$ is subset of G, cl $(T(D))$ is compact, T is continuous on D and T, g satisfy for all $x, y \in D$*

$$d(Tx, Ty) \leq \max\{d(gx, gy), dist(gx, W(Tx, q, k)), dist(gy, W(Ty, q, k)), \\ dist(gx, W(Ty, q, k)), dist(gy, W(Tx, q, k))\},$$

then there is a common fixed point of $P_M(p)$, T and g.

Remark 2.21.

a. *By comparing Theorem 2.2 of O'Regan and Shahzad [22] with Theorem 2.18, their assumptions $q \in F(g)$, C is closed and q-starshaped, g and T are continuous on C, $T(C) \subseteq g(C)$, (g, T) is an R-subweakly commuting on C and g is affine are replaced with $F(g)$ is q-starshaped, cl $(T(F(g))) \subseteq F(g)$ and T is continuous and the spaces undertaken are convex metric spaces.*

b. *Theorem 2.18 and Corollaries 2.19 and 2.20 are proved in normed linear spaces by Al-Thagafi and Shahzad [3].*

Theorem 2.22. *Let (X, d) be a convex metric space with Property (I) and T, g be self mappings of X. If $p \in X$ and $M \in G_\circ$ such that $T(M_p) \subseteq M$, cl $(T(M_p))$ is compact, and $d(Tx, p) \leq d(x, p)$ for all $x \in M_p$, then $P_M(p)$ is nonempty, closed and convex with $T(P_M(p)) \subseteq P_M(p)$. If, in addition, D is a subset of $P_M(p)$, $G = D \cap F(g)$ is q-starshaped, cl $(T(G)) \subseteq G$, and T is continuous on D, and T, g satisfy for all $x, y \in D$*

$$d(Tx, Ty) \leq \max\{d(gx, gy), dist(gx, W(Tx, q, k)), dist(gy, W(Ty, q, k)), \\ dist(gx, W(Ty, q, k)), dist(gy, W(Tx, q, k))\},$$

then there is a common fixed point of $P_M(p)$, T and g.

Proof. Proceeding as in Theorem 2.14 and using Corollary 2.20, we get the result. $\square$

Remark 2.23. *Theorem 2.22 extends and generalizes corresponding results of Al-Thagafi [1], Al-Thagafi and Shahzad [2], Chandok [11], Chandok and Narang [8], Hicks and Humphires [18], Narang and Chandok [20], O'Regan and Shahzad [22].*

Theorem 2.24. *Let D be a subset of a convex metric space (X, d), $I : D \to D$ and $T : D \to 2^D$. If $F(I)$ is q-starshaped, $cl\ (T(F(I))) \subseteq F(I)$, $cl\ (T(D))$ is compact, and $S : D \to D$ is nonexpansive T-selector, then $F(I,T)$ is nonempty.*

Proof. Since S is T-selector on D, $cl\ (S(F(I))) \subseteq cl\ (T(F(I))) \subseteq F(I)$ and $cl\ (S(F(I)))$ is compact. As S is nonexpansive on D, it is nonexpansive on $F(I)$. It follows from Theorem 2.1 that $F(I, S)$ is nonempty, so there exists $z \in F(I)$ such that $z = Iz = Sz \in Tz$. Therefore, $F(I,T)$ is nonempty. $\square$

If $F(I)$ is closed and (T, I) is a Banach operator pair, we have $cl(T(F(I))) \subseteq F(I)$ and so above theorem gives the following result.

Corollary 2.25. *Let D be a subset of a convex metric space (X, d), $I : D \to D$ and $T : D \to 2^D$. If $F(I)$ is q-starshaped, closed, (T, I) is a Banach operator pair, $cl\ (T(D))$ is compact, and $S : D \to D$ is nonexpansive T-selector, then $F(I,T)$ is nonempty.*

Remark 2.26.

a. *Theorem 2.24 and Corollary 2.25 are proved in normed linear spaces by Al-Thagafi and Shahzad [3].*

b. *It may be noted that the assumption of linearity or affinity for I is necessary in almost all known results about common fixed points of maps T, I such that T is I-nonexpansive under the conditions of commuting, weakly commuting, R-subweakly commuting or compatibility (see [1, 6, 13, 20, 22, 24] and the literature cited therein), but our results in this paper are independent of the linearity or affinity.*

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