Thai Journal of Mathematics
Volume 12 (2014) Number 2 : 397–430
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

A Common Minimum-norm Solution of a Generalized Equilibrium Problem and a Fixed Point Problem for a Countable Family of Nonexpansive Mappings[1]

U. Inprasit, W. Nilsrakoo and M. Wattanataweekul[2]

†Department of Mathematics, Statistics and Computer,
Faculty of Science, Ubon Ratchathani University,
Ubon Ratchathani 34190, Thailand
‡Centre of Excellence in Mathematics CHE,
Si Ayutthaya Rd. Bangkok 10400, Thailand
e-mail: nilsrakoo@hotmail.com (W. Nilsrakoo)
schathwa@ubu.ac.th (M. Wattanataweekul)

Abstract : We introduce a new iterative scheme for finding a common minimum-norm solution of a generalized equilibrium problem and a fixed point problem for a countable family of nonexpansive mappings and then establish strong convergence theorems of the scheme by using the NST-condition. Our result extends and improves the main result of Cai et al. [1], Chen [2] and Yao et al. [3] and many others. Finally, we apply our results to solve the problem of finding common zeros of maximal monotone operators, the common minimizer problem, the multiple-sets split feasibility problem.

Keywords : common minimizer problem; common minimum-norm; equilibrium problem; fixed point; multiple-sets split feasibility problem; maximal monotone operator; nonexpansive mapping

2000 Mathematics Subject Classification : 47H09; 47H10.

---

[1]This research was supported by the Centre of Excellence in Mathematics and the Commission on Higher Education, Thailand.

[2]Corresponding author.

1 Introduction

We denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of all positive integers and all real numbers, respectively. Let H be a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and a norm $\|\cdot\|$. Let C be a closed convex subset of H. A mapping $T : C \to H$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. The set of fixed points of T is denoted by $F(T)$, i.e., $F(T) = \{x \in C : x = Tx\}$.

Many problems in nonlinear analysis can be reformulated as a problem of finding a fixed point of a nonexpansive mapping and the iterative algorithms have been extensively investigated by many authors. Two classical iterative schemes are Mann iterative method [4] which is defined as follows: $x_1 \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad \forall n \in \mathbb{N}, \tag{1.1}$$

and Halpern iterative method [5] which is defined as follows: $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) T x_n, \quad \forall n \in \mathbb{N}, \tag{1.2}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$. It is known that under appropriate conditions the Mann iterative method converges only weakly to a fixed point of T but the Halpern iterative method converges strongly to a fixed point of T.

In 2007, Aoyama et al. [6] extended iteration (1.1) to obtain weak convergence to a common fixed point of a countable family of nonexpansive mappings $\{T_n\}_{n=1}^{\infty}$ by the following iteration:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T_n x_n, \quad \forall n \in \mathbb{N}, \tag{1.3}$$

where $x_1 \in C$ and $\{\alpha_n\}$ is a sequence in $(0, 1)$. On the other hand, Aoyama et al. [7] extended iteration (1.2) to obtain strong convergence to a common fixed point of a countable family of nonexpansive mappings $\{T_n\}_{n=1}^{\infty}$ by the following iteration:

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) T_n x_n, \quad \forall n \in \mathbb{N}, \tag{1.4}$$

where $x_1 = x \in C$ and $\{\alpha_n\}$ is a sequence in $(0, 1)$. In the literature, the schemes (1.3) and (1.4) have been widely studied and extended in [8, 9, 10, 11, 12, 13, 14, 15, 16] and references therein.

In many practical problems, such as optimization problems, finding the minimum norm fixed point of nonexpansive mappings is quite important. In general, the nonexpansive mapping may have more than one fixed point. It is known that if C is closed and convex and $T : C \to H$ is nonexpansive, then $F(T)$ is closed and convex and hence there exists a unique $x^* \in F(T)$ satisfies:

$$\|x^*\| = \min\{\|x\| : x \in F(T)\}.$$

That is, x^* is the minimum-norm fixed point of T. In other words, x^* is the metric projection of the origin on $F(T)$. It is an interesting thing to construct iterative sequence to find the minimum-norm fixed point of a nonexpansive mapping T (see

[1, 2, 3, 17, 18, 19, 20, 21, 22]). In 2009, Yao et al. [3] introduced the following iterative scheme for a nonexpansive mapping T on C:

$$\begin{cases} x_1 \in C, \\ y_n = P_C(1-\alpha_n)x_n, \\ x_{n+1} = (1-\beta_n)y_n + \beta_n T y_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. They proved that $\{x_n\}$ defined by (1.5) converges strongly to a minimum-norm fixed point of T if $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Moreover, Cui and Liu [17] and Yao and Xu [22] independently proposed the following iterative scheme:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = P_C((1-\alpha_n)Tx_n), \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.6}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$. Recently, Cai et al. [1] relaxed the iterative scheme (1.6) as follows:

$$\begin{cases} x_1 \in C, \\ y_n = \beta x_n + (1-\beta)Tx_n, \\ x_{n+1} = P_C((1-\alpha_n)y_n), \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.7}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\beta \in (0,1)$. However, the scheme (1.7) is a spacial case of the scheme (1.5) because the scheme (1.5) and the following iterative scheme:

$$\begin{cases} x_1 \in C, \\ y_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = P_C((1-\alpha_n)y_n), \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.8}$$

are equivalent. To see this, let us set $y_1 = \beta_1 x_1 + (1-\beta_1)Tx_1$, $z_n \equiv x_{n+1}$, and $\widehat{\beta_n} \equiv \beta_{n+1}$. Then it follows from (1.8) that

$$\begin{cases} y_1 \in C, \\ z_n = P_C((1-\alpha_n)y_n), \\ y_{n+1} = \widehat{\beta_n} z_n + (1-\widehat{\beta_n})Tz_n, \quad \forall n \in \mathbb{N}. \end{cases} \tag{1.9}$$

Then the scheme (1.7) is the scheme (1.8) with $\beta_n \equiv \beta$. On the other hand, Chen [2] extended the iterative scheme (1.6) for finding a common minimum-norm fixed point of a countable family of nonexpansive mappings $\{T_n\}_{n=1}^{\infty}$ as follows:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = P_C((1-\alpha_n)W_n x_n), \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.10}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$ and W_n is a W-mapping defined by Takahashi [23] (see also [24, 25]).

The equilibrium problem of a bifunction $\varphi : C \times C \to \mathbb{R}$ is to find $x \in C$ such that $\varphi(x, y) \geq 0$ for all $y \in C$. The set of such $x \in C$ is denoted by $EP(\varphi)$, i.e.,

$$EP(\varphi) = \big\{x \in C : \varphi(x, y) \geq 0, \quad \forall y \in C\big\}.$$

The equilibrium problems include fixed point problems, optimization problems, variational inequality problems and Nash equilibrium problems as special cases. Iterative methods have been proposed to solve the equilibrium problems (see, for instance, [26, 27, 28]). For example, Combettes and Hirstoaga [27] introduced an iterative scheme of finding the best approximation to the initial data when $EP(\varphi)$ is nonempty and they also proved a strong convergence theorem in a real Hilbert space.

For a bifunction $\varphi : C \times C \to \mathbb{R}$ and a nonlinear mapping $A : C \to H$, we consider the following generalized equilibrium problem. Find $z \in C$ such that

$$\varphi(z, y) + \langle Az, y - z\rangle \geq 0, \quad \forall y \in C.$$

The set of such $z \in C$ is denoted by $EP(\varphi, A)$, i.e.,

$$EP(\varphi, A) = \{z \in C : \varphi(z, y) + \langle Az, y - z\rangle \geq 0, \quad \forall y \in C\}. \tag{1.11}$$

If $A \equiv 0$, then this problem coincides with the equilibrium problem.

Iterative methods for finding a common element of the set of solutions for an (a generalized) equilibrium problem and the set of fixed points of a nonexpansive mapping are studied. For instance, Tada and Takahashi [29] introduced a modified Mann iterative method as follows:

$$\begin{cases} x_1 \in H, \\ \varphi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tu_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.12}$$

Takahashi and Takahashi [30] introduced a modified Halpern method as follows:

$$\begin{cases} x_1 \in H, \\ \varphi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n x + (1 - \alpha_n)Tu_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.13}$$

Takahashi and Takahashi [31] introduced an iterative method for a generalized equilibrium problem in the following way:

$$\begin{cases} x_1 = x \in C, \\ \varphi(u_n, y) + \langle Ax_n, y - z_n\rangle + \frac{1}{r_n}\langle y - z_n, u_n - x_n\rangle \geq 0 \quad \forall y \in C, \\ y_n = \alpha_n x + (1 - \alpha_n)u_n, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)Ty_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.14}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ and $\{r_n\}$ is a sequence in $(0, \infty)$. It is known that under appropriate conditions the iterative method (1.12) converges weakly to an element of $F(T) \cap EP(\varphi)$ but the iterative method (1.13) and

(1.14) converge strongly to an element of $F(T) \cap EP(\varphi)$ and $F(T) \cap EP(\varphi, A)$, respectively.

Recently, there are many authors introduced and studied the iterative methods for finding a common element of the set of solutions for a generalized equilibrium problem and the set of fixed points of a countable family of nonexpansive mappings (see [13, 14, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42]).

In this paper, we introduce a new iterative scheme for finding a common minimum-norm solution of a generalized equilibrium problem and a fixed point problem for a countable family of nonexpansive mappings and then establish strong convergence theorems of the scheme by using the NST-condition introduced by Nakajo et al. [11]. We give some examples of a generated family of nonexpansive mappings satisfying the NST-condition from a given family of nonexpansive mappings with a common fixed point. Our results extend and improve the main result of Cai et al. [1], Chen [2] and Yao et al. [3] and many others. At the end, we apply our results to solve the problem of finding common zeros of a family of maximal monotone operators, the common minimizer problem and the multiple-sets split feasibility problem.

2 Preliminaries

Let H be a real Hilbert space. It is well-known that for all $x, y \in H$ and all $\lambda \in (0,1)$,

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \tag{2.1}$$

and

$$\|x+y\|^2 = \|x\|^2 - \|y\|^2 + 2\langle x+y, y\rangle.$$

We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that $\{x_n\}$ converges strongly to x. It is also known that H satisfies *Opial's condition* [43] if for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

Let C be a subset of H and $L > 0$. A mapping $T : C \to H$ is said to be

- *firmly nonexpansive* if

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle, \quad \forall x, y \in C;$$

- *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C;$$

- *L-Lipschitzian* if

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

Notice that every firmly nonexpansive mapping is nonexpansive and every nonexpansive mapping is 1-Lipschitzian. If C is nonempty closed and convex, then for every point $x \in H$, there exists a unique nearest point of C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

Such a P_C is called the *metric projection* from H onto C. We know that P_C is a firmly nonexpansive mapping from H onto C. Furthermore, for any $x \in H$ and $z \in C$, $z = P_C x$ if and only if

$$\langle x - z, \ z - y\rangle \geq 0, \quad \forall y \in C.$$

Let C be a nonempty closed convex subset of a real Hilbert space H and A a mapping of C into H. The classical variational inequality is to find $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \quad \forall y \in C.$$

The set of solutions of the classical variational inequality is denoted by $VI(C, A)$. We know that

$$u \in VI(C, A) \quad \Leftrightarrow \quad u = P_C(u - \lambda Au)$$

for all $\lambda > 0$, where I is the identity mapping (see [44]). Let $\alpha > 0$. A mapping A of C into H is said to be

- *monotone* if
$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C;$$
- *α-strongly-monotone* if
$$\langle Ax - Ay, x - y\rangle \geq \alpha\|x - y\|^2, \quad \forall x, y \in C;$$
- *α-inverse-strongly monotone* if
$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

The following lemma is needed for proving the main results.

Lemma 2.1 ([45, Lemma 2.1] and [46, Lemma 2.6]). *Let $\{a_n\}$ be a sequence in $[0, \infty)$, $\{\zeta_n\}$ be a sequence in $(0, 1)$ such that $\sum_{n=1}^{\infty} \zeta_n = \infty$, and $\{\delta_n\}$ be a sequence of real numbers. Suppose that*

$$a_{n+1} \leq (1 - \zeta_n)a_n + \zeta_n\delta_n, \quad \forall n \in \mathbb{N},$$

and one of the following holds:

(i) $\sum_{n=1}^{\infty} \zeta_n\delta_n < \infty$;

(ii) $\limsup_{k\to\infty} \delta_{n_k} \leq 0$ *for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying*

$$\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \geq 0.$$

Then $\lim_{n\to\infty} a_n = 0$.

For solving the equilibrium problem for a bifunction $\varphi : C \times C \to \mathbb{R}$, let us assume that φ satisfies the following conditions (see [26, 27, 29, 30, 31]):

(A1) $\varphi(x,x) = 0$ for all $x \in C$;

(A2) φ is monotone, i.e., $\varphi(x,y) + \varphi(y,x) \le 0$ for all $x, y \in C$;

(A3) $\lim\limits_{t\to 0} \varphi(tz + (1-t)x, y) \le \varphi(x,y)$ for all $x, y, z \in C$;

(A4) for each $x \in C$, $y \mapsto \varphi(x,y)$ is convex and lower semicontinuous.

The following lemma gives a characterization of a solution of an equilibrium problem proved by [26, Corollary 1] and [27, Lemma 2.12].

Lemma 2.2. *Let C be a closed convex subset of a real Hilbert space H, let φ be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying the conditions* (A1)-(A4) *and let $r > 0$. Define a mapping $T_r : H \to C$ by $T_r(x) = x^*$ where x^* is the unique element in C such that*

$$\varphi(x^*, y) + \frac{1}{r}\langle y - x^*, x^* - x\rangle \ge 0 \quad \forall y \in C.$$

Such a mapping T_r is called the resolvent of φ for r. Then, the followings hold:

(i) *T_r is firmly nonexpansive;*

(ii) $F(T_r) = EP(\varphi)$;

(iii) *$EP(\varphi)$ is closed and convex.*

Remark 2.3. *Some well-known examples of resolvents of bifunctions satisfying the conditions* (A1)-(A4) *are presented in [27, Lemma 2.15].*

Recall that a mapping $T : C \to H$ is *demi-closed* at y, if $x_n \rightharpoonup x$ and $Tx_n \to y$, then $Tx = y$. We note that $I - T$ is demi-closed at 0 if T is a nonexpansive mapping.

To deal with a family of mappings, the following conditions of a family of mappings are introduced. Let C be a closed convex subset of a real Hilbert space H, let $\{T_n\}$ be a family of mappings of C into H with $\cap_{n=1}^{\infty} F(T_n) \ne \varnothing$ and $\omega_w\{z_n\}$ denote the set of all weak subsequential limits of a bounded sequence $\{z_n\}$ in C. $\{T_n\}$ is said to satisfy

(a) the *AKTT-condition (I)* [7] if for each bounded subset B of C,

$$\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty;$$

(b) the *AKTT-condition (II)* [6] if for each bounded subset B of C and each increasing sequence $\{n_i\}$ of $\mathbb{N}$, there exists a mapping $T : C \to H$ with $I - T$ is demi-closed at 0 and a subsequence $\{n_{i_j}\}$ of $\{n_i\}$ such that

$$\lim_{j\to\infty} \sup\{\|T_{n_{i_j}} z - Tz\| : z \in B\} = 0 \quad \text{and} \quad F(T) = \cap_{n=1}^{\infty} F(T_n);$$

(c) the *NST-condition* [11] if for each bounded sequence $\{z_n\}$ in C,

$$\lim_{n\to\infty} \|z_n - T_n z_n\| = 0 \quad \text{implies} \quad \omega_w\{z_n\} \subset \cap_{n=1}^{\infty} F(T_n).$$

Remark 2.4. *If there is a mapping $T : C \to H$ such that*

$$Tx = \lim_{n\to\infty} T_n x, \quad \forall x \in C, \tag{2.2}$$

$I - T$ is demi-closed at 0 and $F(T) = \bigcap_{n=1}^{\infty} F(T_n) \neq \varnothing$, then $\{T_n\}$ satisfies the AKTT-condition (II).

Lemma 2.5 ([7, Lemma 3.2])**.** *Let C be a nonempty closed subset of a real Hilbert space H and let $\{T_n\}$ be a family of mappings of C into H which satisfies the AKTT-condition (I). Then there is a mapping $T : C \to H$ satisfying* (2.2)*. In particular, if $I - T$ is demi-closed at 0 and $F(T) = \bigcap_{n=1}^{\infty} F(T_n) \neq \varnothing$, then $\{T_n\}$ satisfies the AKTT-condition (II).*

Lemma 2.6 ([14, Lemma 2.9])**.** *Let C be a closed convex subset of a real Hilbert space H and let $\{T_n\}$ be a family of mappings of C into itself with $\cap_{n=1}^{\infty} F(T_n) \neq \varnothing$. If $\{T_n\}$ satisfies the AKTT-condition (II), then $\{T_n\}$ satisfies the NST-condition.*

3 Main Results

In this section, we give a strong convergence theorem for finding the common minimum-norm element of the set of solutions of the equilibrium problem and the set of fixed points of a countable family of nonexpansive mappings in a Hilbert space. To this end, the following iterative schemes are introduced:

$$\begin{cases} x_1 \in C, \\ \varphi(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C((1-\alpha_n)u_n), \\ x_{n+1} = \beta_n y_n + (1-\beta_n) T_n y_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where C is a closed convex subset of a real Hilbert space H, $\varphi : C \times C \to \mathbb{R}$ is a bifunction, A is an α-inverse-strongly monotone mapping of C into H, $\{T_n\}$ is a sequence of nonexpansive mappings of C into itself, $\{r_n\}$ is a sequence in $(0, \infty)$, and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$.

Theorem 3.1. *Let C be a closed convex subset of a real Hilbert space H and let $\varphi : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions* (A1)-(A4). *Let A be an α-inverse-strongly monotone mapping of C into H and let $\{T_n\}$ be a family of nonexpansive mappings of C into itself satisfying the NST-condition and $\mathfrak{F} := \cap_{n=1}^{\infty} F(T_n) \cap EP(\varphi, A) \neq \varnothing$. Let $\{r_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 2\alpha)$, and let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0, 1)$ satisfying the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then $\{u_n\}, \{y_n\}$ and $\{x_n\}$ defined by (3.1) *converge strongly to the common minimum-norm element of $\mathfrak{F}$.*

Proof. Note that u_n can be rewritten as $u_n = T_{r_n}(x_n - r_n A x_n)$ for each $n \in \mathbb{N}$. Since $\mathfrak{F}$ is nonempty closed convex, put $w = P_{\mathfrak{F}}\theta$, where θ is a zero element in H. That is,

$$\|w\| = \min\{\|z\| : z \in \mathfrak{F}\}.$$

Since $a \leq r_n \leq b < 2\alpha$, we obtain

$$\begin{aligned}
\|u_n - w\|^2 &= \|T_{r_n}(x_n - r_n A x_n) - T_{r_n}(w - r_n A w)\|^2 \\
&\leq \|(x_n - r_n A x_n) - (w - r_n A w)\|^2 \\
&= \|(x_n - w) - r_n(Ax_n - Aw)\|^2 \\
&= \|x_n - w\|^2 - 2r_n\langle x_n - w, Ax_n - Aw\rangle + r_n^2\|Ax_n - Aw\|^2 \\
&\leq \|x_n - w\|^2 - 2r_n\alpha\|Ax_n - Aw\|^2 + r_n^2\|Ax_n - Aw\|^2 \\
&= \|x_n - w\|^2 + r_n(r_n - 2\alpha)\|Ax_n - Aw\|^2 \\
&\leq \|x_n - w\|^2 - a(2\alpha - b)\|Ax_n - Aw\|^2. \qquad (3.2)
\end{aligned}$$

In particular,

$$\|u_n - w\| \leq \|x_n - w\|. \qquad (3.3)$$

By the definition of y_n, we have

$$\begin{aligned}
\|y_n - w\| &= \|P_C((1 - \alpha_n)u_n) - P_C w\| \\
&\leq \|(1 - \alpha_n)u_n - w\| \\
&= \|(1 - \alpha_n)(u_n - w) - \alpha_n w\| \\
&\leq (1 - \alpha_n)\|u_n - w\| + \alpha_n\|w\| \\
&\leq (1 - \alpha_n)\|x_n - w\| + \alpha_n\|w\|. \qquad (3.4)
\end{aligned}$$

So, we obtain

$$\begin{aligned}
\|x_{n+1} - w\| &= \|\beta_n y_n + (1 - \beta_n)T_n y_n - w\| \\
&= \|(1 - \beta_n)(T_n y_n - w) + \beta_n(y_n - w)\| \\
&\leq (1 - \beta_n)\|T_n y_n - w\| + \beta_n\|y_n - w\| \\
&\leq \|y_n - w\|. \qquad (3.5)
\end{aligned}$$

By induction on n, putting $K = \max\{\|x_1 - w\|, \|w\|\}$, we obtain $\|x_n - w\| \leq K$ for all $n \in \mathbb{N}$. In fact, it is obvious that $\|x_1 - w\| \leq K$. Suppose that $\|x_k - w\| \leq K$ for some $k \in \mathbb{N}$. Then, we have

$$\begin{aligned}\|x_{k+1} - w\| &\leq (1-\alpha_k)\|x_k - w\| + \alpha_k\|w\| \\ &\leq (1-\alpha_k)K + \alpha_k K = K.\end{aligned}$$

Hence, the sequence $\{x_n\}$ is bounded and so are $\{y_n\}, \{T_n y_n\}$ and $\{u_n\}$. By (3.2) and (3.5), we obtain

$$\begin{aligned}&\|x_{n+1} - w\|^2 \\ &\leq \|y_n - w\|^2 \\ &\leq \|(1-\alpha_n)(w - u_n) + \alpha_n w\|^2 \\ &\leq (1-\alpha_n)^2\|u_n - w\|^2 - \alpha_n^2\|w\|^2 + 2\langle \alpha_n w, (1-\alpha_n)(w-u_n) + \alpha_n w\rangle \qquad (3.6) \\ &\leq (1-\alpha_n)\big(\|x_n - w\|^2 - a(2\alpha - b)\|Ax_n - Aw\|^2\big) + \alpha_n\|w\|^2 \\ &\quad + 2\alpha_n(1-\alpha_n)\langle w, w - u_n\rangle \\ &\leq (1-\alpha_n)\|x_n - w\|^2 - (1-\alpha_n)a(2\alpha - b)\|Ax_n - Aw\|^2 + \alpha_n\delta_n, \qquad (3.7)\end{aligned}$$

where $\delta_n = \alpha_n\|w\|^2 + 2(1-\alpha_n)\langle w, w - u_n\rangle$. It follows that

$$\|x_{n+1} - w\|^2 \leq (1-\alpha_n)\|x_n - w\|^2 + \alpha_n\delta_n. \qquad (3.8)$$

Next, by using Lemma 2.1 (ii), we shall show that $x_n \to w$. It means that we want to show

$$\limsup_{k\to\infty} \delta_{n_k} = 2\limsup_{k\to\infty}\langle w, w - u_{n_k}\rangle \leq 0$$

for every subsequence $\{n_k\}$ of $\{n\}$ satisfying

$$\liminf_{k\to\infty}(\|x_{n_k+1} - w\| - \|x_{n_k} - w\|) \geq 0.$$

To do this, given a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\liminf_{k\to\infty}(\|x_{n_k+1} - w\| - \|x_{n_k} - w\|) \geq 0.$$

From (3.4), (3.5) and $\alpha_n \to 0$, we obtain

$$\limsup_{n\to\infty}\big(\|x_{n+1} - w\| - \|x_n - w\|\big) \leq \limsup_{n\to\infty}\big(\|y_n - w\| - \|x_n - w\|\big) \leq 0.$$

It follows that

$$\lim_{k\to\infty}\big(\|x_{n_k+1} - w\| - \|x_{n_k} - w\|\big) = 0 = \lim_{k\to\infty}\big(\|y_{n_k} - w\| - \|x_{n_k} - w\|\big). \qquad (3.9)$$

From (3.7), we have

$$(1-\alpha_n)a(2\alpha - b)\|Ax_n - Aw\|^2 \leq \|x_n - w\|^2 - \|x_{n+1} - w\|^2 + \alpha_n\delta_n - \alpha_n\|x_n - w\|^2.$$

Since $\alpha_n \to 0$ and (3.9),

$$\|Ax_{n_k} - Aw\| \to 0. \tag{3.10}$$

Using Lemma 2.2 (ii), we obtain

$$\begin{aligned}
&2\|u_n - w\|^2 \\
&= 2\|T_{r_n}(x_n - r_nAx_n) - T_{r_n}(w - r_nAw)\|^2 \\
&\leq 2\langle (x_n - r_nAx_n) - (w - r_nAw), u_n - w\rangle \\
&= \|(x_n - r_nAx_n) - (w - r_nAw)\|^2 + \|u_n - w\|^2 - \|(x_n - u_n) - r_n(Ax_n - Aw)\|^2 \\
&\leq \|x_n - w\|^2 + \|u_n - w\|^2 - \|(x_n - u_n) - r_n(Ax_n - Aw)\|^2 \\
&= \|x_n - w\|^2 + \|u_n - w\|^2 - \|x_n - u_n\|^2 \\
&\quad + 2r_n\langle x_n - u_n, Ax_n - Aw\rangle - r_n^2\|Ax_n - Aw\|^2.
\end{aligned}$$

So, we have

$$\begin{aligned}
\|u_n - w\|^2 \leq\ & \|x_n - w\|^2 - \|x_n - u_n\|^2 \\
& + 2r_n\langle x_n - u_n, Ax_n - Aw\rangle - r_n^2\|Ax_n - Aw\|^2.
\end{aligned} \tag{3.11}$$

From (3.6) and (3.11), we obtain

$$\begin{aligned}
&\|x_{n+1} - w\|^2 \\
&\leq \|x_n - w\|^2 - (1-\alpha_n)^2\|x_n - u_n\|^2 + 2(1-\alpha_n)^2 r_n\|x_n - u_n\|\|Ax_n - Aw\| \\
&\quad + \alpha_n^2\|w\|^2 + 2\alpha_n(1-\alpha_n)\|u_n - w\|\|w\|
\end{aligned} \tag{3.12}$$

Using $\alpha_n \to 0$, (3.9) and (3.10), we get

$$\|x_{n_k} - u_{n_k}\| \to 0. \tag{3.13}$$

Since $\alpha_n \to 0$, we obtain

$$\begin{aligned}
\|y_n - u_n\| &= \|P_C((1-\alpha_n)u_n) - P_Cu_n\| \\
&\leq \|(1-\alpha_n)u_n - u_n\| \\
&= \alpha_n\|u_n\| \to 0.
\end{aligned} \tag{3.14}$$

By (2.1), we have

$$\begin{aligned}
\|x_{n+1} - w\|^2 &= \beta_n\|y_n - w\|^2 + (1-\beta_n)\|T_ny_n - w\|^2 - \beta_n(1-\beta_n)\|y_n - T_ny_n\|^2 \\
&\leq \|y_n - w\|^2 - \beta_n(1-\beta_n)\|y_n - T_ny_n\|^2.
\end{aligned}$$

Then

$$\beta_n(1-\beta_n)\|y_n - T_ny_n\|^2 \leq \|y_n - w\|^2 - \|x_{n+1} - w\|^2.$$

Since (3.9) and (ii), we obtain

$$\|y_{n_k} - T_{n_k} y_{n_k}\| \to 0. \tag{3.15}$$

We shall show that

$$\limsup_{k\to\infty} \langle w, w - u_{n_k}\rangle \le 0. \tag{3.16}$$

To show this inequality, take a subsequence $\{u_{n_i}\}$ of $\{u_{n_k}\}$ such that $u_{n_i} \rightharpoonup u$ and

$$\limsup_{k\to\infty} \langle w, w - u_{n_k}\rangle = \lim_{i\to\infty} \langle w, w - u_{n_i}\rangle. \tag{3.17}$$

Let us show $u \in \mathfrak{F}$. From (3.14), (3.15) and $\{T_n\}$ satisfies the NST-condition, we get $u \in \cap_{n=1}^{\infty} F(T_n)$. Next, we shall show $u \in EP(\varphi, A)$. For any $y \in C$, we have from (3.1) that

$$\varphi(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n,\ u_n - x_n\rangle \ge 0,$$

and by the condition (A2), we obtain

$$\langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n,\ u_n - x_n\rangle \ge \varphi(y, u_n).$$

Replacing n by n_i, we get

$$\langle Ax_{n_i}, y - u_{n_i}\rangle + \langle y - u_{n_i},\ \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \ge \varphi(y, u_{n_i}). \tag{3.18}$$

Put $z_t = ty + (1-t)u$ for all $t \in (0, 1]$ and $y \in C$. Then, we have $z_t \in C$. So, from (3.18) we have

$$\begin{aligned}
\langle z_t - u_{n_i}, Az_t\rangle &\ge \langle z_t - u_{n_i}, Az_t\rangle - \langle z_t - u_{n_i}, Ax_{n_i}\rangle \\
&\quad - \left\langle z_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle + \varphi(z_t, u_{n_i}) \\
&= \langle z_t - u_{n_i}, Az_t - Au_{n_i}\rangle + \langle z_t - u_{n_i}, Au_{n_i} - Ax_{n_i}\rangle \\
&\quad - \left\langle z_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle + \varphi(z_t, u_{n_i}).
\end{aligned}$$

Since $r_n \ge a > 0$, $\|u_{n_i} - x_{n_i}\| \to 0$, we have $\|Au_{n_i} - Ax_{n_i}\| \to 0$. Furthermore, from monotonicity of A, we obtain $\langle z_t - u_{n_i}, Az_t - Au_{n_i}\rangle \ge 0$. So, from the condition (A4) we get

$$\langle z_t - u, Az_t\rangle \ge \varphi(z_t, u), \tag{3.19}$$

as $i \to \infty$. From the conditions (A1), (A4) and (3.19), we also have

$$\begin{aligned}
0 = \varphi(z_t, z_t) &\le t\varphi(z_t, y) + (1-t)\varphi(z_t, u) \\
&\le t\varphi(z_t, y) + (1-t)\langle z_t - u, Az_t\rangle \\
&= t\varphi(z_t, y) + t(1-t)\langle y - u, Az_t\rangle
\end{aligned}$$

and hence

$$0 \leq \varphi(z_t, y) + (1-t)\langle y - u, Az_t \rangle.$$

Letting $t \to 0$, we have, for each $y \in C$,

$$0 \leq \varphi(u, y) + \langle y - u, Au \rangle.$$

This implies $u \in EP(\varphi, A)$ and hence $u \in \mathfrak{F}$. From (3.17), $\lim_{n\to\infty} \alpha_n = 0$ and the property of metric projection, we have

$$\limsup_{k\to\infty} \langle w, w - u_{n_k} \rangle = \lim_{i\to\infty} \langle w, w - u_{n_i} \rangle = \langle w, w - u \rangle \leq 0.$$

Thus $\limsup_{k\to\infty} \delta_{n_k} \leq 0$. From Lemma 2.1 and $\sum_{n=1}^{\infty} \alpha_n = \infty$, we obtain

$$\lim_{n\to\infty} \|x_n - w\| = 0.$$

We conclude that $x_n \to w$, where $w \in \mathfrak{F}$ such that

$$\|w\| = \min\{\|z\| : z \in \mathfrak{F}\}.$$

From (3.3) and (3.4), we obtain $u_n \to w$ and $y_n \to w$, respectively. This completes the proof. $\square$

Remark 3.2. *We can find some iterations which are equivalent to* (3.1) *such as*

$$\begin{cases} x_1 \in C, \\ y_n = \beta_n x_n + (1-\beta_n) T_n x_n, \\ \varphi(u_n, y) + \langle Ay_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - y_n \rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = P_C((1-\alpha_n)u_n), \quad \forall n \in \mathbb{N}. \end{cases} \tag{3.20}$$

To see this, let us set $y_1 = \beta_1 x_1 + (1-\beta_1)T_1 x_1$, $z_n \equiv x_{n+1}$, $S_n \equiv T_{n+1}$ and $\widehat{\beta_n} \equiv \beta_{n+1}$. Then it follows from (3.20) *that*

$$\begin{cases} y_1 \in C, \\ \varphi(u_n, y) + \langle Ay_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - y_n \rangle \geq 0, \quad \forall y \in C, \\ z_n = P_C((1-\alpha_n)u_n), \\ y_{n+1} = \widehat{\beta_n} z_n + (1-\widehat{\beta_n}) S_n z_n \quad \forall n \in \mathbb{N}. \end{cases} \tag{3.21}$$

Employing the idea in [31], we obtain the following strong convergence theorem by the modified viscosity method, whose proof is left for the reader to verify, for finding the common element of the set of solutions of the equilibrium problem and the set of fixed points of a countable family of nonexpansive mappings in a Hilbert space.

Theorem 3.3. *Let C be a closed convex subset of a real Hilbert space H and let $\varphi : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions* (A1)-(A4). *Let A be an α-inverse-strongly monotone mapping of C into H and let $\{T_n\}$ be a family of nonexpansive mappings of C into itself satisfying the NST-condition and $\mathfrak{F} := \cap_{n=1}^{\infty} F(T_n) \cap EP(\varphi, A) \neq \varnothing$. Let $\{r_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 2\alpha)$, and let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0, 1)$ satisfying the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Let $\{u_n\}, \{y_n\}$ and $\{x_n\}$ be sequences defined by

$$\begin{cases} x_1 \in C, \\ \varphi(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C(\alpha_n f(x_n) + (1 - \alpha_n)u_n), \\ x_{n+1} = \beta_n y_n + (1 - \beta_n)T_n y_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{3.22}$$

where $f : C \to H$ is an α-contraction with $\alpha \in [0, 1)$, that is, $\|f(x) - f(y)\| \leq \alpha\|x - y\|$ for all $x, y \in C$. Then $\{u_n\}, \{y_n\}$ and $\{x_n\}$ defined by (3.22) converge strongly to $z \in \mathfrak{F}$ where $z = P_{\mathfrak{F}} f(z)$.

Next, we use our Theorem 3.1 to obtain strong convergence theorems by generating a sequence $\{S_n\}$ of nonexpansive mappings satisfying the *NST-condition* from a general sequence $\{T_n\}$ of nonexpansive mappings with a common fixed point (see [7, 8, 11, 14, 48]). We now give some examples of a family of mappings satisfying the NST-condition.

Let $\{T_n\}$ be a sequence of mappings of C into itself and let $\lambda_1^n, \lambda_2^n, \lambda_3^n$ be real numbers for all $n \in \mathbb{N}$ such that $0 < \lambda_i^n \leq 1$ for $i = 1, 2, 3$ with $\lambda_1^n + \lambda_2^n + \lambda_3^n = 1$. Then for any $n \in \mathbb{N}$, we define a mapping S_n of C into itself [37] as follow:

$$\begin{aligned} U_{n,n+1} \quad &= I, \\ U_{n,n} \quad &= \lambda_1^n T_n U_{n,n+1} + \lambda_2^n U_{n,n+1} + \lambda_3^n I, \\ U_{n,n-1} \quad &= \lambda_1^{n-1} T_{n-1} U_{n,n} + \lambda_2^{n-1} U_{n,n} + \lambda_3^{n-1} I, \\ &\vdots \\ U_{n,k} \quad &= \lambda_1^k T_k U_{n,k+1} + \lambda_2^k U_{n,k+1} + \lambda_3^k I, \\ &\vdots \\ U_{n,2} \quad &= \lambda_1^2 T_2 U_{n,3} + \lambda_2^2 U_{n,3} + \lambda_3^2 I, \\ S_n = U_{n,1} \quad &= \lambda_1^1 T_1 U_{n,2} + \lambda_2^1 U_{n,2} + \lambda_3^1 I. \end{aligned} \tag{3.23}$$

Such a mapping S_n is called the *S-mapping* generated by $T_1, T_2, \ldots, T_n$ and $\lambda_1^n, \lambda_2^n, \lambda_3^n$ for all $n \in \mathbb{N}$. It is obvious that S_n and $U_{n,k}$ are nonexpansive for every $n \geq k$ if T_n is nonexpansive. We will use the following lemma for this paper. The method of our proof is similar to that of [25].

Lemma 3.4. *Let C be a closed convex subset of a real Hilbert space H. Let $\{T_n\}$ be a sequence of nonexpansive mappings with $\cap_{i=1}^{\infty} F(T_i)$ is nonempty. Let $\{\lambda_i^n : i = 1, 2, 3\}$ be real numbers in $(0, 1)$ such that $\lambda_1^n + \lambda_2^n + \lambda_3^n = 1$ and $\lambda_3^n \geq a > 0$ for all $n \in \mathbb{N}$. Then the sequence $\{S_n\}$ of S-mapping defined by (3.23) satisfies the AKTT-condition (II) and hence it also satisfies the NST-condition.*

Proof. We shall show that $\{S_n\}$ of S-mapping defined by (3.23) satisfies the AKTT-condition (II). To this end, we show that there is a nonexpansive mapping $S : C \to C$ such that $Sx = \lim_{n\to\infty} S_n x$ for all $x \in C$ and $F(S) = \cap_{n=1}^{\infty} F(T_n)$. Let $x \in C$ and $w \in \cap_{n=1}^{\infty} F(T_n)$. Then $U_{j,k}w = w$ for all $i, j \in \mathbb{N}$. Without loss of generality, we may assume $x \neq w$. Fix $k \in \mathbb{N}$. Then, for any $n \in \mathbb{N}$ with $n \geq k$, we obtain

$$\begin{aligned}
&\|U_{n+1,k}x - U_{n,k}x\| \\
&= \|\lambda_1^k(T_kU_{n+1,k+1}x - T_kU_{n,k+1}x) + \lambda_2^k(U_{n+1,k+1}x - U_{n,k+1}x)\| \\
&\leq \lambda_1^k\|T_kU_{n+1,k+1}x - T_kU_{n,k+1}x\| + \lambda_2^k\|U_{n+1,k+1}x - U_{n,k+1}x\| \\
&\leq \lambda_1^k\|U_{n+1,k+1}x - U_{n,k+1}x\| + \lambda_2^k\|U_{n+1,k+1}x - U_{n,k+1}x\| \\
&= (1-\lambda_3^k)\|U_{n+1,k+1}x - U_{n,k+1}x\| \\
&\leq (1-\lambda_3^k)(\lambda_1^{k+1}\|T_{k+1}U_{n+1,k+2}x - T_{k+1}U_{n,k+2}x\| + \lambda_2^{k+1}\|U_{n+1,k+2}x - U_{n,k+2}x\|) \\
&\leq (1-\lambda_3^k)(1-\lambda_3^{k+1})\|U_{n+1,k+2}x - U_{n,k+2}x\| \\
&\ \ \vdots \\
&\leq \prod_{i=k}^{n}(1-\lambda_3^i)\|U_{n+1,n+1}x - U_{n,n+1}x\| \\
&\leq \prod_{i=k}^{n}(1-\lambda_3^i)(\|U_{n+1,n+1}x - w\| + \|U_{n,n+1}x - w\|) \\
&\leq 2\prod_{i=k}^{n}(1-\lambda_3^i)\|x - w\|.
\end{aligned}$$

Let $\varepsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ with $n_0 \geq k$ such that

$$\sum_{j=n_0}^{\infty}(1-a)^{j-k+1} < \frac{\varepsilon}{2\|x-w\|}.$$

It follows from $\lambda_3^n \geq a > 0$ for every $n \in \mathbb{N}$ that for every m, n with $m > n > n_0$, we obtain

$$\begin{aligned}
\|U_{m,k}x - U_{n,k}x\| &\leq \sum_{j=n}^{m-1}\|U_{j+1,k}x - U_{j,k}x\| \\
&\leq \sum_{j=n}^{m-1}\left\{2\left(\prod_{i=k}^{j}(1-\lambda_3^i)\right)\|x-w\|\right\} \\
&\leq 2\|x-w\|\sum_{j=n}^{m-1}(1-a)^{j-k+1} \\
&< \epsilon.
\end{aligned}$$

Then $\{U_{n,k}x\}$ is a Cauchy sequence, and hence $\lim_{n\to\infty} U_{n,k}x$ exists.

For $k \in \mathbb{N}$, we define nonexpansive mappings $U_{\infty,k}$ and S of C into itself as follows:

$$U_{\infty,k}x := \lim_{n\to\infty} U_{n,k}x$$

and

$$Sx := \lim_{n\to\infty} S_n x = \lim_{n\to\infty} U_{n,1}x$$

for every $x \in C$. It is easy to see that $\cap_{n=1}^{\infty} F(T_n) \subset F(S)$. Next, we shall show that

$$F(S) \subset \cap_{n=1}^{\infty} F(T_n).$$

Let $x \in F(S)$ and $y \in \cap_{n=1}^{\infty} F(T_n)$. Then we obtain

$$\begin{aligned}
\|S_n x - S_n y\| &= \|U_{n,1}x - U_{n,1}y\| \\
&\le \lambda_1^1 \|T_1 U_{n,2}x - T_1 U_{n,2}y\| + \lambda_2^1 \|U_{n,2}x - U_{n,2}y\| + \lambda_3^1 \|x - y\| \\
&= (1 - \lambda_3^1)\|U_{n,2}x - U_{n,2}y\| + \lambda_3^1 \|x - y\| \\
&\ \ \vdots \\
&\le \prod_{i=1}^{k-1}(1-\lambda_3^i)\|U_{n,k}x - U_{n,k}y\| + \left(1 - \prod_{i=1}^{k-1}(1-\lambda_3^i)\right)\|x - y\| \\
&\ \ \vdots \\
&\le \prod_{i=1}^{n}(1-\lambda_3^i)\|U_{n,n+1}x - U_{n,n+1}y\| + \left(1 - \prod_{i=1}^{n}(1-\lambda_3^i)\right)\|x - y\| \\
&= \|x - y\|.
\end{aligned}$$

So, we have, as $n \to \infty$,

$$\begin{aligned}
\|x - y\| &= \|Sx - Sy\| \\
&\le \prod_{i=1}^{k-1}(1-\lambda_3^i)\|U_{\infty,k}x - U_{\infty,k}y\| + \left(1 - \prod_{i=1}^{k-1}(1-\lambda_3^i)\right)\|x - y\| \\
&\le \|x - y\|.
\end{aligned}$$

Therefore,

$$\|U_{\infty,k+1}x - y\| = \|U_{\infty,k}x - U_{\infty,k}y\| = \|x - y\|.$$

Since $U_{n,k} = \lambda_1^k T_k U_{n,k+1} + \lambda_2^k U_{n,k+1} + \lambda_3^k I$, we get

$$\|\lambda_1^k(T_k U_{\infty,k+1}x - y) + \lambda_2^k(U_{\infty,k+1}x - y) + \lambda_3^k(x - y)\| = \|x - y\|. \tag{3.24}$$

It is easy to see that

$$\|T_k U_{\infty,k+1}x - y\| \le \|U_{\infty,k+1}x - y\| \le \|x - y\|. \tag{3.25}$$

From (3.24), (3.25) and since $\lambda_i^k > 0$, it follows that

$$\|T_k U_{\infty,k+1}x - x\| = 0 = \|U_{\infty,k+1}x - x\|.$$

That is, $T_k U_{\infty,k+1}x = x$ and $U_{\infty,k+1}x = x$. Therefore $T_k x = x$. This implies that $x \in \cap_{n=1}^{\infty} F(T_n)$ and hence $F(S) = \cap_{n=1}^{\infty} F(T_n)$. □

Using Theorem 3.1 and Lemma 3.4, we get the following result.

Theorem 3.5. *Let C be a closed convex subset of a real Hilbert space H and let $\varphi : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions* (A1)-(A4)*. Let A be an α-inverse-strongly monotone mapping of C into H and let $\{T_n\}$ be a sequence of nonexpansive mappings of C into itself such that $\mathfrak{F} := \cap_{n=1}^{\infty} F(T_n) \cap EP(\varphi, A) \neq \varnothing$. For given $x_1 \in C$ arbitrary, let the sequences $\{u_n\}$, $\{y_n\}$ and $\{x_n\}$ be generated iteratively by*

$$\begin{cases} \varphi(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C, \\ y_n = P_C((1-\alpha_n)u_n), \\ x_{n+1} = \beta_n y_n + (1-\beta_n)S_n y_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

where S_n is the S-mapping defined by (3.23)*, $\{r_n\} \subset [a,b]$ for some $a, b \in (0, 2\alpha)$, and $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$. Assume in addition that*

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

Then $\{u_n\}, \{y_n\}$ and $\{x_n\}$ converge strongly to the common minimum-norm element of $\mathfrak{F}$.

If $\lambda_2^n \equiv 0$ in (3.23), then $S_n \equiv W_n$ which is called the W-mapping defined by Takahashi [23] (see also [24, 25]) and we get the following corollary.

Corollary 3.6. *Let C be a closed convex subset of a real Hilbert space H and let $\varphi : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions* (A1)-(A4)*. Let A be an α-inverse-strongly monotone mapping of C into H and let $\{T_n\}$ be a sequence of nonexpansive mappings of C into itself such that $\mathfrak{F} := \cap_{n=1}^{\infty} F(T_n) \cap EP(\varphi, A) \neq \varnothing$. For given $x_1 \in C$ arbitrary, let the sequences $\{u_n\}$, $\{y_n\}$ and $\{x_n\}$ be generated iteratively by*

$$\begin{cases} \varphi(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C, \\ y_n = P_C((1-\alpha_n)u_n), \\ x_{n+1} = \beta_n y_n + (1-\beta_n)W_n y_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

where W_n is the W-mapping, $\{r_n\} \subset [a,b]$ for some $a, b \in (0, 2\alpha)$, and $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$. Assume in addition that

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then $\{u_n\}, \{y_n\}$ and $\{x_n\}$ converge strongly to the common minimum-norm element of $\mathfrak{F}$.

Proof. As in the proof of [25, Lemmas 3.2 and 3.3], we get that $\{W_n\}$ satisfies the NST-condition and the proof is finished by using Theorem 3.1. □

Consequently, we can generate a sequence $\{S_n\}$ of nonexpansive mappings satisfying NST-condition by using convex combination of a general sequence $\{T_n\}$ of nonexpansive mappings with a common fixed point.

Theorem 3.7. *Let C be a closed convex subset of a real Hilbert space H and let $\varphi : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions* (A1)-(A4)*. Let A be an α-inverse-strongly monotone mapping of C into H and let $\{\gamma_n^k\}$ be a family of nonnegative real numbers where $n, k \in \mathbb{N}$ with $k \leq n$. Let $\{T_n\}$ be a sequence of nonexpansive mappings of C into itself such that $\mathfrak{F} := \cap_{n=1}^{\infty} F(T_n) \cap EP(\varphi, A) \neq \varnothing$. For given $x_1 \in C$ arbitrary, let the sequences $\{u_n\}, \{y_n\}$ and $\{x_n\}$ be generated iteratively by*

$$\begin{cases} \varphi(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C((1-\alpha_n)u_n), \\ x_{n+1} = \beta_n y_n + (1-\beta_n)\sum_{k=1}^{n} \gamma_n^k T_k y_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\} \subset [a, b]$ for some $a, b \in (0, 2\alpha)$ and $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$. Assume in addition that

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iii) $\lim_{n\to\infty} \gamma_n^k > 0,\ \forall k \in \mathbb{N},\ \sum_{k=1}^{n} \gamma_n^k = 1,\ \forall n \in \mathbb{N}$;

(iv) $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\gamma_{n+1}^k - \gamma_n^k| < \infty$.

Then $\{u_n\}, \{y_n\}$ and $\{x_n\}$ converge strongly to the common minimum-norm element of $\mathfrak{F}$.

Proof. In Theorem 3.1, put $S_n = \sum_{k=1}^{n} \gamma_n^k T_k$. As in [7, Theorem 4.1] and Lemma 2.6, we obtain that $\{T_n\}$ satisfies the NST-condition and the desired result. □

Using Theorem 3.1 and [11, Lemma 3.2 (i)], we obtain the following result.

Corollary 3.8. *Let C be a closed convex subset of a real Hilbert space H and let $\varphi : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions* (A1)-(A4)*. Let A be an α-inverse-strongly monotone mapping of C into H. Let S and T be nonexpansive* *mappings of C into itself such that $\mathfrak{F} := F(S) \cap F(T) \cap EP(\varphi, A) \neq \varnothing$. For given $x_1 \in C$ arbitrary, let the sequences $\{u_n\}, \{y_n\}$ and $\{x_n\}$ be generated iteratively by*

$$\begin{cases} \varphi(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C((1-\alpha_n)u_n), \\ x_{n+1} = \beta_n y_n + (1-\beta_n)(\gamma_n S + (1-\gamma_n)T)y_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\} \subset [a, b]$ for some $a, b \in (0, 2\alpha)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1)$. Assume in addition that

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iii) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$.

Then $\{u_n\}, \{y_n\}$ and $\{x_n\}$ converge strongly to the common minimum-norm element of $\mathfrak{F}$.

If we put $T_n = T$ and use Theorem 3.1, then we get the following result.

Corollary 3.9. *Let C be a closed convex subset of a real Hilbert space H and let $\varphi : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions* (A1)-(A4). *Let A be an α-inverse-strongly monotone mapping of C into H and let T be a nonexpansive mapping of C into itself such that $F(T) \cap EP(\varphi, A) \neq \varnothing$. For given $x_1 \in C$ arbitrary, let the sequences $\{u_n\}$, $\{y_n\}$ and $\{x_n\}$ be generated iteratively by*

$$\begin{cases} \varphi(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C((1-\alpha_n)u_n), \\ x_{n+1} = \beta_n y_n + (1-\beta_n)Ty_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\} \subset [a, b]$ for some $a, b \in (0, 2\alpha)$ and $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$. Assume in addition that

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then $\{u_n\}, \{y_n\}$ and $\{x_n\}$ converge strongly to the common minimum-norm element of $F(T) \cap EP(\varphi, A)$.

If we put $\varphi \equiv 0$ and $A \equiv 0$ and use Theorem 3.1, then we get the following result.

Corollary 3.10. *Let C be a closed convex subset of a real Hilbert space H and let $\{T_n\}$ be a sequence of nonexpansive mappings of C into itself such that $F := \cap_{n=1}^{\infty} F(T_n) \neq \varnothing$. Let $x_1 \in C$ be chosen arbitrary. Let $\{x_n\}$ and $\{y_n\}$ be sequences of C defined as follows:*

$$\begin{cases} y_n = P_C((1-\alpha_n)x_n), \\ x_{n+1} = \beta_n y_n + (1-\beta_n)T_n y_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\} \subset (0, 1)$ satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ and $\{y_n\}$ converge strongly to the common minimum-norm element of F.

Remark 3.11. *Corollary 3.10 extends and improves [2, Corollary 3.2] by setting $T_n \equiv W_n$, the W-mapping.*

If $T_n \equiv T$ in Corollary 3.10, then we get the following result.

Corollary 3.12 ([3, Thoerem 3.1])**.** *Let C be a closed convex subset of a real Hilbert space H and let T be a nonexpansive mapping of C into itself such that $F(T) \neq \varnothing$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0,1)$ satisfy the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ and $\{y_n\}$ defined by (1.5) *converge strongly to the minimum-norm element of $F(T)$.*

Remark 3.13. *Since the scheme* (1.5) *and the scheme* (1.8) *are equivalent, Our Corollary 3.12 extends and improves [1, Theorem 3.2] in the following ways:*

(i) *The closed convex cone ($\alpha x \in C$ for all $x \in C$ and $\alpha \geq 0$) imposed on C is replaced by the more general closed convex.*

(ii) *The restriction $\beta_n \equiv \beta \in (0,1)$ is weakened and replaced by $\{\beta_n\} \subset (0,1)$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.*

Finally, we relax the scheme (3.1) to the iterative scheme with perturbations which is related to Yao and Shahzad [21].

Theorem 3.14. *Let C be a closed convex subset of a real Hilbert space H and let $\varphi : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions* (A1)-(A4). *Let A be an α-inverse-strongly monotone mapping of C into H and let $\{T_n\}$ be a sequence of nonexpansive mappings of C into itself satisfying the NST-condition and $\mathfrak{F} := \cap_{n=1}^{\infty} F(T_n) \cap EP(\varphi, A) \neq \varnothing$. For given $x_1 \in C$ arbitrary, let the sequences $\{u_n\}$, $\{y_n\}$ and $\{x_n\}$ be generated iteratively by*

$$\begin{cases} \varphi(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C((1-\alpha_n)u_n + e_n), \\ x_{n+1} = \beta_n y_n + (1-\beta_n)T_n y_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\} \subset [a,b]$ for some $a, b \in (0, 2\alpha)$, $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$, and the computation error sequence $\{e_n\}$ in H. Assume in addition that

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(ii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;

(iii) $\sum_{n=1}^{\infty} \|e_n\| < \infty$ *or* $\lim_{n\to\infty} \frac{\|e_n\|}{\alpha_n} = 0$.

Then $\{u_n\}, \{y_n\}$ *and* $\{x_n\}$ *converge strongly to the common minimum-norm element of* $\mathfrak{F}$.

Proof. Let $\{\widehat{u}_n\}$, $\{\widehat{y}_n\}$ and $\{\widehat{x}_n\}$ be sequences of C defined as follows:

$$\begin{cases} \widehat{x}_1 = x_1 \in C, \\ \varphi(\widehat{u}_n, y) + \langle A\widehat{x}_n, y - \widehat{u}_n\rangle + \frac{1}{r_n}\langle y - \widehat{u}_n, \widehat{u}_n - \widehat{x}_n\rangle \ge 0, \quad \forall y \in C, \\ \widehat{y}_n = P_C((1-\alpha_n)\widehat{u}_n), \\ \widehat{x}_{n+1} = \beta_n \widehat{y}_n + (1-\beta_n)T_n\widehat{y}_n, \quad \forall n \in \mathbb{N}. \end{cases}$$

Using Theorem 3.1, we get that $\{\widehat{u}_n\}$, $\{\widehat{y}_n\}$ and $\{\widehat{x}_n\}$ converge strongly to w, the common minimum-norm point of $\mathfrak{F}$. Notice that $u_n \equiv T_{r_n}(I - r_nA)x_n$ and $\widehat{u}_n \equiv T_{r_n}(I - r_nA)\widehat{x}_n$. Then

$$\|\widehat{u}_n - u_n\| \le \|\widehat{x}_n - x_n\|$$

and so

$$\begin{aligned} \|\widehat{x}_{n+1} - x_{n+1}\| &\le \beta_n\|\widehat{y}_n - y_n\| + (1-\beta_n)\|T_n\widehat{y}_n - T_ny_n\| \\ &\le \|\widehat{y}_n - y_n\| \\ &\le (1-\alpha_n)\|\widehat{u}_n - u_n\| + \|e_n\| \\ &\le (1-\alpha_n)\|\widehat{x}_n - x_n\| + \|e_n\|. \end{aligned}$$

By Lemma 2.1, we get $\widehat{x}_n - x_n \to 0$. It follows that $\{u_n\}, \{y_n\}$ and $\{x_n\}$ converge strongly to w as desired. □

4 Applications

From now on, we apply the proposed methods for approximating the minimum-norm solution of the problem of finding a common zeros of a family of maximal monotone operators, the common minimizer problem and the multiple-sets split feasibility problem. In view of a unified treatment of considered examples, we give some additional preliminaries.

Lemma 4.1 ([14, Lemma 2.10])**.** *Let* C *and* K *be closed convex subsets of a real Hilbert space* H. *Let* $\{T_n\}$ *be a family of nonexpansive mappings of* C *into* K *and* $F(\mathcal{T}) := \cap_{n=1}^{\infty} F(T_n) \ne \varnothing$ *and let* $\{R_n\}$ *be a family of nonexpansive mappings of* K *into* C *with* $F(\mathcal{R}) := \cap_{n=1}^{\infty} F(R_n) \ne \varnothing$ *and*

$$\|R_nx - z\|^2 \le \|x - z\|^2 - a_n\|R_nx - x\|^2, \quad \forall x \in K, z \in F(\mathcal{R}) \text{ and } n \in \mathbb{N},$$

where $\{a_n\}$ *is a sequence in* $[a, \infty) \subset (0, \infty)$ *which includes a family of firmly nonexpansive mappings as a special case. Let* $\{V_n\}$ *be a family of mappings defined by*

$$V_n = T_nR_n, \quad \forall n \in \mathbb{N}.$$

If $\{T_n\}$ and $\{R_n\}$ satisfy the NST-condition and $F(\mathcal{T}) \cap F(\mathcal{R}) \neq \varnothing$, then $\{V_n\}$ is a family of nonexpansive mappings of K into itself satisfying the NST-condition and $\bigcap_{n=1}^{\infty} F(V_n) = F(\mathcal{T}) \cap F(\mathcal{R})$.

4.1 The common zeros of maximal monotone operators

We first consider the problem of finding a zero of a maximal monotone operator which plays an important role in many branches of applied sciences. One of interesting methods for solving this problem is the proximal point method which has been studied and developed by many researchers (see [9, 49, 50, 51, 44, 52]).

An operator $B : H \to 2^H$ is said to be a *monotone operator* on H if

$$\langle x - y, x' - y' \rangle \geq 0, \quad \forall x, y \in \mathrm{dom}(B), x' \in Bx \text{ and } y' \in By$$

where 2^H denotes the set of all subsets of H and $\mathrm{dom}(B)$ is the effective domain of B, that is, $\mathrm{dom}(B) = \{x \in H : Bx \neq \varnothing\}$. A monotone operator B on H is said to be *maximal* if its graph is not properly contained in the graph of any other monotone operator B' on H. See more detail about maximal monotone operators in Takahashi's book [44].

To deal with a maximal monotone operator, the concept of the resolvent is introduced and it plays an important role in our problem. Let B be a maximal monotone operator on H and $\lambda > 0$, the resolvent of B for λ denoted by J_λ^B, is a mapping of H into $\mathrm{dom}(B)$ given by $J_\lambda^B x = (I + \lambda B)^{-1} x$ for all $x \in H$. It is known that the resolvent J_λ^B is firmly nonexpansive and $F(J_\lambda^B) = B^{-1}0 := \{x \in H : 0 \in Bx\}$.

Using theorems of the previous sections we will prove a strong convergence theorem to the common minimum-norm element of the set of solutions of the problem of finding a zero of a maximal monotone operator, the set of solutions of the equilibrium problem and the set of fixed points of a countable family of nonexpansive mappings in a Hilbert space. To this end, we need the following lemma.

Lemma 4.2 ([9, Lemma 5.1] and [50, Lemma 3.4])**.** *Let B be a maximal monotone operator on H with $B^{-1}0 \neq \varnothing$. Assume that $\{\lambda_n\}$ is a sequence in $[c, \infty)$ for some $c > 0$. Then $\{J_{\lambda_n}^B\}$ is a sequence of firmly nonexpansive mappings with $F(J_{\lambda_n}^B) = B^{-1}0$ for all $n \in \mathbb{N}$ and satisfies the NST-condition.*

Theorem 4.3. *Let C be a closed convex subset of a real Hilbert space H and let $\varphi : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions* (A1)-(A4)*. Let A be an α-inverse-strongly monotone mapping of C into H and let B be a maximal monotone operator on H with* $\mathrm{dom}(B) \subset C$*. Let $\{T_n\}$ be a sequence of nonexpansive mappings of C into itself satisfying the NST-condition and $\mathfrak{F} := \cap_{n=1}^{\infty} F(T_n) \cap EP(\varphi, A) \cap B^{-1}0 \neq \varnothing$. For given $x_1 \in C$ arbitrary, let the sequences $\{u_n\}$, $\{y_n\}$ and $\{x_n\}$ be generated* *iteratively by*

$$\begin{cases} \varphi(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C((1-\alpha_n)u_n + e_n), \\ x_{n+1} = \beta_n y_n + (1-\beta_n)T_n J^B_{\lambda_n} y_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\} \subset [a, b]$ *for some* $a, b \in (0, 2\alpha)$, $\{\lambda_n\} \subset [c, \infty)$ *for some* $c > 0$, $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$, *and the computation error sequence* $\{e_n\} \subset H$. *Assume in addition that*

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iii) $\sum_{n=1}^{\infty} \|e_n\| < \infty$ *or* $\lim_{n\to\infty} \frac{\|e_n\|}{\alpha_n} = 0$.

Then $\{u_n\}, \{y_n\}$ *and* $\{x_n\}$ *converge strongly to the common minimum-norm element of* $\mathfrak{F}$.

Proof. Let $V_n \equiv T_n J^B_{\lambda_n}$. By Lemmas 4.1 and 4.2, we have that $\{V_n\}$ is a family of nonexpansive mappings with $\bigcap_{n=1}^{\infty} F(V_n) = \bigcap_{n=1}^{\infty} F(T_n) \cap \Omega$. Applying Theorem 3.1, we obtain the result. $\square$

If T_n is the identity mapping in Theorem 4.3, then we get the following result.

Corollary 4.4. *Let* C *be a closed convex subset of a real Hilbert space* H *and let* $\varphi : C \times C \to \mathbb{R}$ *be a bifunction satisfying conditions* (A1)-(A4). *Let* A *be an* α*-inverse-strongly monotone mapping of* C *into* H *and let* B *be a maximal monotone operator on* H *such that* $\mathrm{dom}(B) \subset C$ *and* $\mathfrak{F} := EP(\varphi, A) \cap B^{-1}0 \neq \varnothing$. *For given* $x_1 \in C$ *arbitrary, let the sequences* $\{u_n\}$, $\{y_n\}$ *and* $\{x_n\}$ *be generated iteratively by*

$$\begin{cases} \varphi(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C((1-\alpha_n)u_n + e_n), \\ x_{n+1} = \beta_n y_n + (1-\beta_n) J^B_{\lambda_n} y_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\} \subset [a, b]$ *for some* $a, b \in (0, 2\alpha)$, $\{\lambda_n\} \subset [c, \infty)$ *for some* $c > 0$, $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$, *and the computation error sequence* $\{e_n\} \subset H$. *Assume in addition that*

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iii) $\sum_{n=1}^{\infty} \|e_n\| < \infty$ *or* $\lim_{n\to\infty} \frac{\|e_n\|}{\alpha_n} = 0$.

Then $\{u_n\}, \{y_n\}$ *and* $\{x_n\}$ *converge strongly to the common minimum-norm element of* $\mathfrak{F}$.

If $\varphi \equiv 0$ and $A \equiv 0$ in Corollary 4.4, then we get the following result.

Corollary 4.5. *Let B be a maximal monotone operator on a Hilbert space H such that $B^{-1}0 \neq \varnothing$. For given $x_1 \in H$ arbitrary, let the sequences $\{x_n\}$ and $\{y_n\}$ be generated iteratively by*

$$\begin{cases} y_n = (1-\alpha_n)x_n + e_n, \\ x_{n+1} = \beta_n y_n + (1-\beta_n)J^B_{\lambda_n} y_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\lambda_n\} \subset [c,\infty)$ for some $c > 0$, $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$, and the computation error sequence $\{e_n\} \subset H$. Assume in addition that

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iii) $\sum_{n=1}^{\infty} \|e_n\| < \infty$ *or* $\lim_{n\to\infty} \frac{\|e_n\|}{\alpha_n} = 0$.

Then $\{x_n\}$ and $\{y_n\}$ converge strongly to the common minimum-norm element of $\mathfrak{F}$.

Next we consider a more general problem. We present strong convergence theorems to the common minimum-norm element of the set of solutions of the problem of finding a common zeros of a family of maximal monotone operators, the set of solutions of the equilibrium problem and the set of fixed points of a countable family of nonexpansive mappings in a Hilbert space.

Using Theorem 3.5, we have the following theorem.

Theorem 4.6. *Let C be a closed convex subset of a real Hilbert space H and let $\varphi : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions* (A1)-(A4). *Let A be an α-inverse-strongly monotone mapping of C into H and let $\{B_n\}$ be a family of maximal monotone operators on H with $\mathrm{dom}(B_n) \subset C$. Let $\{T_n\}$ be a sequence of nonexpansive mappings of C into itself such that $\mathfrak{F} := \cap_{n=1}^{\infty} F(T_n) \cap EP(\varphi, A) \cap \left(\cap_{n=1}^{\infty} B_n^{-1}0\right) \neq \varnothing$. For given $x_1 \in C$ arbitrary, let the sequences $\{u_n\}$, $\{y_n\}$ and $\{x_n\}$ be generated iteratively by*

$$\begin{cases} \varphi(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C((1-\alpha_n)u_n + e_n), \\ x_{n+1} = \beta_n y_n + (1-\beta_n)S_n y_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

where S_n is the S-mapping generated by $T_1 J^{B_1}_{\lambda_1}, T_2 J^{B_2}_{\lambda_2}, \ldots, T_n J^{B_n}_{\lambda_n}$ and $\lambda_1^n, \lambda_2^n, \lambda_3^n$ for all $n \in \mathbb{N}$, $\{r_n\} \subset [a,b]$ for some $a, b \in (0, 2\alpha)$, $\{\lambda_n\} \subset (0,\infty)$, $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$, and the computation error sequence $\{e_n\} \subset H$. Assume in addition that

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iii) $\sum_{n=1}^{\infty} \|e_n\| < \infty$ *or* $\lim_{n\to\infty} \frac{\|e_n\|}{\alpha_n} = 0$.

Then $\{u_n\}, \{y_n\}$ and $\{x_n\}$ converge strongly to the common minimum-norm element of $\mathfrak{F}$.

If T_n is the identity mapping in Theorem 4.6, then we get the following corollary.

Corollary 4.7. *Let C be a closed convex subset of a real Hilbert space H and let $\varphi : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions* (A1)-(A4). *Let A be an α-inverse-strongly monotone mapping of C into H and let $\{B_n\}$ be a family of maximal monotone operators on H with* $\mathrm{dom}(B_n) \subset C$ *and $\mathfrak{F} := EP(\varphi, A) \cap \big(\cap_{n=1}^{\infty} B_n^{-1}0\big) \neq \varnothing$. For given $x_1 \in C$ arbitrary, let the sequences $\{u_n\}$, $\{y_n\}$ and $\{x_n\}$ be generated iteratively by*

$$\begin{cases} \varphi(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C((1-\alpha_n)u_n + e_n), \\ x_{n+1} = \beta_n y_n + (1-\beta_n)S_n y_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

where S_n is the S-mapping generated by $J_{\lambda_1}^{B_1}, J_{\lambda_2}^{B_2}, \ldots, J_{\lambda_n}^{B_n}$ and $\lambda_1^n, \lambda_2^n, \lambda_3^n$ for all $n \in \mathbb{N}$, $\{r_n\} \subset [a,b]$ for some $a, b \in (0, 2\alpha)$, $\{\lambda_n\} \subset [c, \infty)$ for some $c > 0$, $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$, and the computation error sequence $\{e_n\} \subset H$. Assume in addition that

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iii) $\sum_{n=1}^{\infty} \|e_n\| < \infty$ *or* $\lim_{n\to\infty} \frac{\|e_n\|}{\alpha_n} = 0$.

Then $\{u_n\}, \{y_n\}$ and $\{x_n\}$ converge strongly to the common minimum-norm element of $\mathfrak{F}$.

Using Theorem 3.7, we have the following theorem.

Theorem 4.8. *Let C be a closed convex subset of a real Hilbert space H and let $\varphi : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions* (A1)-(A4). *Let A be an α-inverse-strongly monotone mapping of C into H and let $\{B_n\}$ be a family of maximal monotone operators on H with* $\mathrm{dom}(B_n) \subset C$. *Let $\{\gamma_n^k\}$ be a family of nonnegative real numbers where $n, k \in \mathbb{N}$ with $k \leq n$. Let $\{T_n\}$ be a sequence of nonexpansive mappings of C into itself such that $\mathfrak{F} := \cap_{n=1}^{\infty} F(T_n) \cap EP(\varphi, A) \cap \big(\cap_{n=1}^{\infty} B_n^{-1}0\big) \neq \varnothing$. For given $x_1 \in C$ arbitrary, let the sequences $\{u_n\}, \{y_n\}$ and $\{x_n\}$ be generated iteratively by*

$$\begin{cases} \varphi(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C((1-\alpha_n)u_n + e_n), \\ x_{n+1} = \beta_n y_n + (1-\beta_n)\sum_{k=1}^{n} \gamma_n^k T_k J_{\lambda_k}^{B_k} y_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\} \subset [a,b]$ for some $a, b \in (0, 2\alpha)$, $\{\lambda_n\} \subset (0, \infty)$ and $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$. Assume in addition that

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iii) $\lim_{n\to\infty}\gamma_n^k > 0,\ \forall k \in \mathbb{N},\ \sum_{k=1}^n \gamma_n^k = 1\ \forall n \in \mathbb{N};$

(iv) $\sum_{n=1}^\infty \sum_{k=1}^n |\gamma_{n+1}^k - \gamma_n^k| < \infty.$

Then $\{u_n\}, \{y_n\}$ *and* $\{x_n\}$ *converge strongly to the common minimum-norm element of* $\mathfrak{F}$.

4.2 The common minimizer problem

Let C be a closed convex subset of a real Hilbert space H and let f be a continuously differentiable and convex function on C. The minimizer problem is to find $x^* \in C$ such that

$$f(x^*) = \min_{x\in C} f(x). \tag{4.1}$$

For more details, see [53, 54]. Denote the solution set of (4.1) by $\Omega(C, f)$. It is known that $\Omega(C, f) = VI(C, \nabla f)$, where ∇f denotes the gradient of f (see, *e.g.,* [53, Proposition 3.1]), that is,

$$x^* \in \Omega(C, f) \quad \Leftrightarrow \quad \langle \nabla f(x^*), x - x^*\rangle \ge 0, \quad \forall x \in C.$$

If the gradient ∇f is L-Lipschitzian on C, then ∇f is $1/L$-inverse-strongly monotone mapping.

Lemma 4.9 ([47, Lemma 2.3] and [48, Lemma 4.14])**.** *Let C be a closed convex subset of a real Hilbert space H. Let $\alpha > 0$ and let A be an α-inverse-strongly monotone mapping of C into H with $VI(C, A) \ne \varnothing$. Then*

$$\|P_C(I - \lambda A)x - z\|^2 \le \|x - z\|^2 - \frac{2\alpha - \lambda}{2\alpha}\|P_C(I - \lambda A)x - x\|^2$$

for all $\lambda > 0$, $x \in C$ and $z \in VI(C, A)$. Furthermore, if $\{\lambda_n\}$ is a sequences in $[c, d]$ for some $c, d \in (0, 2\alpha)$, then $\{P_C(I - \lambda_n A)\}$ is a family of nonexpansive mapping of C into itself satisfying the NST-condition.

Using Theorem 3.1, Lemma 4.1 and Lemma 4.9, we have the following theorem.

Theorem 4.10. *Let C be a closed convex subset of a real Hilbert space H and let $\varphi : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions* (A1)-(A4)*. Let A be an α-inverse-strongly monotone mapping of C into H and f be a continuously differentiable and convex function on C and its gradient ∇f be L-Lipschitzian. Let $\{T_n\}$ be a sequence of nonexpansive mappings of C into itself satisfying the NST-condition and $\mathfrak{F} := \cap_{n=1}^\infty F(T_n) \cap EP(\varphi, A) \cap \Omega(C, f) \ne \varnothing$. For given $x_1 \in C$ arbitrary, let the sequences $\{u_n\}, \{y_n\}$ and $\{x_n\}$ be generated iteratively by*

$$\begin{cases} \varphi(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C, \\ y_n = P_C((1 - \alpha_n)u_n), \\ x_{n+1} = \beta_n y_n + (1 - \beta_n)T_n P_C(y_n - \lambda_n \nabla f(y_n)), \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\} \subset [a, b]$ for some $a, b \in (0, 2\alpha)$, $\{\lambda_n\} \subset [c, d]$ for some $c, d \in (0, 2/L)$, and $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

Then $\{u_n\}, \{y_n\}$ *and* $\{x_n\}$ *converge strongly to the common minimum-norm element of* $\mathfrak{F}$.

Remark 4.11. *Theorem 4.10 extends and improves [1, Theorem 4.1].*

Next we consider a more general problem. Let C be a closed convex subset of a real Hilbert space H and let $\{C_n\}$ be a sequence of nonempty closed convex subsets of C with $\bigcap_{n=1}^{\infty} C_n \neq \varnothing$. Let $\{f_n\}$ be a sequence of continuously differentiable and convex functions on C. The common minimizer problem [54] is to find $x^* \in \cap_{n=1}^{\infty} C_n$ such that

$$f_n(x^*) = \min_{x\in C_n} f_n(x), \quad \forall n \in \mathbb{N}. \tag{4.2}$$

Denote the solution set of (4.2) by Ω, that is, $\Omega = \cap_{n=1}^{\infty}\Omega(C_n, f_n)$. In [54], the problem was studied when $\{C_n\}_{n=1}^{N}$ is a finite family of closed convex subsets of H with $\bigcap_{n=1}^{N} C_n \neq \varnothing$ and $\{f_n\}_{n=1}^{N}$ is a finite family of continuously differentiable convex functions on C. Obviously, if $N = 1$ then the problem is nothing but the well-known minimizer problem.

The following theorems give a strong convergence theorem to the common minimum-norm element of the set of solutions of the common minimizer problem, the set of solutions of the equilibrium problem and the set of fixed points of a countable family of nonexpansive mappings in a Hilbert space by using Theorems 3.5 and 3.7, respectively.

Theorem 4.12. *Let* C *be a closed convex subset of a real Hilbert space* H *and let* $\{C_n\}$ *be a sequence of nonempty closed convex subsets of* C *with* $\bigcap_{n=1}^{\infty} C_n \neq \varnothing$. *Let* $\{f_n\}$ *be a sequence of continuously differentiable and convex functions on* C *with* $\Omega \neq \varnothing$ *and its gradient* ∇f_n *be L-Lipschitzian. Let* $\varphi : C \times C \to \mathbb{R}$ *be a bifunction satisfying conditions* (A1)-(A4) *and let* A *be an* α*-inverse-strongly monotone mapping of* C *into* H. *Let* $\{T_n\}$ *be a sequence of nonexpansive mappings of* C *into itself such that* $\mathfrak{F} := \cap_{n=1}^{\infty} F(T_n) \cap EP(\varphi, A) \cap \Omega \neq \varnothing$. *For given* $x_1 \in C$ *arbitrary, let the sequences* $\{u_n\}, \{y_n\}$ *and* $\{x_n\}$ *be generated iteratively by*

$$\begin{cases} \varphi(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C, \\ y_n = P_C((1-\alpha_n)u_n), \\ x_{n+1} = \beta_n y_n + (1-\beta_n) S_n y_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{4.3}$$

where S_n *is the S-mapping generated by* $T_1 P_{C_1}(I - \gamma_1\nabla f_1), T_2 P_{C_2}(I - \gamma_2 \nabla f_2), \ldots, T_n P_{C_n}(I - \gamma_n \nabla f_n)$ *and* $\lambda_1^n, \lambda_2^n, \lambda_3^n$ *for all* $n \in \mathbb{N}$, $\{r_n\} \subset [a, b]$ *for some* $a, b \in (0, 2\alpha)$, $\{\lambda_n\} \subset (0, 2/L)$, *and* $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ *satisfying the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

Then $\{u_n\}, \{y_n\}$ and $\{x_n\}$ converge strongly to the common minimum-norm element of $\mathfrak{F}$.

Theorem 4.13. *Let C be a closed convex subset of a real Hilbert space H and let $\{C_n\}$ be a sequence of nonempty closed convex subsets of C with $\bigcap_{n=1}^{\infty} C_n \neq \varnothing$. Let $\{f_n\}$ be a sequence of continuously differentiable and convex functions on C with $\Omega \neq \varnothing$ and its gradient ∇f_n be L-Lipschitzian. Let $\varphi : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions* (A1)-(A4) *and let A be an α-inverse-strongly monotone mapping of C into H. Let $\{T_n\}$ be a sequence of nonexpansive mappings of C into itself such that $\mathfrak{F} := \cap_{n=1}^{\infty} F(T_n) \cap EP(\varphi, A) \cap \Omega \neq \varnothing$. For given $x_1 \in C$ arbitrary, let the sequences $\{u_n\}, \{y_n\}$ and $\{x_n\}$ be generated iteratively by*

$$\begin{cases} \varphi(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C((1-\alpha_n)u_n), \\ x_{n+1} = \beta_n y_n + (1-\beta_n)\sum_{k=1}^{n} \gamma_n^k T_k P_{C_k}(y_n - \lambda_k \nabla f_k y_n), \quad \forall n \in \mathbb{N}, \end{cases} \tag{4.4}$$

where $\{r_n\} \subset [a, b]$ for some $a, b \in (0, 2\alpha)$, $\{\lambda_n\} \subset (0, 2/L)$, and $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$. Assume in addition that

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iii) $\lim_{n\to\infty} \gamma_n^k > 0,\ \forall k \in \mathbb{N}, \sum_{k=1}^{n} \gamma_n^k = 1, \quad \forall n \in \mathbb{N}$;

(iv) $\sum_{n=1}^{\infty}\sum_{k=1}^{n} |\gamma_{n+1}^k - \gamma_n^k| < \infty$.

Then $\{u_n\}, \{y_n\}$ and $\{x_n\}$ converge strongly to the common minimum-norm element of $\mathfrak{F}$.

4.3 The multiple-sets split feasibility problem

We first consider the split feasibility problem. Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $\Psi : H_1 \to H_2$ be a linear bounded operator. The split feasibility problem [55] is to find a point x^* satisfying the property:

$$x^* \in C \quad \text{and} \quad \Psi x^* \in Q. \tag{4.5}$$

Denote the solution set of (4.5) by $\Upsilon(C, Q, \Psi)$. Various iterative schemes have been invented to solve it (see [56, 57, 58, 59] and reference therein). For example, Byrne [56] introduced the following iterative scheme: $x_1 \in C$ and

$$x_{n+1} = P_C((I - \lambda\Psi^*(I - P_Q)\Psi))x_n, \quad \forall n \in \mathbb{N}, \tag{4.6}$$

where $0 < \lambda < 2\rho(\Psi\Psi^*)$ and $\rho(\Psi\Psi^*)$ denotes the spectral radius of the self-adjoint operator $\Psi\Psi^*$. He proved that $\{x_n\}$ converges weakly to a solution of Υ. In 2010, Wang and Xu [58] extended the iterative scheme (4.6) to strongly convergence theorem for finding a minimum-norm solution $\Upsilon(C, Q, \Psi)$ as follows: $x_1 \in C$ and

$$x_{n+1} = P_C((1-\alpha_n)(I - \lambda\Psi^*(I - P_Q)\Psi))x_n, \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, and $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$ or $\lim_{n\to\infty}\frac{\alpha_n}{\alpha_{n+1}} = 1$.

Notice that the split feasibility problem is equivalent to the minimizer problem by setting f to be the proximity function defined by

$$f(x) = \frac{1}{2}\|\Psi x - P_Q\Psi x\|^2.$$

Therefore, $x^* \in \Upsilon(C,\Psi)$ if and only if x^* solves the minimizer problem

$$f(x^*) = \min_{x\in C} f(x) = \min_{x\in C}\frac{1}{2}\|\Psi x - P_Q\Psi x\|^2 = 0.$$

Moreover, it is known [57, Lemma 8.1] that $\nabla f = \Psi^*(I-P_Q)\Psi$ is $\rho(\Psi\Psi^*)$-Lipschitzian.

Now, we give an application of Theorem 4.10 as follows:

Theorem 4.14. *Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $\Psi : H_1 \to H_2$ be a linear bounded operator. Let $\varphi : C\times C \to \mathbb{R}$ be a bifunction satisfying conditions* (A1)-(A4) *and let A be an α-inverse-strongly monotone mapping of C into H_1. Let $\{T_n\}$ be a sequence of nonexpansive mappings of C into itself satisfying the NST-condition and $\mathfrak{F} := \cap_{n=1}^{\infty}F(T_n)\cap EP(\varphi,A)\cap\Upsilon(C,Q,\Psi) \neq \varnothing$. For given $x_1 \in C$ arbitrary, let the sequences $\{u_n\},\{y_n\}$ and $\{x_n\}$ be generated iteratively by*

$$\begin{cases} \varphi(u_n,y) + \langle Ax_n, y-u_n\rangle + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \quad \forall y\in C,\\ y_n = P_C((1-\alpha_n)u_n),\\ x_{n+1} = \beta_n y_n + (1-\beta_n)T_nP_C((y_n - \lambda_n\Psi^*(I-P_Q)\Psi y_n)), \quad \forall n\in\mathbb{N}, \end{cases}$$

where $\{r_n\}\subset[a,b]$ for some $a,b\in(0,2\alpha)$, $\{\lambda_n\}\subset[c,d]$ for some $c,d\in(0,2\rho(\Psi\Psi^*))$, and $\{\alpha_n\},\{\beta_n\}\subset(0,1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty}\alpha_n = 0, \sum_{n=1}^{\infty}\alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$.

Then $\{u_n\},\{y_n\}$ and $\{x_n\}$ converge strongly to the common minimum-norm element of $\mathfrak{F}$.

Remark 4.15. *Theorem 4.14 extends and improves [1, Theorem 4.2].*

Next we consider a more general problem. Let $\{C_n\}$ and $\{Q_n\}$ be two families of nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $\Psi : H_1 \to H_2$ be a linear bounded operator. The multiple-sets split feasibility problem [60] is to find a point x^* satisfying the property:

$$x^* \in \cap_{n=1}^{\infty}C_n \quad\text{and}\quad \Psi x^* \in \cap_{n=1}^{\infty}Q_n. \tag{4.7}$$

Denote the solution set of (4.7) by Υ (see also [61, 62]). Notice that the multiple-sets split feasibility problem is equivalent to the common minimizer problem by setting each f_n to be the proximity function defined by

$$f_n(x) = \frac{1}{2}\|\Psi x - P_{Q_n}\Psi x\|^2. \tag{4.8}$$

Let us observe that in the special case of Theorems 4.12 and 4.13 when each f_n defined by (4.8), the iterative methods (4.3) and (4.4) converge strongly to converge strongly to the common minimum-norm point of $\mathfrak{F} = \cap_{n=1}^{\infty} F(T_n) \cap EP(\varphi, A) \cap \Upsilon$, respectively.

Acknowledgements

I would like to thank the referee(s) for his comments and suggestions on the manuscript. This research was supported by the Centre of Excellence in Mathematics and the Commission on Higher Education, Thailand.

References

[1] Y. Cai, Y. Tang, L. Liu, Iterative algorithms for minimum-norm fixed point of non-expansive mapping in Hilbert space, Fixed Point Theory Appl. 2012, 2012: 49, 10 pp.

[2] R.Chen, Iterative schemes for fixed point computation of nonexpansive mappings, Abstr. Appl. Anal. 2012, Art. ID 469270, 12 pp.

[3] Y. Yao, H. Zhou, Y. C. Liou, Strong convergence of a modified Krasnoselski-Mann iterative algorithm for non-expansive mappings, J. Appl. Math. Comput. 29 (2009) 383–389.

[4] W. R. Mann, Mean value methods in iteration, Proc. Amer. Math. Soc. 4 (1953) 506–510.

[5] B. Halpern, Fixed points of nonexpanding maps, Bull. Amer. Math. Soc. 73 (1967) 957–961.

[6] K. Aoyama, Y. Kimura, W. Takahashi, M. Toyoda, Finding common fixed points of a countable family of nonexpansive mappings in a Banach space, Sci. Math. Jpn. 66 (2007) 89–99.

[7] K. Aoyama, Y. Kimura, W. Takahashi, M. Toyoda, Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space, Nonlinear Anal. 67 (2007) 2350–2360.

[8] K. Aoyama, Y. Kimura, Strong convergence theorems for strongly nonexpansive sequences, Appl. Math. Comput. 217 (2011) 7537–7545.

[9] K. Aoyama, Y. Kimura, W. Takahashi, M. Toyoda, On a strongly nonexpansive sequence in Hilbert spaces, J. Nonlinear Convex Anal. 8 (2007) 471–489.

[10] C. Klin-eam, S. Suantai, Strong convergence of composite iterative schemes for a countable family of nonexpansive mappings in Banach spaces, Nonlinear Anal. 73 (2010) 431–439.

[11] K. Nakajo, K. Shimoji, W. Takahashi, Strong convergence to common fixed points of families of nonexpansive mappings in Banach spaces, J. Nonlinear Convex Anal. 8 (2007) 11–34.

[12] K. Nakajo, K. Shimoji, W. Takahashi, Strong convergence theorems of Halpern's type for families of nonexpansive mappings in Hilbert spaces, Thai J. Math. 7 (2009) 49–67.

[13] W. Nilsrakoo, S. Saejung, Weak and strong convergence theorems for countable Lipschitzian mappings and its applications, Nonlinear Anal. 69 (2008) 2695–2708.

[14] W. Nilsrakoo, S. Saejung, Weak convergence theorems for a countable family of Lipschitzian mappings, J. Comput. Appl. Math. 230 (2009) 451–462.

[15] S. Plubtieng, R. Wangkeeree, Strong convergence of modified Mann iterations for a countable family of nonexpansive mappings, Nonlinear Anal. 70 (2009) 3110–3118.

[16] S. Wang, C. Hu, Two new iterative methods for a countable family of nonexpansive mappings in Hilbert spaces, Fixed Point Theory Appl. 2010, Art. ID 852030, 12 pp.

[17] Y. L. Cui, X. Liu, Notes on Browder's and Halpern's methods for nonexpansive mappings, Fixed Point Theory 10 (2009) 89–98.

[18] Y. Yao, R. Chen, H. K. Xu, Schemes for finding minimum-norm solutions of variational inequalities, Nonlinear Anal. 72 (2010) 3447–3456.

[19] Y. Yao, Y. C. Liou, G. Marino, Strong convergence of two iterative algorithms for nonexpansive mappings in Hilbert spaces, Fixed Point Theory Appl. 2009, Art. ID 279058, 7 pp.

[20] X. Yang, Y. C. Liou, Y. Yao, Finding minimum norm fixed point of nonexpansive mappings and applications, Math. Probl. Eng. 2011, Art. ID 106450, 13 pp.

[21] Y. Yao, N. Shahzad, New methods with perturbations for non-expansive mappings in Hilbert spaces, Fixed Point Theory Appl. 2011, 2011:79, 9 pp.

[22] Y. Yao, H. K. Xu, Iterative methods for finding minimum-norm fixed points of nonexpansive mappings with applications, Optimization 60 (2011) 645–658.

[23] W. Takahashi, Weak and strong convergence theorems for families of nonexpansive mappings and their applications, Ann. Univ. Mariae Curie-Sklodowska Sect. A 51 (1997) 277–292.

[24] W. Takahashi, K. Shimoji, Convergence theorems for nonexpansive mappings and feasibility problems, Math. Comput. Modelling 32 (2000) 1463–1471.

[25] K. Shimoji, W. Takahashi, Strong convergence to common fixed points of infinite nonexpansive mappings and applications, Taiwanese J. Math. 5 (2001) 387–404.

[26] E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, Math. Student 63 (1994) 123–145.

[27] P. L. Combettes, S. A. Hirstoaga, Equilibrium programming in Hilbert spaces, J. Nonlinear Convex Anal. 6 (2005) 117–136.

[28] S. D. Flåm, A. S. Antipin, Equilibrium programming using proximal-like algorithms, Math. Programming Ser. A 78 (1997) 29–41.

[29] A. Tada, W. Takahashi, Weak and strong convergence theorems for a nonexpansive mapping and an equilibrium problem, J. Optim. Theory Appl. 133 (2007) 359–370.

[30] S. Takahashi, W. Takahashi, Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces, J. Math. Anal. Appl. 331 (2007) 506–515.

[31] S. Takahashi, W. Takahashi, Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space, Nonlinear Anal. 69 (2008) 1025–1033.

[32] S. Chaiyasil, S. Suantai, Approximation method for generalized mixed equilibrium problems and fixed point problems for a countable family of nonexpansive mappings, J. Nonlinear Anal. Optim. 2 (2011) 337–353.

[33] I. Inchan, Viscosity iteration method for generalized equilibrium problems and fixed point problems of finite family of nonexpansive mappings, Appl. Math. Comput. 219 (2012) 2949–2959.

[34] J. S. Jung, A general composite iterative method for equilibrium problems and fixed point problems, J. Comput. Anal. Appl. 12 (2010) 124–140.

[35] U. Kamraksa, R. Wangkeeree, Generalized equilibrium problems and fixed point problems for nonexpansive semigroups in Hilbert spaces, J. Global Optim. 51 (2011) 689–714.

[36] A. Kangtunyakarn, S. Suantai, A new mapping for finding common solutions of equilibrium problems and fixed point problems of finite family of nonexpansive mappings, Nonlinear Anal. 71 (2009) 4448–4460.

[37] A. Kangtunyakarn, S. Suantai, Hybrid iterative scheme for generalized equilibrium problems and fixed point problems of finite family of nonexpansive mappings, Nonlinear Anal. Hybrid Syst. 3 (2009) 296–309.

[38] N. Onjai-uea, C. Jaiboon, P. Kumam, A strong convergence theorem based on a relaxed extragradient method for generalized equilibrium problems, Appl. Math. Sci. (Ruse) 4 (2010) 691–708.

[39] A. Razani, M. Yazdi, A new iterative method for generalized equilibrium and fixed point problems of nonexpansive mappings, Bull. Malays. Math. Sci. Soc. (2) 35 (2012), no. 4, 1049–1061.

[40] S. Wang, SG. Marino, F. Wang, Strong convergence theorems for a generalized equilibrium problem with a relaxed monotone mapping and a countable family of nonexpansive mappings in a Hilbert space, Fixed Point Theory Appl. 2010, Art. ID 230304, 22 pp.

[41] R. Wangkeeree, T. Bantaojai, A general composite algorithms for solving general equilibrium problems and fixed point problems in Hilbert spaces, Thai J. Math. 9 (2011) 197–218.

[42] H. Zhang, Y. Su, Strong convergence theorem for generalized equilibrium problems and countable family of nonexpansive mappings, Int. J. Optim. Theory Methods Appl. 1 (2009) 87–101.

[43] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, Bull. Amer. Math. Soc. 73 (1967) 591–597.

[44] W. Takahashi, Nonlinear functional analysis. Fixed point theory and its applications. Yokohama Publishers, Yokohama, 2000.

[45] H. K. Xu, Another control condition in an iterative method for nonexpansive mappings, Bull. Austral. Math. Soc. 65 (2002) 109–113.

[46] S. Saejung, P. Yotkaew, Approximation of zeros of inverse strongly monotone operators in Banach spaces. Nonlinear Anal. 75 (2012) 742–750.

[47] K. Nakajo, K. Shimoji, W. Takahashi, Strong convergence theorems by the hybrid method for families of nonexpansive mappings in Hilbert spaces, Taiwanese J. Math. 10 (2006) 339–360.

[48] W. Nilsrakoo, S. Saejung, Strong convergence theorems for a countable family of quasi-Lipschitzian mappings and its applications, J. Math. Anal. Appl. 356 (2009) 154167.

[49] S. Kamimura, W. Takahashi, Approximating solutions of maximal monotone operators in Hilbert spaces, J. Approx. Theory 106 (2000) 226–240.

[50] S. Saejung, A supplement to a regularization method for the proximal point algorithm, J. Global Optim. 56 (2013) 121–129.

[51] Y. Song, C. Yang, A note on a paper "A regularization method for the proximal point algorithm" [J. Global Optim. 36 (2006) 115–125] by H. K. Xu., J. Global Optim. 43 (2009) 171–174.

[52] H. K. Xu, A regularization method for the proximal point algorithm, J. Global Optim. 36 (2006) 115–125.

[53] D. P. Bertsekas, J.N. Tsitsiklis, Parallel and Distributed Computation: Numerical Methods, Prentice-Hall, Englwood Cliffs (1989)

[54] Y. Censor, A. Gibali, S. Reich, S. Sabach, Common solutions to variational inequalities, Set-Valued Var. Anal. 20 (2012) 229–247.

[55] Y. Censor, T. Elfving, A multiprojection algorithm using Bregman projections in a product space, Numer. Algorithms 8 (1994) 221–239.

[56] C. Byrne, Iterative oblique projection onto convex sets and the split feasibility problem, Inverse Problems 18 (2002) 441–453.

[57] C. Byrne, A unified treatment of some iterative algorithms in signal processing and image reconstruction, Inverse Problems 20 (2004) 103–120.

[58] F. Wang, H. K. Xu, Approximating curve and strong convergence of the CQ algorithm for the split feasibility problem. J. Inequal. Appl. 2010, Art. ID 102085, 13 pp.

[59] H. K. Xu, A variable Krasnosel'skii-Mann algorithm and the multiple-set split feasibility problem. Inverse Problems 22 (2006) 2021–2034.

[60] Y.Censor, T. Elfving, N. Kopf, T. Bortfeld, The multiple-sets split feasibility problem and its applications for inverse problems. Inverse Problems 21 (2005) 2071–2084.

[61] H. He, S. Liu, M. A. Noor, Some Krasnonsel'skii-Mann algorithms and the multiple-set split feasibility problem. Fixed Point Theory Appl. 2010, Art. ID 513956, 12 pp.

[62] M. Eslamian, A. Latif, General split feasibility problems in Hilbert spaces. Abst. Appl. Anal. 2013, Art. ID 805104, 6 pp.

(Received 10 January 2014)
(Accepted 10 April 2014)