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The k-star Property for Permutation Groups

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Abstract : For an integer k at least 2, a permutation group G has the *k-star property* if, for every k-subset of points, G contains an element that fixes it setwise but not pointwise. This property holds for all k-transitive, generously k-transitive, and almost generously k-transitive permutation groups. Study of the k-star property was motivated by recent work on the case $k = 3$ by P. M. Neumann and the second author. The paper focuses on intransitive groups with the k-star property, studying properties of their transitive constituents, and relationships between the k-star and m-star properies for $k \neq m$. Several open problems are posed.

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1 Introduction

For an integer k at least 2, a permutation group G acting on a set Ω will be said to be a *k-star group* if it has the following property, called the *k-star property* :

> *for every k-subset Θ of Ω the permutation group G^Θ induced on Θ by its setwise stabiliser G_Θ in G is non-trivial.*

This condition may be regarded as a measure of the transitivity of the group G although, as we will see, when $k \geq 3$ some k-star groups are intransitive. Since a k-transitive group G on Ω is, by definition, transitive on ordered k-tuples of distinct points, for such groups we have $G^\Theta = \mathrm{Sym}(\Theta)$ for each k-subset Θ of Ω, and hence every k-transitive group is a k-star group. The k-star property is also a generalisation of the notions of generous and almost generous transitivity studied by Peter Neumann in [2], and the 3-star property was studied by Neumann and the second author in [3]. In contrast with [3], we will allow Ω to contain less than k points. All such groups vacuously satisfy the condition and hence are k-star groups. We will say that a k-star group G on Ω is *trivial* if either G is k-transitive or $|\Omega| < k$.

Let $G \leq \mathrm{Sym}(\Omega)$ with $|\Omega| > k \geq 1$. Then, as defined in [2], G is *generously k-transitive* if $G^\Theta = \mathrm{Sym}(\Theta)$ for all $(k+1)$-subsets Θ of Ω and *almost generously k-transitive* if $G^\Theta \geq \mathrm{Alt}(\Theta)$ for all $(k+1)$-subsets Θ of Ω. Thus the generously 1-transitive groups (usually called simply generously transitive) are precisely the

2-star groups. They are those transitive groups in which all suborbits are self-paired (see, for example [1, Section 3.2]). Also each almost generously 2-transitive group is a 3-star group. It was shown in [2] that an almost generously 2-transitive group is in fact 2-transitive, while in [3] it was proved that finite primitive 3-star groups have rank at most 3, and that there are infinite primitive 3-star groups of arbitrary rank. (The rank of a transitive permutation group is the number of its orbits on ordered pairs of not necessarily distinct points.)

In this paper we will focus on intransitive k-star groups, and possible relationships between the k-star and m-star properties for $k \neq m$. We begin with a few observations.

Remark 1.1

1. For any positive integers k, ℓ, n such that $\ell < k \leq n$, there exists a k-star permutation group on a set of n points having ℓ orbits. See Example 2.1.
2. Moreover, for an arbitrary k-star group G, if Ω' is any non-empty union of G-orbits, then the permutation group induced by G on Ω' is also a k-star group. This follows immediately from the definition of a k-star group.

Intuitively the property of being a k-star group may seem to be stronger than that for a $(k-1)$-star group. Indeed, trivial k-star groups are also $(k-1)$-star groups, and this is true also for some other families of k-star groups as shown in Theorem 1.3 below. However this is not always true for non-trivial k-star groups. We make several observations about the possible relationships between the k-star and m-star properties for unequal k, m in the next proposition.

Proposition 1.2 *Let k, m be integers such that $2 \leq m < k$.*

(a) There exists a non-trivial k-star group that is not an m-star group.

(b) If $m \leq 5 < k$, then there exists a non-trivial m-star group that is not a k-star group.

(c) If either $m \geq 3$ or $k \leq 5$, then there exists a non-trivial k-star group that is also a non-trivial m-star group.

In [3] it was observed that a 3-star group can have at most two orbits, and a similar restriction holds for k-star groups in general.

Theorem 1.3 *Let k, ℓ be positive integers with $k \geq 2$. Suppose that $G \leq \mathrm{Sym}(\Omega)$ and that G is a k-star group with orbits $\Omega_1, \ldots, \Omega_\ell$. Then*

(a) $\ell \leq k-1$, and there are examples for each k, ℓ;

(b) for each $i \leq \ell$ and each integer $m \in [k+1-\ell, k]$, the group G^{Ω_i} is an m-star group;

(c) the number of integers $i \leq \ell$ such that G^{Ω_i} is not a 2-star group is at most $\min\{\lfloor \frac{k-1}{2} \rfloor, k-\ell-1\}$, and this bound can be attained for each $k \geq 2$.

Examples demonstrating existence in Theorem 1.3 (a), and attainment of the bound in part (c), are given in Examples 2.1 and 2.2 respectively. The proof of the theorem is in Section 3. Part (c) suggests the following problem.

Problem 1.4 Given $\ell, m \geq 2$ and $\ell, m < k$, find a function $f(k, \ell, m)$ such that, for each k-star group G with orbits $\Omega_1, \ldots, \Omega_\ell$, all but at most $f(k, \ell, m)$ of the induced groups G^{Ω_i} are m-star groups, and there exists such a group for which the bound $f(k, \ell, m)$ is attained.

By Theorem 1.3, $f(k, \ell, 2) = \min\{\lfloor \frac{k-1}{2} \rfloor, k - \ell - 1\}$. The proof of Proposition 1.2 is given in Section 3. The examples given to prove Proposition 1.2 (a), and to prove part (c) for $m \geq 3$, are all intransitive. Also those given to prove part (c) with $m = 2$ are transitive, but are not 6-star groups. We ask the following.

Question 1.5

(a) For which $k \geq 6$, does there exist a non-trivial k-star group that is also a 2-star group?

(b) For which integers m, k with $2 \leq m < k$, does there exist a transitive k-star group that is also a non-trivial m-star group?

Finally, although we have not studied the primitive case in this paper, it is of special interest, given the main result of [3] that a finite primitive 3-star group has rank at most 3. In [3, Final Note], it was observed that all the infinite permutation groups in the family constructed in the proof of [3, Observation 3.8] were k-star groups for each finite k, and the family contained groups of all ranks. Thus we ask the following:

Question 1.6 Is there an upper bound, depending on k, on the rank of a finite primitive k-star group?

2 Examples

In this section we give several families of examples that will be used in the proofs of Proposition 1.2 and Theorem 1.3.

Example 2.1 Let k, ℓ, n be positive integers such that $\ell < k \leq n$, let Ω be a set of n points, and let $\{\Omega_1 \mid \Omega_2 \mid \cdots \mid \Omega_\ell\}$ be a partition of Ω with ℓ non-empty parts. Let G denote the stabiliser in $\mathrm{Sym}(\Omega)$ of this partition, that is, $G = \mathrm{Sym}(\Omega_1) \times \cdots \times \mathrm{Sym}(\Omega_\ell)$. Since $\ell < k$, each k-subset Δ of Ω contains at least two points, say α and β, in the same part Ω_i, for some i, and the 2-cycle (α, β) lies in G and fixes Δ setwise. Thus G is a k-star permutation group on Ω, and is non-trivial if $\ell \geq 2$.

Example 2.2 Let $k \geq 2$, and set $m := \lfloor \frac{k}{2} \rfloor - 1$. For $i = 1, \ldots, m$, let $\Omega_i := \{3i-2, 3i-1, 3i\}$, let $\Omega_{m+1} := \{3m+1, 3m+2\}$ and $\Omega_{m+2} := \{3m+3\}$. Let

$$\ell := \begin{cases} m+1 & \text{if } k \text{ is even} \\ m+2 & \text{if } k \text{ is odd} \end{cases} \quad \text{and note that} \quad k = \begin{cases} 2m+2 & \text{if } k \text{ is even} \\ 2m+3 & \text{if } k \text{ is odd.} \end{cases}$$

Let $\Omega := \bigcup_{i=1}^{\ell} \Omega_i$ and $G := (\prod_{i=1}^{m} \mathrm{Alt}(\Omega_i)) \times (\prod_{i=m+1}^{\ell} \mathrm{Sym}(\Omega_i))$. For each k-subset $\Delta \subset \Omega$, either Δ contains Ω_i for some $i \leq m$, or Δ contains Ω_{m+1}. In either case, $G^{\Delta} \neq 1$, and hence G is a k-star group.

Moreover, G^{Ω_i} is a 2-star group if and only if $m < i \leq \ell$, so that G^{Ω_i} fails to be a 2-star group for exactly m of the G-orbits Ω_i. Note that $m = \min\{\lfloor \frac{k-1}{2} \rfloor, k - \ell - 1\}$, the upper bound of Theorem 1.3 (c).

Example 2.3 Let $k \geq 2$ and consider the symmetric group $G = \mathrm{Sym}(\Omega_0)$ acting on the set $\Omega = \Omega_0^{\{2\}}$ of unordered pairs from Ω_0. For simplicity take $|\Omega_0| \geq k$. Each k-subset Θ of Ω can be thought of as the edge set of a graph with vertex set Ω_0, and a non-trivial element of G^{Θ} as an automorphism of the graph permuting the edges non-trivially.

Moreover, it is straightforward to check that every graph with at most 5 edges possesses an automorphism that moves some edge, so that G is a non-trivial k-star group if $k \leq 5$. This was observed in [3, Example 3.4 and Final Note]. However if $k \geq 6$, then there is a graph with n vertices and k edges for which no automorphism permutes the edges non-trivially, so G is not a k-star group if $k \geq 6$.

To complete the argument, we give a simple example, suggested to us by Gábor Ivanyos, of a graph $\Gamma(n,k)$ that has exactly n vertices and k edges, where $6 \leq k < n$, and no automorphism permutes the edges non-trivially. Take the vertex set of $\Gamma(n,k)$ as the set $V := \{1, 2, \ldots, n\}$, and define the k edges to be the pairs $\{i, i+1\}$, for $1 \leq i \leq k-1$, and $\{3, k+1\}$. Let g be an automorphism of $\Gamma(n,k)$, that is, $g \in \mathrm{Sym}(V)$ and maps edges to edges. Since the only vertex lying on three edges is 3, g fixes the vertex 3, and since the three paths in $\Gamma(n,k)$ starting at 3 have different lengths $1, 2, k-3$, it follows that g fixes each of the vertices $1, 2, \ldots, k+1$, and hence g fixes each of the edges.

3 Proofs

Proof of Proposition 1.2. (a) Consider a k-star group $\hat{G}$ on a set $\hat{\Omega}$, constructed as in Example 2.1, having $k-1$ orbits $\Omega_1, \ldots, \Omega_{k-1}$, and assume that $|\Omega_1| > k$. Let $\Omega := \bigcup_{i=1}^{m} \Omega_i$ and let G be the permutation group induced by $\hat{G}$ on Ω. By Remark 1.1.2, G on Ω is a k-star group, and since $|\Omega_1| > k$ and G has $m \geq 2$ orbits, G is a non-trivial k-star group. Also, since G has $m \geq 2$ orbits, it follows from Theorem 1.3 (a) that G is not an m-star group.

(b) This part follows from Example 2.3.

(c) If $m = 2$ and $k \leq 5$, then the groups given in Example 2.3 have the required properties, and further examples with $m = 2, k = 3$ are given in [3, Examples 3.5

and 3.6]. Thus we may assume that $3 \leq m < k$. Consider the group $\hat{G}$ defined in the proof of part (a). This time let $\Omega := \bigcup_{i=1}^{m-1} \Omega_i$ and let G be the permutation group induced by $\hat{G}$ on Ω. As in part (a), G is a non-trivial k-star group. Let Θ be an m-subset of Ω. For $m \leq i \leq k-1$ choose a point $\alpha_i \in \Omega_i$, and set $\hat{\Theta} := \Theta \cup \{\alpha_m, \ldots, \alpha_{k-1}\}$. Then $|\hat{\Theta}| = k$, and as $\hat{G}$ is a k-star group, there exists $\hat{g} \in \hat{G}$ such that $\hat{g}$ fixes $\hat{\Theta}$ setwise and acts non-trivially on it. Now $\hat{g}$ fixes $\Omega_i \cap \hat{\Theta} = \{\alpha_i\}$ for $m \leq i \leq k-1$, and hence $\hat{g}$ fixes Θ setwise and acts non-trivially on it. Thus $G^{\Theta} \neq 1$. It follows that G is a non-trivial m-star group. □

Proof of Theorem 1.3. (a) If $\ell \geq k$, then a k-subset $\Delta \subset \Omega$ consisting of one point from each of $\Omega_1, \ldots, \Omega_k$ is such that $G^{\Delta} = 1$, contradicting the fact that G is a k-star group. Thus $\ell \leq k-1$.

(b) Choose $i \leq \ell$ and $m \in [k+1-\ell, k]$. If $|\Omega_i| < m$ then G^{Ω_i} is a trivial m-star group. So assume that $|\Omega_i| \geq m$ and consider an arbitrary m-subset $\Delta' \subseteq \Omega_i$. Since $k - m \leq \ell - 1$, we may form a k-subset Δ of Ω consisting of Δ' together with one point from each of $k-m$ orbits Ω_j different from Ω_i. Since G is a k-star group, we have $G^{\Delta} \neq 1$, and since the setwise stabiliser G_{Δ} fixes each $\Omega_j \cap \Delta$ setwise, it follows that G_{Δ} fixes $\Delta \setminus \Delta'$ pointwise, and hence $G_{\Delta} \leq G_{\Delta'}$ and acts non-trivially on Δ'. In particular $G^{\Delta'} \neq 1$, and so G is an m-star group.

(c) If $k = 2$ then by the second observation in the introduction each G^{Ω_i} is a 2-star group. Thus we may assume that $k \geq 3$. Let m be the number of G-orbits Ω_i such that G^{Ω_i} is not a 2-star group. If $m \leq 1$ then the bound of part (c) holds since $\ell < k$ and $k \geq 3$. So assume that $m \geq 2$, and without loss of generality assume that G^{Ω_i} is not a 2-star group for $1 \leq i \leq m$. Then, for each $i \leq m$, there exist two points $\alpha_i, \beta_i \in \Omega_i$ such that no element of G interchanges α_i and β_i. For $m < i \leq \ell$ choose a point $\alpha_i \in \Omega_i$. If $m \geq \frac{k}{2}$, set $\Delta' := \{\alpha_i, \beta_i \,|\, 1 \leq i \leq \lfloor \frac{k}{2} \rfloor\}$. Consider the k-subset $\Delta := \Delta'$ if k is even, or $\Delta' \cup \{\alpha_\ell\}$ if k is odd. The stabiliser G_{Δ} fixes setwise $\Omega_i \cap \Delta$ for each i. If $i \leq m$ then $\Omega_i \cap \Delta = \{\alpha_i, \beta_i\}$, and as no element of G interchanges α_i and β_i it follows that G_{Δ} fixes α_i and β_i. Thus $G^{\Delta} = 1$, which is a contradiction, so $m < k/2$ and hence $m \leq \lfloor \frac{k-1}{2} \rfloor$. Next suppose that $m \geq k - \ell$, and set $\Delta := \{\alpha_i, \beta_i \,|\, 1 \leq i \leq m\} \cup \{\alpha_{m+1}, \ldots, \alpha_{k-m}\}$. Then Δ is a k-subset, and arguing as in the previous case, $G^{\Delta} = 1$, which is a contradiction. So $m \leq k - \ell - 1$. Thus we have proved that $m \leq \min\{\lfloor \frac{k-1}{2} \rfloor, k - \ell - 1\}$, and groups for which this upper bound is attained, for any given $k \geq 2$, can be found in Example 2.2. □

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