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On a Quarter-Symmetric Metric Connection in an LP-Sasakian Manifold

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Abstract : The object of the present paper is to study a quarter-symmetric metric connection in an LP-Sasakian manifold. We study some curvature properties of an LP-Sasakian manifold with respect to the quarter-symmetric metric connection.

Keywords : quarter-symmetric metric connection; LP-Sasakain manifold; locally ϕ-symmetric; ϕ-recurrent; locally projective ϕ-symmetric; ϕ-projectively flat; η-Einstein manifold.
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1 Introduction

The quarter-symmetric connection generalizes the semi-symmetric connection. The semi-symmetric metric connection is important in the geometry of Riemannian manifolds having also physical application; for instance, the displacement on the earth surface following a fixed point is metric and semi-symmetric ([1]).

In 1975, Golab ([2]) defined and studied quarter-symmetric connection in a differentiable manifold.

A linear connection $\tilde{\nabla}$ on an n-dimensional Riemannian manifold (M^n, g) is said to be a quarter-symmetric connection ([2]) if its torsion tensor $\tilde{T}$ defined by

$$\tilde{T}(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y],$$

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is of the form

$$\tilde{T}(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y, \tag{1.1}$$

where η is 1-form and ϕ is a tensor of type (1, 1). In addition, a quarter-symmetric linear connection $\tilde{\nabla}$ satisfies the condition

$$(\tilde{\nabla}_X g)(Y,Z) = 0 \tag{1.2}$$

for all X, Y, Z $\in$ TM, where TM is the Lie algebra of vector fields of the manifold M^n, then $\tilde{\nabla}$ is said to be quarter-symmetric metric connection. In particular, if $\phi X = X$ and $\phi Y = Y$, then the quarter-symmetric connection reduces to a semi-symmetric connection ([3]).

After Golab ([2]), Rastogi ([4, 5]) continued the systematic study of quarter-symmetric metric connection. In 1980, Mishra and Pandey ([6]) studied quarter-symmetric metric connection in a Riemannian, Kaehlerian and Sasakian manifold. In 1982, Yano and Imai ([7]) studied quarter-symmetric metric connection in Hermition and Kaehlerian manifolds. In 1991, Mukhopadhyay et al. ([8]) studied quarter-symmetric metric connection on a Riemannian manifold with an almost complex structure ϕ. Quarter-symmetric metric connection are also studied by De and Biswas ([9]), Singh ([10]), De and Mondal ([11]), De and De ([12]) and many others.

On the other hand, there is a class of almost paracontact metric manifolds, namely Lorentzian para-Sasakian manifolds. In 1989, Matsumoto ([13]) introduced the notion of LP-Sasakian manifolds. Then, Mihai and Rosca ([14]) introduced the same notion independently and obtained many interesting results. LP-Sasakian manifolds are also studied by De et al. ([15]), Mihai et al. ([16]), Saikh and Baishya ([17]), Singh et al. ([18]) and others. The paper is organized as follows

In this paper, we study a quarter-symmetric metric connection in an Lorentzian para-Sasakian manifold. In Section 2, some preliminary results are recalled. In Section 3, we find the expression for curvature tensor (resp. Ricci tensor) with respect to quarter-symmetric metric connection and investigate relations between curvature tensor (resp. Ricci tensor) with respect to quarter-symmetric metric connection and curvature tensor (resp. Ricci tensor) with respect to Levi-Civita connection. Section 4 deals with locally ϕ-symmetric LP-Sasakian manifold with respect to quarter-symmetric metric connection. ϕ-recurrent LP-Sasakian manifold admitting quarter-symmetric metric connection are studied in Section 5 and it is obtained that if an LP-Sasakian manifold is ϕ-recurrent with respect to quarter-symmetric metric connection then (M^n, g) is an η-Einstein manifold with respect to Levi-Civita connection. Section 6 contains locally projective ϕ-symmetric LP-Sasakian manifold with respect to quarter-symmetric metric connection. Section 7 is devoted to study of ϕ-projectively flat LP-Sasakian manifold with respect to quarter-symmetric metric connection. In the last section, we study $R.\tilde{R} = 0$ and obtained (M^n, g) is an η-Einstein manifold.

2 Preliminaries

An n-dimensional, $(n = 2m+1)$, differentiable manifold M^n is called Lorentzian para-Sasakian (briefly, LP-Sasakian) manifold ([13, 19]), if it admits a (1, 1)-tensor field ϕ, a contravariant vector field ξ, a 1-form η and a Lorentzian metric g which satisfy

$$\eta(\xi) = -1, \tag{2.1}$$

$$\phi^2 X = X + \eta(X)\xi, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.3}$$

$$g(X, \xi) = \eta(X), \tag{2.4}$$

$$\nabla_X \xi = \phi X, \tag{2.5}$$

$$(\nabla_X \phi)(Y) = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.6}$$

where ∇ denotes the covariant differentiation with respect to Lorentzian metric g.

It can be easily seen that in an LP-Sasakian manifold the following relations hold:

$$\phi\xi = 0, \eta(\phi) = 0, \tag{2.7}$$

$$rank(\phi) = n - 1. \tag{2.8}$$

If we put

$$\Phi(X, Y) = g(X, \phi Y) \tag{2.9}$$

for any vector field X and Y, then the tensor field $\Phi(X, Y)$ is a symmetric (0, 2)-tensor field ([13]). Also since the 1-form η is closed in an LP-Sasakian manifold, we have ([13, 15])

$$(\nabla_X \eta)(Y) = \Phi(X, Y), \Phi(X, \xi) = 0, \tag{2.10}$$

for all X, Y $\in$ TM.

Also in an LP-Sasakian manifold, the following relations hold ([15, 19])

$$g(R(X, Y)Z, \xi) = \eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \tag{2.11}$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.12}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.13}$$

$$R(\xi, X)\xi = X + \eta(X)\xi, \tag{2.14}$$

$$S(X, \xi) = (n - 1)\eta(X), \tag{2.15}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n - 1)\eta(X)\eta(Y) \tag{2.16}$$

for any vector field X, Y and Z, where R and S are the Riemannian curvature tensor and Ricci tensor of the manifold respectively.

3 Curvature Tensor of an LP-Sasakian Manifold with respect to Quarter-Symmetric Metric Connection

Let $\tilde{\nabla}$ be the linear connection and ∇ be Riemannian connection of an almost contact metric manifold such that

$$\tilde{\nabla}_X Y = \nabla_X Y + H(X,Y), \tag{3.1}$$

where H is the tensor field of type (1, 1). For $\tilde{\nabla}$ to be a quarter-symmetric metric connection in M^n, we have ([2])

$$H(X,Y) = \frac{1}{2}[\tilde{T}(X,Y) + \tilde{T}'(X,Y) + \tilde{T}'(Y,X)] \tag{3.2}$$

and

$$g(\tilde{T}'(X,Y),Z) = g(\tilde{T}(Z,X),Y). \tag{3.3}$$

In view of equations (1.1) and (3.3), we have

$$\tilde{T}'(X,Y) = \eta(X)\phi Y - g(\phi X,Y)\xi. \tag{3.4}$$

Now, using equations (1.1) and (3.4) in equation (3.2), we get

$$H(X,Y) = \eta(Y)\phi X - g(\phi X,Y)\xi. \tag{3.5}$$

Hence a quarter-symmetric metric connection in an LP-Sasakian manifold is given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X - g(\phi X,Y)\xi. \tag{3.6}$$

Thus the above equation is the relation between quarter-symmetric metric connection and the Levi-Civita connection.

The curvature tensor $\tilde{R}$ of M^n with respect to quarter-symmetric metric connection $\tilde{\nabla}$ is defined by

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X\tilde{\nabla}_Y Z - \tilde{\nabla}_Y\tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]}Z. \tag{3.7}$$

In view of equation (3.6), above equation takes the form

$$\begin{aligned}\tilde{R}(X,Y)Z = {}& R(X,Y)Z + g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X \\ & + \eta(Z)\{\eta(Y)X - \eta(X)Y\} + \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi,\end{aligned} \tag{3.8}$$

where $\tilde{R}$ and R are the Riemannian curvature tensor with respect to $\tilde{\nabla}$ and ∇ respectively.
From equation (3.8) it follows that

$$\tilde{S}(Y,Z) = S(Y,Z) + (n-1)\eta(Y)\eta(Z), \tag{3.9}$$

where $\tilde{S}$ and S are the Ricci tensor of the connection $\tilde{\nabla}$ and ∇ respectively. Contracting above equation, we get

$$\tilde{r} = r - (n-1), \tag{3.10}$$

where $\tilde{r}$ and r are the scalar curvature tensor of the connection $\tilde{\nabla}$ and ∇ respectively.

4 Locally ϕ-Symmetric LP-Sasakian Manifold with respect to Quarter-Symmetric Metric Connection

Definition 4.1. An LP-Sasakian manifold M^n is said to be locally ϕ-symmetric if

$$\phi^2((\nabla_W R)(X,Y)Z) = 0, \tag{4.1}$$

for all vector fields X, Y, Z, W orthogonal to ξ. This notion was introduced by Takahashi for Sasakian manifolds ([20]).

Definition 4.2. An LP-Sasakian manifold M^n is said to be ϕ-symmetric if

$$\phi^2((\nabla_W R)(X,Y)Z) = 0, \tag{4.2}$$

for arbitrary vector fields X, Y, Z, W.

Analogous to the definition of locally ϕ-symmetric LP-Sasakian manifolds with respect to Levi-Civita connection, we define a locally ϕ-symmetric LP-Sasakian manifolds with respect to the quarter-symmetric metric connection by

$$\phi^2((\tilde{\nabla}_W \tilde{R})(X,Y)Z) = 0, \tag{4.3}$$

for all vector fields X, Y, Z, W orthogonal to ξ.
In view of equations (3.6) and (3.8), we have

$$\begin{aligned}(\tilde{\nabla}_W \tilde{R})(X,Y)Z = (\nabla_W \tilde{R})(X,Y)Z + \eta(\tilde{R}(X,Y)Z)\phi W \\ - g(\phi W, \tilde{R}(X,Y)Z)\xi.\end{aligned} \tag{4.4}$$

Now differentiating equation (3.8) covariantly with respect to W, we get

$$\begin{aligned}&(\nabla_W \tilde{R})(X,Y)Z \\ &\quad = (\nabla_W R)(X,Y)Z + g((\nabla_W \phi)X, Z)\phi Y + g(\phi X, Z)(\nabla_W \phi)(Y) \\ &\qquad - g((\nabla_W \phi)Y, Z)\phi X - g(\phi Y, Z)(\nabla_W \phi)(X) + (\nabla_W \eta)(Z)\{\eta(Y)X \\ &\qquad - \eta(X)Y\} + \eta(Z)\{(\nabla_W \eta)(Y)X - (\nabla_W \eta)(X)Y\} + \{g(Y,Z)\eta(X) \\ &\qquad - g(X,Z)\eta(Y)\}(\nabla_W \xi) + \{g(Y,Z)(\nabla_W \eta)(X) - g(X,Z)(\nabla_W \eta)(Y)\}\xi,\end{aligned} \tag{4.5}$$

which on using equations (2.6), (2.9) and (2.10) reduces to

$$\begin{aligned}&(\nabla_W \tilde{R})(X,Y)Z \\ &\quad = (\nabla_W R)(X,Y)Z - \{g(Y,W)\eta(Z) + g(Z,W)\eta(Y) \\ &\qquad + 2\eta(Y)\eta(W)\eta(Z)\}\phi X + \{g(X,W)\eta(Z) + g(Z,W)\eta(X) \\ &\qquad + 2\eta(X)\eta(W)\eta(Z)\}\phi Y + \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\phi W \\ &\qquad + \{\eta(Y)g(\phi X, Z) - \eta(X)g(\phi Y, Z)\}W + \{g(W,\phi Z)\eta(Y) \\ &\qquad + g(W,\phi Y)\eta(Z)\}X - \{g(W,\phi Z)\eta(X) + g(W,\phi X)\eta(Z)\}Y \\ &\qquad + \{g(\phi X, Z)g(Y,W) - g(\phi Y, Z)g(X,W) + 2\eta(Y)\eta(W)g(\phi X, Z) \\ &\qquad - 2\eta(X)\eta(W)g(\phi Y, Z) + g(\phi X, W)g(Y,Z) - g(\phi Y, W)g(X,Z)\}\xi.\end{aligned} \tag{4.6}$$

Now taking the inner product of equation (3.8) with ξ and using equations (2.1), (2.7) and (2.11), we obtain

$$\eta(\tilde{R}(X,Y)Z) = 0. \tag{4.7}$$

By virtue of equations (4.6), (4.7) and (2.7), equation (4.4) takes the form

$$\begin{aligned}
&\phi^2((\tilde{\nabla}_W \tilde{R})(X,Y)Z) \\
&\quad = \phi^2((\nabla_W R)(X,Y)Z) + \{g(W,\phi Z)\eta(Y) + g(W,\phi Y)\eta(Z)\}\phi^2 X \\
&\qquad - \{g(W,\phi Z)\eta(X) + g(W,\phi X)\eta(Z)\}\phi^2 Y + \{\eta(Y)g(\phi X,Z) \\
&\qquad - \eta(X)g(\phi Y,Z)\}\phi^2 W - \{g(Y,W)\eta(Z) + g(Z,W)\eta(Y) \\
&\qquad + 2\eta(Y)\eta(W)\eta(Z)\}\phi^2(\phi X) + \{g(X,W)\eta(Z) + g(Z,W)\eta(X) \\
&\qquad + 2\eta(X)\eta(W)\eta(Z)\}\phi^2(\phi Y) + \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\phi^2(\phi W).
\end{aligned} \tag{4.8}$$

Consider X, Y, Z and W are orthogonal to ξ, then equation (4.8) yields

$$\phi^2((\tilde{\nabla}_W \tilde{R})(X,Y)Z) = \phi^2((\nabla_W R)(X,Y)Z). \tag{4.9}$$

Hence we can state the following

Theorem 4.3. *In an LP-Sasakian manifold the quarter-symmetric metric connection $\tilde{\nabla}$ is locally ϕ-symmetric iff the Levi-Civita connection is so.*

5 ϕ-Recurrent LP-Sasakian Manifold with respect to Quarter-Symmetric Metric Connection

Definition 5.1. An n-dimensional LP-Sasakian manifold M^n is said to be ϕ-recurrent if there exists a non-zero 1-form A such that

$$\phi^2((\nabla_W R)(X,Y)Z) = A(W)R(X,Y)Z, \tag{5.1}$$

for arbitrary vector fields X, Y, Z, W.

If X, Y, Z, W are orthogonal to ξ then the manifold is called locally ϕ-recurrent manifold.

If the 1-form A vanishes, then the manifold is reduces to ϕ-symmetric manifold ([20]).

Definition 5.2. An n-dimensional LP-Sasakian manifold M^n is said to be ϕ-recurrent with respect to quarter-symmetric metric connection if there exists a non-zero 1-form A such that

$$\phi^2((\tilde{\nabla}_W \tilde{R})(X,Y)Z) = A(W)\tilde{R}(X,Y)Z, \tag{5.2}$$

for arbitrary vector fields X, Y, Z, W.

Suppose M^n is ϕ-recurrent with respect to quarter-symmetric metric connection, then in view of equations (2.2) and (5.2), we can write

$$g((\tilde{\nabla}_W \tilde{R})(X,Y)Z,U) + \eta((\tilde{\nabla}_W \tilde{R})(X,Y)Z)\eta(U) = A(W)g(\tilde{R}(X,Y)Z,U). \quad (5.3)$$

By virtue of equations (4.4) and (4.7) above equation reduces to

$$g((\nabla_W \tilde{R})(X,Y)Z,U) + \eta((\nabla_W \tilde{R})(X,Y)Z)\eta(U) = A(W)g(\tilde{R}(X,Y)Z,U), \quad (5.4)$$

which on using equation (4.6) takes the form

$$\begin{aligned}
&g((\nabla_W R)(X,Y)Z,U) + \{g(X,W)\eta(Z) + g(Z,W)\eta(X)\\
&+ 2\eta(X)\eta(W)\eta(Z)\}g(\phi Y,U) - \{g(Y,W)\eta(Z) + g(Z,W)\eta(Y)\\
&+ 2\eta(Y)\eta(W)\eta(Z)\}g(\phi X,U) + \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}g(\phi W,U)\\
&+ \{\eta(Y)g(\phi X,Z) - \eta(X)g(\phi Y,Z)\}g(W,U) + \{g(W,\phi Z)\eta(Y)\\
&+ g(W,\phi Y)\eta(Z)\}g(X,U) - \{g(W,\phi Z)\eta(X) + g(W,\phi X)\eta(Z)\}g(Y,U)\\
&+ \eta((\nabla_W R)(X,Y)Z)\eta(U) + \{\eta(Y)\eta(W)\eta(U)g(\phi X,Z)\\
&- \eta(X)\eta(W)\eta(U)g(\phi Y,Z) + \eta(X)\eta(Z)\eta(U)g(\phi Y,W)\\
&- \eta(Y)\eta(Z)\eta(U)g(\phi X,W)\}\\
&= A(W)g(R(X,Y)Z,U) + A(W)\{g(\phi X,Z)g(\phi Y,U) - g(\phi Y,Z)g(\phi X,U)\\
&\quad + \eta(Z)(\eta(Y)g(X,U) - \eta(X)g(Y,U)) + \eta(U)(g(Y,Z)\eta(X)\\
&\quad - g(X,Z)\eta(Y))\}.
\end{aligned} \quad (5.5)$$

Putting $Z = \xi$ in above equation and using equations (2.1) and (2.7), we get

$$\begin{aligned}
&g((\nabla_W R)(X,Y)\xi,U) - \{g(X,W)g(\phi Y,U) + \eta(X)\eta(W)g(\phi Y,U)\}\\
&+ \{g(Y,W)g(\phi X,U) + \eta(Y)\eta(W)g(\phi X,U)\} + \{g(Y,Z)\eta(X)\\
&- g(X,Z)\eta(Y)\}g(\phi W,U) + \{\eta(Y)g(\phi X,Z) - \eta(X)g(\phi Y,Z)\}g(W,U)\\
&+ \{g(W,\phi X)g(Y,U) - g(W,\phi Y)g(X,U)\} + \eta((\nabla_W R)(X,Y)\xi)\eta(U)\\
&+ \{\eta(Y)\eta(U)g(\phi X,W) - \eta(X)\eta(U)g(\phi Y,W)\}\\
&= A(W)g(R(X,Y)\xi,U) - A(W)\{(\eta(Y)g(X,U) - \eta(X)g(Y,U)).
\end{aligned} \quad (5.6)$$

Now, putting $X = U = e_i$ in above equation and taking summation over i, $1 \le i \le n$, we get

$$\begin{aligned}
(\nabla_W S)(Y,\xi) &+ \sum_{i=1}^{n} g((\nabla_W R)(e_i,Y)\xi,\xi)g(e_i,\xi)\\
&= A(W)S(Y,\xi) + ng(\phi Y,W) - (n-1)A(W)\eta(Y).
\end{aligned} \quad (5.7)$$

Let us denote the second term of left hand side of equation (5.7) by E. In this case E vanishes. Namely, we have

$$\begin{aligned}
g((\nabla_W R)(e_i,Y)\xi,\xi) &= g(\nabla_W R(e_i,Y)\xi,\xi) - g(R(\nabla_W e_i,Y)\xi,\xi)\\
&\quad - g(R(e_i,\nabla_W Y)\xi,\xi) - g(R(e_i,Y)\nabla_W \xi,\xi)
\end{aligned} \quad (5.8)$$

at p ∈ M. In local co-ordinates $\nabla_W e_i = W^j \Gamma^h_{ji} e_h$, where Γ^h_{ji} are the Christoffel symbols. Since $\{e_i\}$ is an orthonormal basis, the metric tensor $g_{ij} = \delta_{ij}$, δ_{ij} is the Kronecker delta and hence the Christoffel symbols are zero. Therefore $\nabla_W e_i = 0$. Since R is skew-symmetric, we have

$$g(R(e_i, \nabla_W Y)\xi, \xi) = 0. \tag{5.9}$$

Using equation (5.9) and $\nabla_W e_i = 0$ in equation (5.8), we get

$$g((\nabla_W R)(e_i, Y)\xi, \xi) = g(\nabla_W R(e_i, Y)\xi, \xi) - g(R(e_i, Y)\nabla_W \xi, \xi). \tag{5.10}$$

In view of $g(R(e_i, Y)\xi, \xi) = -g(R(\xi, \xi)e_i, Y) = 0$ and $(\nabla_W g) = 0$, we have

$$g(\nabla_W R(e_i, Y)\xi, \xi) - g(R(e_i, Y)\xi, \nabla_W \xi) = 0, \tag{5.11}$$

which implies

$$g((\nabla_W R)(e_i, Y)\xi, \xi) = -g(R(e_i, Y)\xi, \nabla_W \xi) - g(R(e_i, Y)\nabla_W \xi, \xi).$$

Since R is skew-symmetric, we have

$$g((\nabla_W R)(e_i, Y)\xi, \xi) = 0. \tag{5.12}$$

Using equation (5.12) in equation (5.7), we get

$$(\nabla_W S)(Y, \xi) = A(W)S(Y, \xi) + ng(\phi Y, W) - (n-1)\eta(Y)A(W). \tag{5.13}$$

Now, we have

$$(\nabla_W S)(Y, \xi) = \nabla_W S(Y, \xi) - S(\nabla_W Y, \xi) - S(Y, \nabla_W \xi,$$

which on using equations (2.5), (2.9),(2.10) and (2.15) takes the form

$$(\nabla_W S)(Y, \xi) = (n-1)g(W, \phi Y) - S(Y, \phi W). \tag{5.14}$$

Form equations (5.13) and (5.14), we have

$$S(Y, \phi W) + g(Y, \phi W) = 0. \tag{5.15}$$

Replacing W by ϕW in above equation and using equation (2.2), we get

$$S(Y, W) = -[g(Y, W) + n\eta(Y)\eta(W)], \tag{5.16}$$

which shows that M^n is an $\eta-$Einstein manifold. Thus we state as follows:

Theorem 5.3. *If an LP-Sasakian manifold is ϕ-recurrent with respect to the quarter-symmetric metric connection then the manifold is an η-Einstein manifold with respect to the Levi-Civita connection.*

6 Locally Projective ϕ-Symmetric LP-Sasakian Manifold with respect to Quarter-Symmetric Metric Connection

Definition 6.1. An n-dimensional LP-Sasakian manifold M^n is said to be locally projective ϕ-symmetric if

$$\phi^2((\nabla_W P)(X,Y)Z) = 0, \tag{6.1}$$

for all vector fields X, Y, Z, W orthogonal to ξ, where P is the projective curvature tensor defined as

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{(n-1)}[S(Y,Z)X - S(X,Z)Y]. \tag{6.2}$$

Equivalently

Definition 6.2. An n-dimensional LP-Sasakian manifold M^n is said to be locally projective ϕ-symmetric with respect to the quarter-symmetric metric connection if

$$\phi^2((\tilde{\nabla}_W \tilde{P})(X,Y)Z) = 0, \tag{6.3}$$

for all vector fields X, Y, Z, W orthogonal to ξ, where $\tilde{P}$ is the projective curvature tensor with respect to the quarter-symmetric metric connection given by

$$\tilde{P}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{1}{(n-1)}[\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y], \tag{6.4}$$

where $\tilde{R}$ and $\tilde{S}$ are the Riemannian curvature tensor and Ricci tensor with respect to quarter-symmetric metric connection $\tilde{\nabla}$.

Using equation (3.6), we can write

$$\begin{aligned}(\tilde{\nabla}_W \tilde{P})(X,Y)Z = (\nabla_W \tilde{P})(X,Y)Z + \eta(\tilde{P}(X,Y)Z)\phi W \\ - g(\phi W, \tilde{P}(X,Y)Z)\xi.\end{aligned} \tag{6.5}$$

Now differentiating equation (6.4) with respect to W, we get

$$\begin{aligned}(\nabla_W \tilde{P})(X,Y)Z = (\nabla_W \tilde{R})(X,Y)Z \\ - \frac{1}{(n-1)}[(\nabla_W \tilde{S})(Y,Z)X - (\nabla_W \tilde{S})(X,Z)Y].\end{aligned} \tag{6.6}$$

In view of equations (4.6) and (3.9) above equation reduces to

$$\begin{aligned}
(\nabla_W \tilde{P})(X,Y)Z \\
&= (\nabla_W R)(X,Y)Z - \{g(Y,W)\eta(Z) + g(Z,W)\eta(Y) \\
&\quad + 2\eta(Y)\eta(W)\eta(Z)\}\phi X + \{g(X,W)\eta(Z) + g(Z,W)\eta(X) \\
&\quad + 2\eta(X)\eta(W)\eta(Z)\}\phi Y + \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\phi W \\
&\quad + \{\eta(Y)g(\phi X,Z) - \eta(X)g(\phi Y,Z)\}W + \{g(\phi X,Z)g(Y,W) \\
&\quad - g(\phi Y,Z)g(X,W) + 2\eta(Y)\eta(W)g(\phi X,Z) - 2\eta(X)\eta(W)g(\phi Y,Z) \\
&\quad + g(\phi X,W)g(Y,Z) - g(\phi Y,W)g(X,Z)\}\xi \\
&\quad - \frac{1}{(n-1)}\{(\nabla_W S)(Y,Z)X - (\nabla_W S)(X,Z)Y\}
\end{aligned} \tag{6.7}$$

which on using equation (6.2) reduces to

$$\begin{aligned}
(\nabla_W \tilde{P})(X,Y)Z \\
&= (\nabla_W P)(X,Y)Z - \{g(Y,W)\eta(Z) + g(Z,W)\eta(Y) \\
&\quad + 2\eta(Y)\eta(W)\eta(Z)\}\phi X + \{g(X,W)\eta(Z) + g(Z,W)\eta(X) \\
&\quad + 2\eta(X)\eta(W)\eta(Z)\}\phi Y + \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\phi W \\
&\quad + \{\eta(Y)g(\phi X,Z) - \eta(X)g(\phi Y,Z)\}W + \{g(\phi X,Z)g(Y,W) \\
&\quad - g(\phi Y,Z)g(X,W) + 2\eta(Y)\eta(W)g(\phi X,Z) - 2\eta(X)\eta(W)g(\phi Y,Z) \\
&\quad + g(\phi X,W)g(Y,Z) - g(\phi Y,W)g(X,Z)\}\xi.
\end{aligned} \tag{6.8}$$

Now, using equations (3.8) and (3.9) in equation (6.4), we get

$$\begin{aligned}
\tilde{P}(X,Y)Z = R(X,Y)Z + g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X + [g(Y,Z)\eta(X) \\
- g(X,Z)\eta(Y)]\xi - \frac{1}{(n-1)}[S(Y,Z)X - S(X,Z)Y],
\end{aligned} \tag{6.9}$$

which gives

$$\begin{aligned}
\tilde{P}(X,Y)Z = P(X,Y)Z + g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X \\
+ [g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\xi.
\end{aligned} \tag{6.10}$$

Taking the inner product of equation (6.9) with ξ and using equations (2.1), (2.7) and (2.11), we get

$$\eta(\tilde{P}(X,Y)Z) = -\frac{1}{(n-1)}[S(Y,Z)X - S(X,Z)Y]. \tag{6.11}$$

Now applying equations (2.2), (6.8) and (6.11) in equation (6.5), we get

$$\begin{aligned}
&\phi^2((\tilde{\nabla}_W \tilde{P})(X,Y)Z) \\
&\quad = \phi^2((\nabla_W P)(X,Y)Z) - \{g(Y,W)\eta(Z) + g(Z,W)\eta(Y) \\
&\qquad + 2\eta(Y)\eta(W)\eta(Z)\}\phi^2(\phi X) + \{g(X,W)\eta(Z) + g(Z,W)\eta(X) \\
&\qquad + 2\eta(X)\eta(W)\eta(Z)\}\phi^2(\phi Y) + \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\phi^2(\phi W) \\
&\qquad + \{\eta(Y)g(\phi X,Z) - \eta(X)g(\phi Y,Z)\}\phi^2(W) \\
&\qquad + \frac{1}{n-1}\{S(Y,Z)\eta(X) + S(X,Z)\eta(Y)\}\phi^2(\phi W).
\end{aligned} \tag{6.12}$$

By assuming X, Y, Z, W orthogonal to ξ, above equation reduces to

$$\phi^2((\tilde{\nabla}_W \tilde{P})(X,Y)Z) = \phi^2((\nabla_W P)(X,Y)Z). \tag{6.13}$$

Hence we can state as follows:

Theorem 6.3. *An n-dimensional LP-Sasakian manifold is locally projective ϕ-symmetric with respect to $\tilde{\nabla}$ if and only if it is locally projective ϕ-symmetric with respect to the Levi-Civita connection ∇.*

Again from equations (2.2), (6.7) and (6.11), we have

$$\begin{aligned}
&\phi^2((\tilde{\nabla}_W \tilde{P})(X,Y)Z) \\
&\quad = \phi^2((\nabla_W R)(X,Y)Z) - \{g(Y,W)\eta(Z) + g(Z,W)\eta(Y) \\
&\qquad + 2\eta(Y)\eta(W)\eta(Z)\}\phi^2(\phi X) + \{g(X,W)\eta(Z) + g(Z,W)\eta(X) \\
&\qquad + 2\eta(X)\eta(W)\eta(Z)\}\phi^2(\phi Y) + \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\phi^2(\phi W) \\
&\qquad + \{\eta(Y)g(\phi X,Z) - \eta(X)g(\phi Y,Z)\}\phi^2(W) \\
&\qquad - \frac{1}{(n-1)}\{(\nabla_W S)(Y,Z)\eta(X) - (\nabla_W S)(X,Z)\eta(Y)\} \\
&\qquad - \frac{1}{(n-1)}\{S(Y,Z)\eta(X) - S(X,Z)\eta(Y)\}\phi^2(\phi W).
\end{aligned} \tag{6.14}$$

Taking X, Y, Z and W orthogonal to ξ in equation (6.14), we obtain by simple calculation

$$\phi^2((\tilde{\nabla}_W \tilde{P})(X,Y)Z) = \phi^2((\nabla_W R)(X,Y)Z). \tag{6.15}$$

Thus we can state as follows:

Theorem 6.4. *A ϕ-symmetric LP-Sasakian manifold admitting the quarter-symmetric metric connection $\tilde{\nabla}$ is locally projective ϕ-symmetric with respect to the quarter-symmetric metric connection $\tilde{\nabla}$ if and only if it is locally ϕ-symmetric with respect to the Levi-Civita connection ∇.*

7 ϕ-Projectively Flat LP-Sasakian Manifold with respect to Quarter-Symmetric Metric Connection

Definition 7.1. An n-dimensional differentiable manifold (M^n, g) satisfying the equation

$$\phi^2(P(\phi X, \phi Y)\phi Z) = 0 \tag{7.1}$$

is called ϕ-projectively flat. Analogous to the equation (7.1) we define an n-dimensional LP-Sasakian manifold is said to be ϕ-projectively flat with respect to quarter-symmetric metric connection if it satisfies

$$\phi^2(\tilde{P}(\phi X, \phi Y)\phi Z) = 0, \tag{7.2}$$

where $\tilde{P}$ is the projective curvature tensor of the manifold with respect to quarter-symmetric metric connection.

Suppose M^n is ϕ-projectively flat LP-Sasakian manifold with respect to quarter-symmetric metric connection. It is easy to see that $\phi^2(\tilde{P}(\phi X, \phi Y)\phi Z) = 0$ holds if and only if

$$g(\tilde{P}(\phi X, \phi Y)\phi Z, \phi W) = 0, \tag{7.3}$$

for any X, Y, Z, W $\in$ TM. So by the use of equation (6.4) ϕ-projectively flat means

$$g(\tilde{R}(\phi X, \phi Y)\phi Z, \phi W) = \frac{1}{(n-1)}[\tilde{S}(\phi Y, \phi Z)g(\phi X, \phi W) - \tilde{S}(\phi X, \phi Z)g(\phi Y, \phi W)], \tag{7.4}$$

which on using equations (3.8) and (3.9) reduces to

$$\begin{aligned} g(R(\phi X, \phi Y)\phi Z, \phi W) + g(X, \phi Z)g(Y, \phi W) - g(Y, \phi Z)g(X, \phi W) \\ = \frac{1}{(n-1)}[S(\phi Y, \phi Z)g(\phi X, \phi W) - S(\phi X, \phi Z)g(\phi Y, \phi W)]. \end{aligned} \tag{7.5}$$

Let $\{e_1, e_2, \ldots, e_{n-1}, \xi\}$ be a local orthonormal basis of the vector fields in M^n. Using the fact that $\{\phi e_1, \phi e_2, \ldots, \phi e_{n-1}, \xi\}$ is also local orthonormal basis. Putting $X = W = e_i$ in equation (7.5) and summing over i, we get

$$\begin{aligned} \sum_{i=1}^{n-1}[g(R(\phi e_i, \phi Y)\phi Z, \phi e_i) + g(e_i, \phi Z)g(Y, \phi e_i) - g(Y, \phi Z)g(e_i, \phi e_i)] \\ = \frac{1}{(n-1)}\sum_{i=1}^{n-1}[S(\phi Y, \phi Z)g(\phi e_i, \phi e_i) - S(\phi e_i, \phi Z)g(\phi Y, \phi e_i)]. \end{aligned} \tag{7.6}$$

Also, it can be seen that ([21])

$$\sum_{i=1}^{n-1} g(R(\phi e_i, \phi Y)\phi Z, \phi e_i) = S(\phi Y, \phi Z) + g(\phi Y, \phi Z), \tag{7.7}$$

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi Z)S(\phi Y, \phi e_i) = S(\phi Y, \phi Z), \tag{7.8}$$

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) = n - 1 \tag{7.9}$$

and

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi Z)g(\phi Y, \phi e_i) = g(\phi Y, \phi Z). \tag{7.10}$$

Hence by virtue of equations (7.7), (7.8), (7.9) and (7.10) equation (7.6) takes the form

$$S(\phi Y, \phi Z) = -2(n-1)g(\phi Y, \phi Z). \tag{7.11}$$

Now, using equations (2.3) and (2.16) in above equation, we get

$$S(Y, Z) = -(n-1)[g(Y, Z) + 3\eta(X)\eta(Y)].$$

Thus we can state as follows:

Theorem 7.2. *An n-dimensional ϕ-projectively flat LP-Sasakian manifold admitting the quarter-symmetric metric connection is an η-Einstein manifold with respect to the Levi-Civita connection.*

8 An LP-Sasakian Manifold with Quarter-Symmetric Metric Connection satisfying $R.\tilde{R} = 0$

Suppose $(R(\xi, X).\tilde{R})(Y, Z)U = 0$ on M^n. Then it can be written as

$$\begin{aligned} R(\xi, X).\tilde{R}(Y, Z)U - \tilde{R}(R(\xi, X)Y, Z)U - \tilde{R}(Y, R(\xi, X)Z)U \\ - \tilde{R}(Y, Z)R(\xi, X)U = 0. \end{aligned} \tag{8.1}$$

In view of equation (2.12) above equation reduces to

$$\begin{aligned} &'\tilde{R}(Y, Z, U, X)\xi - \eta(\tilde{R}(Y, Z)U)X - g(X, Y)\tilde{R}(\xi, Z)U \\ &+ \eta(Y)\tilde{R}(X, Z)U - g(X, Z)\tilde{R}(Y, \xi)U + \eta(Z)\tilde{R}(Y, X)U \\ &\qquad - g(X, U)\tilde{R}(Y, Z)\xi + \eta(U)\tilde{R}(Y, Z)X = 0. \end{aligned} \tag{8.2}$$

By virtue of equation (3.8) above equation takes the form

$$\begin{aligned}
&'R(Y,Z,U,X)\xi + \{g(\phi Y,U)g(\phi Z,X) - g(\phi Z,U)g(\phi Y,X)\}\xi \\
&+ \eta(U)\{g(X,Y)\eta(Z) - g(X,Z)\eta(Y)\}\xi + \eta(X)\{g(Z,U)\eta(Y) \\
&- g(Y,U)\eta(Z)\}\xi + \eta(Y)[g(\phi X,U)\phi Z - g(\phi Z,U)\phi X \\
&+ \eta(U)\eta(Z)X - \eta(U)\eta(X)Z + \{g(Z,U)\eta(X) - g(X,U)\eta(Z)\}\xi] \\
&+ \eta(Z)[g(\phi Y,U)\phi X - g(\phi X,U)\phi Y + \eta(U)\eta(X)Y \\
&- \eta(U)\eta(Y)X + \{g(X,U)\eta(Y) - g(Y,U)\eta(X)\}\xi] \\
&+ \eta(U)[g(\phi Y,X)\phi Z - g(\phi Z,X)\phi Y + \eta(Z)\eta(X)Y \\
&- \eta(X)\eta(Y)Z + \{g(Z,X)\eta(Y) - g(Y,X)\eta(Z)\}\xi] \\
&+ \eta(Y)R(X,Z)U + \eta(Z)R(Y,X)U + \eta(U)R(Y,Z)X = 0.
\end{aligned} \tag{8.3}$$

Now, taking the inner product of above equation with ξ and using equations (2.1) and (2.4), we get

$$\begin{aligned}
'R(Y,Z,U,X) = {} & g(\phi Z,U)g(\phi Y,X) - g(\phi Y,U)g(\phi Z,X) - \eta(U)\{g(X,Y)\eta(Z) \\
& - g(X,Z)\eta(Y)\} - \eta(X)\{g(Z,U)\eta(Y) - g(Y,U)\eta(Z)\}.
\end{aligned} \tag{8.4}$$

Putting $Y = X = e_i$ in above equation and taking summation over i, we get

$$S(Z,U) = g(Z,U) + (-n)\eta(Z)\eta(U).$$

Thus we can state as follows:

Theorem 8.1. *An n-dimensional LP-Sasakian manifold admitting quarter-symmetric metric connection satisfying $R(\xi,X).\tilde{R} = 0$ is an η-Einstein manifold.*

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