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On the Stability of Quadratic ($*$-) Derivations on ($*$-) Banach Algebras

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Abstract : The stability of quadratic derivations on a Banach algebra is established. The stability and the superstability of quadratic $*$-derivations on Banach $*$-algebras are proved as well. In fact, it is modified the result of Jang and Park [1] on a quadratic $*$-derivation.

Keywords : $*$-derivation; quadratic $*$-derivation; Banach $*$-algebra; stability; superstability.
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1 Introduction

A classical question in the theory of functional equations is that "when is it true that a mapping which approximately satisfies on a functional equation must be somehow close to an its exact solution. Such a problem was formulated by Ulam in [2] and solved for the Cauchy functional equation by Hyers [3].

In [4], Rassias provided a generalization of Hyers theorem which allows the Cauchy difference to be unbounded. Găvruta [5] extended the Hyers-Ulam stability in the spirit of Rassias approach. It gave rise to the stability theory for

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functional equations. In the case that every approximately solution of a functional equation is an exact solution of it, we say that is superstable. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [6–8]).

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

is called a *quadratic* functional equation and every solution of the quadratic functional equation is said to be a *quadratic function*. A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [9] for mappings $f : X \longrightarrow Y$, where X is a normed space and Y is a Banach space. Cholewa [10] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. In [11], Czerwik established the generalized Hyers-Ulam stability of the quadratic functional equation.

The Hyers-Ulam stability of quadratic derivations on Banach algebras was studied in [12] for the first time. After that this is generalized to the stability and the superstability of quadratic $*$-derivations on Banach C^*-algebras in [1].

In [13], Bodaghi et al. proved the generalized Hyers-Ulam stability and the superstability for quadratic double centralizers and quadratic multipliers by using the alternative fixed point (Theorem 3.1) under certain conditions on Banach algebras. This approach also is employed to establish of the stability of ternary quadratic derivations on ternary Banach algebras and C^*-ternary rings in [14] (for more see [15–18]).

In this paper, we bring an example of quadratic derivations on a Banach algebra and then we investigate their stability with directed method. We also prove the stability and the superstability of quadratic $*$-derivations on Banach $*$-algebras.

2 Stability of Quadratic Derivations

Let A be a Banach algebra. A Banach space B which is also a left A-module is called a *left Banach A-module* if there is $k>0$ such that

$$\|a\cdot x\| \leq k\|a\|\|x\|.$$

Similarly, a right Banach A-module and a Banach A-bimodule are defined.

Let A be a Banach algebra and B be a Banach A-bimodule. A mapping $D : A \to B$ is called a *quadratic derivation* if D is a quadratic homogeneous mapping, that is D is quadratic and $L(\mu a)=\mu^2 D(a)$ for all $a \in A$ and $\mu \in \mathbb{C}$, and $D(ab)=D(a)\cdot b^2+a^2\cdot D(b)$ for all $a,b \in A$.

First, we indicate an example of quadratic derivations on a Banach algebra. In fact, this example is taken from [19] with the non trivial module actions while in [19] the left module action is trivial.

Example 2.1. Let $\mathcal{A}$ be a Banach algebra. Set

$$\mathcal{T} := \begin{bmatrix} 0 & \mathcal{A} & \mathcal{A} \\ 0 & 0 & \mathcal{A} \\ 0 & 0 & 0 \end{bmatrix}.$$

Then $\mathcal{T}$ is a Banach algebra with the usual sum and product matrix operations and with the following norm:

$$\left\| \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \right\| = \|a\| + \|b\| + \|c\| \qquad (a, b, c \in \mathcal{A}).$$

So

$$\mathcal{T}^* = \begin{bmatrix} 0 & \mathcal{A}^* & \mathcal{A}^* \\ 0 & 0 & \mathcal{A}^* \\ 0 & 0 & 0 \end{bmatrix},$$

is the dual of $\mathcal{T}$ with the following norm

$$\left\| \begin{bmatrix} 0 & f & g \\ 0 & 0 & h \\ 0 & 0 & 0 \end{bmatrix} \right\| = Max\{\|f\|, \|g\|, \|h\|\} \qquad (f, g, h \in \mathcal{A}^*).$$

Suppose that $\mathfrak{A} = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}, \mathfrak{X} = \begin{bmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix} \in \mathcal{T}$ *and* $\mathcal{F} = \begin{bmatrix} 0 & f & g \\ 0 & 0 & h \\ 0 & 0 & 0 \end{bmatrix} \in \mathcal{T}^*$ in which $f, g, h \in \mathcal{A}^*, a, b, c, x, y, z \in \mathcal{A}$. Consider the module actions of $\mathcal{T}$ on $\mathcal{T}^*$ as follows:

$$\langle \mathcal{F} \cdot \mathfrak{A}, \mathfrak{X} \rangle = f(ax) + g(by) + h(cz),$$
$$\langle \mathfrak{A} \cdot \mathcal{F}, \mathfrak{X} \rangle = f(xa) + g(yb) + h(zc).$$

Obviously, $\mathcal{T}^*$ is a Banach $\mathcal{T}$-module. Let $\mathcal{F}_0 = \begin{bmatrix} 0 & f_0 & g_0 \\ 0 & 0 & h_0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathcal{T}^*$ which is fixed. We define $D : \mathcal{T} \longrightarrow \mathcal{T}^*$ by

$$D(\mathfrak{A}) = \mathcal{F}_0 \cdot \mathfrak{A}^2 - \mathfrak{A}^2 \cdot \mathcal{F}_0 \qquad (\mathfrak{A} \in \mathcal{T}).$$

Given $\mathfrak{A} = \begin{bmatrix} 0 & a_1 & b_1 \\ 0 & 0 & c_1 \\ 0 & 0 & 0 \end{bmatrix}, \mathfrak{B} = \begin{bmatrix} 0 & a_2 & b_2 \\ 0 & 0 & c_2 \\ 0 & 0 & 0 \end{bmatrix} \in \mathcal{T}$, we have

$$\begin{aligned} \langle D(\mathfrak{A} + \mathfrak{B}), \mathfrak{X} \rangle &= \langle \mathcal{F}_0 \cdot (\mathfrak{A} + \mathfrak{B})^2 - (\mathfrak{A} + \mathfrak{B})^2 \cdot \mathcal{F}_0, \mathfrak{X} \rangle \\ &= g_0((a_1 + a_2)(c_1 + c_2)y) - g_0(y(a_1 + a_2)(c_1 + c_2)). \end{aligned} \tag{2.1}$$

Similarly,

$$\langle D(\mathfrak{A} - \mathfrak{B}), \mathfrak{X} \rangle = g_0((a_1 - a_2)(c_1 - c_2)y) - g_0(y(a_1 - a_2)(c_1 - c_2)). \tag{2.2}$$

On the other hand,

$$\langle 2D(\mathfrak{A}), \mathfrak{X}\rangle = \langle 2\mathcal{F}_0 \cdot \mathfrak{A}^2 - 2\mathfrak{A}^2 \cdot \mathcal{F}_0, \mathfrak{X}\rangle = g_0(2a_1c_1y) - g_0(2ya_1c_1). \tag{2.3}$$

Also,

$$\langle 2D(\mathfrak{B}), \mathfrak{X}\rangle = 2g_0(a_2c_2y) + 2g_0(ya_2c_2). \tag{2.4}$$

If follows from (2.1)-(2.4) that

$$D(\mathfrak{A}+\mathfrak{B}) + D(\mathfrak{A}-\mathfrak{B}) = 2D(\mathfrak{A}) + 2D(\mathfrak{B})$$

for all $\mathfrak{A}, \mathfrak{B} \in \mathcal{T}$. This shows that D is quadratic, and thus D is a quadratic homogeneous mapping. It is easy to check that $\mathcal{T}^3 = \{0\}$. Therefore $D(\mathfrak{A}\mathfrak{B}) = D(\mathfrak{A}) \cdot \mathfrak{B}^2 + \mathfrak{A}^2 \cdot D(\mathfrak{B}) = 0$ for all $\mathfrak{A}, \mathfrak{B} \in \mathcal{T}$. Hence, D is a quadratic derivation.

It is proved in [20] that for the vector spaces X and Y and the fixed positive integer m, the map $f : X \longrightarrow Y$ is quadratic if and only if the following equality holds.

$$2f\left(\frac{mx+my}{2}\right) + 2f\left(\frac{mx-my}{2}\right) = m^2 f(x) + m^2 f(y),$$

for all $x, y \in X$. Also, we can show that f is quadratic if and only if for a fixed positive integer m, we have

$$f(mx+my) + f(mx-my) = 2m^2 f(x) + 2m^2 f(y).$$

for all $x, y \in X$.

Throughout this section, we assume that A is a Banach algebra and B is a Banach A-bimodule and denote $\overbrace{A \times A \times \cdots \times A}^{n-times}$ by A^n.

Theorem 2.2. *Suppose that $f : A \longrightarrow B$ is a mapping with $f(0) = 0$. Assume that there exists a functions $\phi : A \times A \longrightarrow [0, \infty)$ such that*

$$\widetilde{\phi}(a,b) := \sum_{n=1}^{\infty} \frac{1}{4^n}\phi(2^n a, 2^n b) < \infty \tag{2.5}$$

and

$$\left\| 2f\left(\frac{\mu a+\mu b}{2}\right) + 2f\left(\frac{\mu a-\mu b}{2}\right) - \mu^2 f(a) - \mu^2 f(b)\right\| \leq \phi(a,b) \tag{2.6}$$

$$\|f(ab) - f(a) \cdot b^2 - a^2 \cdot f(b)\| \leq \phi(a,b) \tag{2.7}$$

for all $\mu \in \mathbb{T}_{\frac{1}{n_0}} = \{e^{i\theta} \; ; \; 0 \leq \theta \leq \frac{2\pi}{n_0}\}$ and all $a, b \in A$ in which n_0 is a natural number. Also, if for each fixed $a \in A$ the mappings $t \mapsto f(ta)$ from $\mathbb{R}$ to B is continuous, then there exists a unique quadratic derivation $D : A \longrightarrow B$ satisfying

$$\|f(a) - D(a)\| \leq \widetilde{\phi}(a, 0) \tag{2.8}$$

for all $a \in A$.

Proof. Setting $b = 0, \mu = 1$ and replacine a by $2a$ in (2.6), we get

$$\left\|\frac{1}{4}f(2a) - f(a)\right\| \leq \frac{1}{4}\phi(2a, 0) \tag{2.9}$$

for all $a \in A$. Now, we use the Rassias' method on inequality (2.9) to show that

$$\left\|\frac{f(2^n a)}{4^n} - f(a)\right\| \leq \sum_{j=1}^{n} \frac{\phi(2^j a, 0)}{4^j} \tag{2.10}$$

for all $a \in A$ and all positive integers n. We can also show that

$$\left\|\frac{f(2^{n+m} a)}{4^{n+m}} - \frac{f(2^m a)}{4^m}\right\| \leq \sum_{j=m+1}^{n+m} \frac{\phi(2^j a, 0)}{4^j} \tag{2.11}$$

for all $a \in A$ and all non-negative integers n and m with $n > m$. It follows from (2.5) and (2.11) that the sequence $\left\{\frac{f(2^n a)}{4^n}\right\}$ is Cauchy. Due to the completness of B, this sequence is convergent. So one can define the mapping $D : A \longrightarrow B$ by

$$D(a) := \lim_{n\to\infty} \frac{f(2^n a)}{4^n}. \tag{2.12}$$

Replacing $2^n a, 2^n b$ by a, b, respectively (2.6) and multiplying both sides by $\frac{1}{4^n}$, we get

$$\begin{aligned}
&\left\|2D\left(\frac{\mu a + \mu b}{2}\right) + 2D\left(\frac{\mu a - \mu b}{2}\right) - \mu^2 D(a) - \mu^2 D(b)\right\| \\
&\quad = \lim_{n\longrightarrow\infty} \frac{1}{4^n}\left\|2f\left(\frac{\mu 2^n(a+b)}{2}\right) + 2f\left(\frac{\mu 2^n(a-b)}{2}\right) - \mu^2 f(2^n a) - \mu^2 f(2^n b)\right\| \\
&\quad \leq \lim_{n\longrightarrow\infty} \frac{\phi(2^n a, 2^n b)}{4^n} = 0
\end{aligned}$$

for all $a, b \in \mathcal{A}$ and $\mu \in \mathbb{T}_{\frac{1}{n_0}}$. So

$$2D\left(\frac{\mu a + \mu b}{2}\right) + 2D\left(\frac{\mu a - \mu b}{2}\right) = \mu^2 D(a) + \mu^2 D(b) \tag{2.13}$$

for all $a, b \in \mathcal{A}$ and $\mu \in \mathbb{T}_{\frac{1}{n_0}}$. Putting $\mu = 1$ in (2.13) we have

$$2D\left(\frac{a+b}{2}\right) + 2D\left(\frac{a-b}{2}\right) = D(a) + D(b) \tag{2.14}$$

for all $a, b \in A$. By [20, Proposition 1], D is a quadratic mapping, and thus by (2.13) we can get

$$D(\mu a + \mu b) + D(\mu a - \mu b) = 2\mu^2 D(a) + 2\mu^2 D(b) \tag{2.15}$$

for all $a, b \in A$ and $\mu \in \mathbb{T}_{\frac{1}{n_0}}$. Letting $b = 0$ in (2.15), we get $D(\mu a) = \mu^2 D(a)$ for all $a \in A$ and $\mu \in \mathbb{T}_{\frac{1}{n_0}}$. Now, let $\mu \in \mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ such that $\mu = e^{i\theta}$ in which $0 \leq \theta < 2\pi$. Set $\mu_1 = e^{\frac{i\theta}{n_0}}$, thus μ_1 belongs to $\mathbb{T}_{\frac{1}{n_0}}$ and $D(\mu a) = D(\mu_1^{n_0} a) = \mu_1^{2n_0} D(a) = \mu^2 D(a)$ for all $a \in A$. Then under the hypothesis that $f(ta)$ is continuous in $t \in \mathbb{R}$ for each fixed $a \in A$, by the same reasoning as in the proof of [11], $D(\mu a) = \mu^2 D(a)$ for all $\mu \in \mathbb{R}$ and $a \in A$. So,

$$D(\mu a) = D\left(\frac{\mu}{|\mu|}|\mu| a\right) = \frac{\mu^2}{|\mu|^2} D(|\mu| a) = \frac{\mu^2}{|\mu|^2}|\mu|^2 D(a) = \mu^2 D(a),$$

for all $a \in A$ and $\mu \in \mathbb{C}$ ($\mu \neq 0$). Therefore, D is a quadratic homogeneous mapping. If we substitute a, b by $2^n a, 2^n b$ respectively in (2.7) and divide both sides by 4^{2n} we have

$$\left\|\frac{f(2^{2n}ab)}{4^{2n}} - \frac{f(2^n a)}{4^n} \cdot b^2 - a^2 \cdot \frac{f(2^n b)}{4^n}\right\| \leq \frac{\phi(2^n a, 2^n b)}{4^{2n}} \leq \frac{\phi(2^n a, 2^n b)}{4^n}$$

for all $a, b \in A$. Taking the limit as n tend to infinity, we get $D(ab) = D(a) \cdot b^2 + a^2 \cdot D(b)$, for all $a, b \in A$. Moreover, (2.10) and (2.12) show that the inequality (2.8) holds. For the uniqueness of D, let $\widetilde{D} : A \longrightarrow B$ be another quadratic derivation satisfying (2.8). Then we have

$$\begin{aligned}
\|D(a) - \widetilde{D}(a)\| &= \frac{1}{4^n}\|D(2^n a) - \widetilde{D}(2^n a)\| \\
&\leq \frac{1}{4^n}(\|D(2^n a) - f(2^n a)\| + \|f(2^n a) - D'(2^n a)\|) \\
&\leq 2\sum_{j=1}^{\infty} \frac{1}{4^{n+j}}\phi(2^{n+j}a, 0) = 2\sum_{j=n}^{\infty} \frac{1}{4^j}\phi(2^j a, 0)
\end{aligned}$$

which $\|D(a) - \widetilde{D}(a)\|$ tends to zero as $n \to \infty$ for all $a \in A$. □

Theorem 2.3. *Suppose that $f : A \longrightarrow B$ is a mapping with $f(0) = 0$ for which there exists a function $\phi : A \times A \to [0, \infty)$ satisfying (2.6), (2.7) and*

$$\widetilde{\phi}(a, b) := \sum_{k=0}^{\infty} 4^k \phi(2^{-k}a, 2^{-k}b) < \infty,$$

for all $a, b \in A$. Also, if for each fixed $a \in A$ the mappings $t \mapsto f(ta)$ from $\mathbb{R}$ to B is continuous, then there exists a unique quadratic derivation $D : A \longrightarrow B$ satisfying

$$\|f(a) - D(a)\| \leq \widetilde{\phi}(a, 0) \tag{2.16}$$

for all $a \in A$.

Proof. Substituting a by $2a$ and putting $b = 0, \mu = 1$ in (2.6), we have

$$\|4f(a) - f(2a)\| \leq \phi(2a, 0) \tag{2.17}$$

for all $a \in A$. Replacing a by $\frac{a}{2}$ in (2.17) to obtain

$$\left\|4f\left(\frac{a}{2}\right) - f(a)\right\| \leq \phi(a, 0) \tag{2.18}$$

Now, we use the triangular inequality and continue this way to get

$$\left\|4^n f\left(\frac{a}{2^n}\right) - f(a)\right\| \leq \sum_{j=0}^{n-1} 4^j \phi\left(\frac{a}{2^j}, 0\right) \tag{2.19}$$

for all $a \in A$ and all positive integers n. So, we have

$$\begin{aligned}\left\|4^m f\left(\frac{a}{2^m}\right) - 4^{m+n} f\left(\frac{a}{2^{m+n}}\right)\right\| &\leq \sum_{j=1}^{n} 4^{j+m} \phi\left(\frac{a}{2^{j+m}}, 0\right) \\ &= \sum_{j=m}^{m+n-1} 4^j \varphi\left(\frac{a}{2^j}, 0\right)\end{aligned}$$

for all $a \in A$ and all non-negative integers n and m with $n > m$. Thus the sequence $\left\{4^n f(\frac{a}{2^n})\right\}$ is Cauchy. Since B is a Banach module, this sequence convergence to the mapping $D : A \longrightarrow B$, that is

$$D(a) := \lim_{n\to\infty} \frac{f(2^n a)}{4^n}. \tag{2.20}$$

Similar to the proof of Theorem 2.2, we can obtain the desired result. □

Corollary 2.4. *Let δ, r be positive real numbers with $r \neq 2$, and let $f : A \longrightarrow B$ be a mapping such that*

$$\left\|2f\left(\frac{\mu a + \mu b}{2}\right) + 2f\left(\frac{\mu a - \mu b}{2}\right) - \mu^2 f(a) - \mu^2 f(b)\right\| \leq \delta(\|a\|^r + \|b\|^r)$$

$$\|f(ab) - f(a) \cdot b^2 - a^2 \cdot f(b)\| \leq \delta(\|a\|^r + \|b\|^r)$$

for all $\mu \in \mathbb{T}_{\frac{1}{n_0}}$ and all $a, b \in A$. Then there exists a unique quadratic derivation $D : A \longrightarrow B$ satisfying

$$\|f(a) - D(a)\| \leq \frac{2^r \delta}{|4 - 2^r|} \|a\|^r \tag{2.21}$$

for all $a \in A$.

Proof. By assumptions, we have $f(0) = 0$. Now, the result follows from Theorems 2.2 and 2.3 by taking $\phi(a, b) = \delta(\|a\|^r + \|b\|^r)$ for all $a, b \in A$. $\square$

In the next corollary, we show that a quadratic derivation under which condition can be superstable.

Corollary 2.5. *Let r, s, δ be non-negative real numbers with $0 < r + s \neq 2$ and let $f : A \to B$ be a mapping such that*

$$\left\|2f\left(\frac{\mu a + \mu b}{2}\right) + 2f\left(\frac{\mu a - \mu b}{2}\right) - \mu^2 f(a) - \mu^2 f(b)\right\| \leq \delta(\|a\|^r \|b\|^s) \tag{2.22}$$

$$\|f(ab) - f(a) \cdot b^2 - a^2 \cdot f(b)\| \leq \delta(\|a\|^r \|b\|^s)$$

for all $\mu \in \mathbb{T}_{\frac{1}{n_0}}$ and all $a, b \in A$. Then f is a quadratic derivation on A.

Proof. Putting $a = b = 0$ in (2.22), we get $f(0) = 0$. Now, if we put $b = 0$, $\mu = 1$ and replace a by $2a$ in (2.22), then we have

$$f(2a) = 4f(a) \tag{2.23}$$

for all $a \in A$. It is easy to see by induction that $f(2^n a) = 4^n f(a)$, and so $f(a) = \frac{f(2^n a)}{4^n}$ for all $a \in A$ and $n \in \mathbb{N}$. On the other hand, if we replace a by $\frac{a}{2}$ in (2.23), then we have $f(a) = 4f(\frac{a}{2})$. Again, by induction we get $f(a) = 4^n f(\frac{a}{2^n})$. In both cases, it follows from Theorems 2.2 and 2.3 that f is a quadratic homogeneous mapping. Now, by putting $\phi(a, b) = \delta(\|a\|^r \|b\|^s)$ in Theorems 2.2 and 2.3, we can obtain the desired result. $\square$

3 Stability of Quadratic $*$-Derivations

To prove the main result in this section, we need the following theorem which has been proved by Diaz and Margolis in [21] (later an extension of the result was given in [22]).

Theorem 3.1 (The Fixed Point Alternative)**.** *Let (Ω, d) be a complete generalized metric space and $\mathcal{T} : \Omega \to \Omega$ be a mapping with Lipschitz constant $L < 1$. Then, for each element $\alpha \in \Omega$, either $d(\mathcal{T}^n\alpha, \mathcal{T}^{n+1}\alpha) = \infty$ for all $n \geq 0$, or there exists a natural number n_0 such that:*

(i) *$d(\mathcal{T}^n\alpha, \mathcal{T}^{n+1}\alpha) < \infty$ for all $n \geq n_0$;*

(ii) *the sequence $\{\mathcal{T}^n\alpha\}$ is convergent to a fixed point β^* of $\mathcal{T}$;*

(iii) *β^* is the unique fixed point of $\mathcal{T}$ in the set $\Lambda = \{\beta \in \Omega : d(\mathcal{T}^{n_0}\alpha, \beta) < \infty\}$;*

(iv) *$d(\beta, \beta^*) \leq \frac{1}{1-L} d(\beta, \mathcal{T}\beta)$ for all $\beta \in \Lambda$.*

Let A, B be Banach $*$-algebras and B be also a Banach A-module. Then we say B is a Banach $*$-module over A. Obviously, every Banach $*$-algebra is a Banach $*$-module over itself.

Throughout this section, we assume that A, B are Banach $*$-algebras and B is also a Banach $*$-module over A.

A quadratic derivation $D : A \to B$ is is called a quadratic $*$-derivation if D satisfies in condition $D(a^*) = D(a)^*$ for all $a \in A$. This definition was introduced in [1]. The following theorem has been proved by Jang and Park in [1, Theorem 4.2].

Theorem 3.2. *Suppose that $f : A \to A$ is a mapping with $f(0) = 0$ for which there exists a function $\varphi : A^4 \to [0, \infty)$ such that*

$$\widetilde{\phi}(a, b, c, d) := \sum_{n=1}^{\infty} \frac{1}{4^n} \phi(2^n a, 2^n b, 2^n c, 2^n d) < \infty,$$

$$\|f(\mu a + \mu b + cd) + f(\mu a - \mu b + cd) - 2\mu^2 f(a) - 2\mu^2 f(b) - 2f(c)d^2 - 2c^2 f(d)\| \\ \leq \varphi(a, b, c, d),$$

$$\|f(a^*) - f(a)^*\| \leq \varphi(a, a, a, a)$$

for all $a, b, c, d \in A$ and $\mu \in \mathbb{T}$. Also, if for each fixed $a \in A$ the mappings $t \mapsto f(ta)$ from $\mathbb{R}$ to A is continuous, then there exists a unique quadratic $*$-derivation $D : A \longrightarrow B$ satisfying

$$\|f(a) - D(a)\| \leq \frac{1}{4} \widetilde{\phi}(a, 0, 0, 0)$$

for all $a \in A$.

For the case $f(x) = x^2$, we have $f(\alpha+\beta)+f(\gamma+\beta) = \alpha^2+2\beta^2+\gamma^2+2\alpha\beta+2\gamma\beta$. In the above theorem, the authors did not consider $2\alpha\beta+2\gamma\beta$. On the other hand, one can show that the quadratic $*$-derivation D in the above theorem must be zero, and thus the result is trivial. Therefore it could be true if we divide the main inequality into two parts: quadratic part and derivation part. In the next theorem we wish to modify Theorem 3.2 and deduce a similar result by using Theorem 3.1.

Theorem 3.3. *Let $f : A \to B$ be a continuous mapping with $f(0) = 0$ and let $\varphi : A \times A \to [0, \infty)$ be a continuous function such that*

$$\left\| 2f\left(\frac{\mu a + \mu b}{2}\right) + 2f\left(\frac{\mu a - \mu b}{2}\right) - \mu^2 f(a) - \mu^2 f(b) \right\| \leq \varphi(a, b) \tag{3.1}$$

$$\|f(ab) - f(a) \cdot b^2 - a^2 \cdot f(b)\| \leq \varphi(a, b) \tag{3.2}$$

$$\|f(a^*) - f(a)^*\| \le \varphi(a,a) \tag{3.3}$$

for all $\mu \in \mathbb{T}_{\frac{1}{n_0}}$ *and* $a, b \in A$*. If there exists a constant* $k \in (0,1)$*, such that*

$$\varphi(2a, 2b) \le 4k\varphi(a,b) \tag{3.4}$$

for all $a, b \in A$*, then there exists a unique* $*$*-derivation* $D : A \to B$ *satisfying*

$$\|f(a) - D(a)\| \le \frac{k}{1-k}\widetilde{\varphi}(a) \quad (a \in A) \tag{3.5}$$

where $\widetilde{\varphi}(a) = \varphi(a,0)$*.*

Proof. It follows from (3.4) that

$$\lim_j \frac{\varphi(2^j a, 2^j b)}{4^j} = 0 \tag{3.6}$$

for all $a, b \in A$. Putting $\mu = 1, b = 0$ and replacing a by $2a$ in (3.1), we have

$$\|4f(a) - f(2a)\| \le \widetilde{\varphi}(2a) \le 4k\widetilde{\varphi}(a) \tag{3.7}$$

for all $a \in A$, and so

$$\left\|f(a) - \frac{1}{4}f(2a)\right\| \le k\widetilde{\varphi}(a) \tag{3.8}$$

for all $a \in A$. We consider the set $\Omega := \{h : A \to A \mid h(0) = 0\}$ and introduce the generalized metric on Ω as follows:

$$d(h_1, h_2) := \inf\{C \in (0,\infty) : \|h_1(a) - h_2(a)\| \le C\widetilde{\varphi}(a),\ \forall a \in \mathcal{A}\},$$

if there exist such constant C, and $d(h_1, h_2) = \infty$, otherwise. One can easily show that (Ω, d) is complete. We now define the linear mapping $\mathcal{T} : \Omega \to \Omega$ by

$$T(h)(a) = \frac{1}{4}h(2a) \tag{3.9}$$

for all $a \in A$. Given $h_1, h_2 \in \Omega$, let $C \in \mathbb{R}^+$ be an arbitrary constant with $d(h_1, h_2) \le C$, that is

$$\|h_1(a) - h_2(a)\| \le C\widetilde{\varphi}(a) \tag{3.10}$$

for all $a \in A$. Substituting a by $2a$ in the inequality (3.10) and using the equalities (3.4) and (3.9), we have

$$\|(\mathcal{T}h_1)(a) - (\mathcal{T}h_2)(a)\| = \frac{1}{4}\|h_1(2a) - h_2(2a)\| \le \frac{1}{4}C\widetilde{\varphi}(2a) \le Ck\widetilde{\varphi}(a),$$

for all $a \in A$, and thus $d(\mathcal{T}h_1, \mathcal{T}h_2) \leq Ck$. Therefore, we conclude that $d(\mathcal{T}h_1, \mathcal{T}h_2) \leq kd(h_1, h_2)$ for all $h_1, h_2 \in \Omega$. It follows from (3.8) that

$$d(\mathcal{T}f, f) \leq k. \tag{3.11}$$

By the part (iv) of Theorem 3.1, the sequence $\{\mathcal{T}^n f\}$ converges to a unique fixed point $D : A \to B$ in the set $\Omega_1 = \{h \in \Omega, d(f, h) < \infty\}$, that is

$$\lim_{n\to\infty} \frac{f(2^n a)}{4^n} = D(a) \tag{3.12}$$

for all $a \in A$. By Theorem 3.1 and (3.11), we have

$$d(f, D) \leq \frac{d(\mathcal{T}f, f)}{1-k} \leq \frac{k}{1-k}.$$

The above inequality shows that (3.5) holds for all $a \in \mathcal{A}$. Replace a, b by $2^n a, 2^n b$, respectively in (3.1). Now, dividing both sides of the resulting inequality by 2^n, and letting n goes to infinity, we obtain

$$2D\left(\frac{\mu a + \mu b}{2}\right) + 2D\left(\frac{\mu a - \mu b}{2}\right) = \mu^2 D(a) + \mu^2 D(b) \tag{3.13}$$

for all $a, b \in \mathcal{A}$ and $\mu \in \mathbb{T}_{\frac{1}{n_0}}$. Putting $\mu = 1$ in (3.13) we have

$$2D\left(\frac{a+b}{2}\right) + 2D\left(\frac{a-b}{2}\right) = D(a) + D(b) \tag{3.14}$$

for all $a, b \in A$. By [20, Proposition 1], D is a quadratic mapping, and thus by (3.13) we can get

$$D(\mu a + \mu b) + D(\mu a - \mu b) = 2\mu^2 D(a) + 2\mu^2 D(b) \tag{3.15}$$

for all $a, b \in A$ and $\mu \in \mathbb{T}_{\frac{1}{n_0}}$. Letting $b = 0$ in (3.15), we get $D(\mu a) = \mu^2 D(a)$ for all $a \in A$ and $\mu \in \mathbb{T}_{\frac{1}{n_0}}$. Similar to the proof of Theorem 2.2, we have $D(\mu a) = \mu^2 D(a)$ for all $a \in A$ and $\mu \in \mathbb{T}$, and thus D is quadratic homogeneous by [13, Theorem 2.2]. If we replace a, b by $2^n a, 2^n b$ respectively, in (3.2), we have

$$\left\| \frac{f(2^{2n}ab)}{4^{2n}} - \frac{f(2^n a)}{4^n} \cdot b^2 - a^2 \cdot \frac{f(2^n b)}{4^n} \right\| \leq \frac{\varphi(2^n a, 2^n b)}{4^{2n}} \leq \frac{\varphi(2^n a, 2^n b)}{4^n}$$

for all $a, b \in A$. Taking the limit as n tend to infinity, we get $D(ab) = D(a) \cdot b^2 + c^2 \cdot D(b)$, for all $a, b \in A$. Substituting a by $2^n a$ in (3.3) and then dividing the both sides of the obtained inequality by 4^n, we get

$$\left\| \frac{f(2^n a^*)}{4^n} - \frac{f(2^n a)^*}{4^n} \right\| \leq \frac{\varphi(2^n a, 2^n a)}{4^n} \tag{3.16}$$

for all $a \in A$. Passing to the limit as $n \to \infty$ in (3.16) and applying (3.6), we conclude that $D(a^*) = D(a)^*$ for all $a \in A$. This shows that D is a quadratic $*$-derivation. □

Corollary 3.4. *Let r, δ be non-negative real numbers with $r < 2$ and let $f : A \to B$ be a mapping such that*

$$\left\|2f\left(\frac{\mu a + \mu b}{2}\right) + 2f\left(\frac{\mu a - \mu b}{2}\right) - \mu^2 f(a) - \mu^2 f(b)\right\| \leq \delta(\|a\|^r + \|b\|^r)$$

$$\|f(ab) - f(a) \cdot b^2 - a^2 \cdot f(b)\| \leq \delta(\|a\|^r + \|b\|^r)$$

$$\|f(a^*) - f(a)^*\| \leq 2\delta\|a\|^r$$

for all $\mu \in \mathbb{T}_{\frac{1}{n_0}}$ and all $a, b \in A$. Then there exists a unique quadratic $*$-derivation $D : A \to B$ satisfying

$$\|f(a) - D(a)\| \leq \frac{2^r\delta}{4 - 2^r}\|a\|^r$$

for all $a \in A$.

Proof. The hypotheses show that $f(0) = 0$. The result follows from Theorem 3.3 by taking $\varphi(a, b) = \delta(\|a\|^r + \|b\|^r)$ for all $a, b \in A$. □

In the following corollary, we show that under some conditions the superstability for quadratic $*$-derivations.

Corollary 3.5. *Let r, δ be non-negative real numbers with $r < 1$ and let $f : A \to B$ be a mapping such that*

$$\left\|2f\left(\frac{\mu a + \mu b}{2}\right) + 2f\left(\frac{\mu a - \mu b}{2}\right) - \mu^2 f(a) - \mu^2 f(b)\right\| \leq \delta(\|a\|^r\|b\|^r\|) \quad (3.17)$$

$$\|f(ab) - f(a) \cdot b^2 - a^2 \cdot f(b)\| \leq \delta(\|a\|^r\|b\|^r)$$

$$\|f(a^*) - f(a)^*\| \leq \delta\|a\|^{2r}$$

for all $\mu \in \mathbb{T}_{\frac{1}{n_0}}$ and all $a, b \in A$. Then f is a quadratic $*$-derivation on A.

Proof. Putting $a = b = 0$ in (3.17), we get $f(0) = 0$. Similar to the proof of Corollary 2.5, we can obtain the desired result from Theorem 3.3 by putting $\varphi(a, b) = \delta(\|a\|^r\|b\|^r)$. □

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