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Semi-Global Aspects of Uryshon's Lemma and Broken Two Sheets Spaces

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Abstract : In this paper we give a semi-global description of what is known as the well known Uryshon's Lemma in classical topology. Giving up the usual notions of open and closeness in a topological space (X, T_X) where T_X is a topology, we deal with a pair (X, S_X) with an arbitrary subset S_X of power set $P(X)$ of X. Thus we deal with a semi-global character where so-called open or closed character in such spaces treated in reference to some specific pseudo continuous types of maps. Keeping this in note, we explore some characteristics with so-called topological flavour in such a space (X, S_X), which may be considered as some sort of analytical description of available classical topological notion.

Keywords : ps-continuity; f_{psc}-normal space; Uryshon's Lemma; f-countably ordered; quasi-continuity.
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1 Introduction

Here we deal with a pseudo (ps)-continuous mapping of the space (X, S_X) [1] and obtain some results upto so-called Uryshon's Lemma (in a classical topology) which reveals what the theorem demands in contrast to global character of the space to carry out analogous notion of the same that we would like to mention here as a semi-global one. The results thus obtained may be regarded that some sort of global character is revealed through continuous embedding of such a space.

And possibly this is the real vigour what is examined in our so-called semi-global Uryshon's Lemma.

We here, as far as possible, try to give-up the well known rigorous global nature of a classical topology and would like to stick to only some semi-global analytical approach taking into consideration of a so-called ps-continuous mapping dealing with inverse transformation of members of S_X in order to make algebraic one.

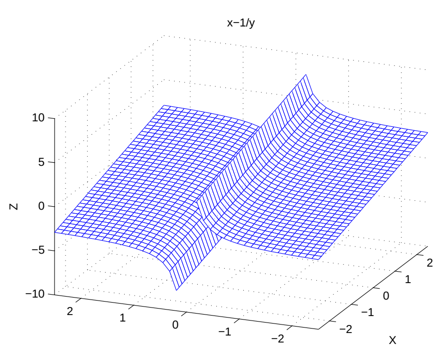
For conformity with global and local character of a continuous map, it might not be linked with characteristically the definition of a topological space so far the belongingness of arbitrary union of open sets is concerned.

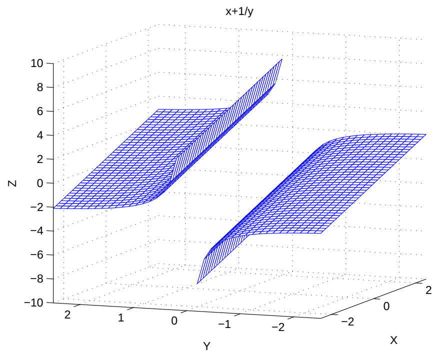
As we do not consider here the union and intersection biased tendency as in a topology, only in this sense, here is a deficit of space biased (position based) notion. And this loss in particular leads us to algebraic flavour of what we are going to discuss here.We also would like to give light on some restricted cases of such type of pseudo space or some may call it a less topological space.

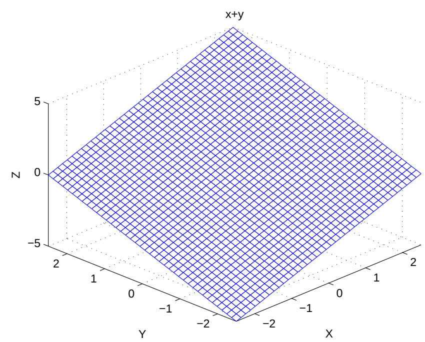
Our study may be confined (whenever necessary) on the notion of a ps-closed set in connection with a definite pseudo-continuous map f instead of studying available notion of closeness in case of a classical topology.

Here also we note the interesting aspect of the non-equivalence of continuity as co-ordinate wise and continuity as a whole as discussed, in Cox and Beidleman [2] may lead us to another dimension in such a ps structure. For the sake of completeness we would like to cite how the example of Klein's 4 group appears as an example justifying the condition imposed in such structures giving a broad viewed algebraic aspect of spaces with topological vigour.

The continuity in a topological near-ring is carried out by internal and external compositions of a group or a near-ring group respectively. It is observed that in case of a topological group, the binary operation is continuous in the product space; the corresponding co-ordinate wise continuity is an obvious character in the so-called one-sheet space. But the converse needs a hard work in the real sense if it happens in the so-called broken two-sheets space. Of course, Beidleman and Cox in [2] are of view that co-ordinate-wise continuity is all that is necessary in many cases. What is stated above may be verified (at least intuitively) with the help of the following graphs with respect to the usual topologies on R^2 (that seems to be self explanatory).







We recall that in the definition of a topological ring, Kaplansky [3] insisted that addition and multiplication be continuous on the product space; however, as defined by Beidleman [2], the authors found that so-called co-ordinate wise continuity is all that is necessary in many cases.

Some careful observation have elegantly reveals what we have attempted and carried out with some sort of rare and alarming beauty, hitherto the so-called continuity of such pseudo structures are concerned.

Keeping aside the concrete so-called topological aspects of what has been explained above , we here dare to review this aspect of above type of algebraic structure from more or less algebraic point of view in a broaden court-yard with a view to play the same game in more sophisticated country of algebra.

For the moment we leave available topological nomenclature, however insisted, embrace some abstract familiar algebraic ways of approach. Undoubtedly everything would be justified with sufficient examples if and when necessary.

1.1 Definitions and Notations

A mapping $f : (X, S_X) \longrightarrow (Y, S_Y)$ is ps-continuous if for every $H \in S_Y$, there exists a collection $\{G_\alpha | G_\alpha \in S_X\}$ such that $f^{-1}(H) = \bigcup G_\alpha$. f is ps-continuous at $p \in X$ if for every $H \in S_Y$ with $f(p) \in H$, there exists a $G \in S_X$ such that

$p \in G \subseteq f^{-1}(H)$. The definition of ps-continuity is in visible difference to that of topological continuity as $f^{-1}(H)$ may not be a member of S_X even though $H \in S_X$. A subset $F \subset Y$ is closed under f (denoted $f_{psc} - closed$) where $f : (X, S_X) \longrightarrow (Y, S_Y)$ is a ps-continuous map if there is a collection $\{G_\alpha | G_\alpha \in S_X\}$ such that $f^{-1}(H) = \bigcap G_\alpha^c$. If S_X is such that for every sub collection $\{G_\alpha | G_\alpha \in S_X\}$, there is an $H \in S_X$ such that $f^{-1}(H) = \bigcup G_\alpha$, then for an $f_{psc}-$ closed set F of X, there exists an $H \in S_X$ such that $f^{-1}(F) = f^{-1}(H^c)$. Moreover interestingly for a f_{psc}-closed set H,-the collection $\{G_\alpha | G_\alpha \in S_X\}$ with $f^{-1}(H) = \bigcap G_\alpha^c$, need not be unique. A subset A of X is f_{psc}-open if its complement A^c is f_{psc}-closed though it is not far from as in classical topology, i.e. if A is f_{psc}-open, then $f^{-1}(A^c) = \bigcap_\alpha H_\alpha^c$ with $\{H_\alpha | H_\alpha \in G_X\}$ which gives $f^{-1}(A) = \bigcup_\alpha H_\alpha$ leading to the fact that every member of S_X is f_{psc}-open as well as every member of S_X^c(the collection of complement of members of S_X) is f_{psc}-closed. De-Morgan's and well-behaved character of f^{-1} gives what we have mentioned. The union of two f_{psc}-closed sets may not be again f_{psc}-closed which is in contrast to that of a closed set in a topology. For f_{psc}-closed set F and E, $f^{-1}(F) = \bigcap_\alpha H_\alpha^c, H_\alpha \in S_X$ and $f^{-1}(E) = \bigcap_\beta G_\beta^c, G_\beta \in S_X$ gives $f^{-1}(F \cap E) = \bigcap_\alpha \bigcap_\beta (H_\alpha \cap G_\beta)^c$. Since $H_\alpha \cap G_\beta$ may not be a member of S_X; this implies that $E \cap F$ need not be f_{psc}-closed. However the arbitrary union of f_{psc}-open sets is f_{psc}-open. An element $x \in X$ is a f_{psc}-limit point of A if for any f_{psc}-open set G containing $x, (G \backslash \{x\}) \cap A \neq \phi$. A subset A of X together with its f_{psc}-limit points $A'$is the f_{psc}- closure of A denoted $\bar{A}$.

1.2 Discussion

Here we note some examples and observations in support of the main results.

Example 1.1. *Let $X = \{a, b, c\}$ and $Y = \{1, 2, 3\}$ be two sets where $S_X = \{\{a, b\}, \{c\}, \{a\}\}(\subseteq P(X))$ and $S_Y = \{\{1, 2\}, \{1\}\}(\subseteq P(Y))$. Consider a mapping $f : (X, S_X) \to (Y, S_Y)$ defined by $f(a) = 1 = f(b) = f(c)$. Here f is ps-continuous and for $F = \{2\} \subset Y$, $f^{-1}\{2\} = G_1^c \cap G_2^c$, where $G_1 = \{a, b\}, G_2 = \{c\}$. Again $f^{-1}\{2\} = G_1^c \cap G_2^c \cap G_3^c$ with $G_3 = \{a\}, G_3^c = \{b, c\}$.Thus $F = \{2\}$ is an f_{psc}-closed set.*

Note: The identity mapping $I : (X, S_X) \to (X, S_X)$ is always a ps-continuous one. Hence the closed set of X (in the general topological notion) is I_{psc}-closed in our context.

Example 1.2. *For two sets $X = \{a, b, c\}$ and $Y = \{\alpha, \beta, \gamma, \delta\}$ where $S_X = \{\{a, b\}\}, \{c\}\}$ and $S_Y = \{\{\alpha\}, \{\beta, \gamma, \delta\}\}$, consider $f : X \to Y$ such that $f(a) = \alpha = f(b), f(c) = \beta$. Here $f^{-1}\{\alpha\} = \{a, b\}, f^{-1}\{\beta, \gamma, \delta\} = \{c\}$. It is easy to see that f is ps-continuous and also ps-continuous at every point of X.*

Example 1.3. *Let $X = \{a, b, c\}$ and $S_X = \{\{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Consider a mapping $f : (X, S_X) \to (X, S_X)$ by $f(a) = a, f(b) = b = f(c)$. f is not ps-continuous as $f^{-1}\{b\} = \{b, c\}$. It is seen that f is not ps-continuous at every* *point of X. As for example, at $c \in X, f(c) = b \in \{b\}$, but there exists no $G \in S_X$ such that $c \in G \subseteq f^{-1}\{b\}$.*

Example 1.4. *Let $X = \{a, b, c\}$ and $Y = \{\alpha, \beta, \delta\}$ be two sets where $S_X = \{\{a, b\}, \{c\}\}$ and $S_Y = \{\{\alpha\}, \{\beta, \gamma, \delta\}\}$. Consider a mapping $f : X \to Y$ by $f(a) = \alpha = f(b), f(c) = \beta$ which is ps-continuous and here $\{\alpha\}, \{\beta\}, \{\gamma\}, \{\delta, \}$ $\{\alpha, \gamma\}, \{\alpha, \delta\}, \{\beta, \gamma\}, \{\beta, \delta\}, \{\beta, \gamma, \delta\}$, etc are the $f_{psc}-$closed sets. But $f^{-1}\{\{\alpha\} \cup \{\beta\}\} = f^{-1}\{\alpha\} \cup f^{-1}\{\beta\} = \{a, b, c\}$, which shows that the union of two $f_{psc}-$closed sets may not $f_{psc}-$closed in turn.*

2 Preliminaries

In the following lemmas we establish the analogous results with semi-global treatment which are very much fundamental in classical topology. Unless otherwise specified throughout this paper, f will mean a ps-continuous map on (X, S_X).It is trivial to note the following lemma.

Lemma 2.1. *A mapping $f : (X, S_X) \to (Y, S_Y)$ is ps-continuous if and only if it is ps-continuous at each point $p \in X$.*

Lemma 2.2. *Let A be a subset of X and $A^{'} \subseteq A$. Then A is $f_{psc}-$ closed.*

Proof. For a $p \in f^{-1}(A^c)$, we get $f(p) \notin A$ which gives $f(p) \notin A^{'}$. Then there exists an f_{psc}-open set G_p containing $f(p)$ such that $(G_p \backslash f(p)) \cap A = \phi$ which gives $f(p) \in G_p \subset A^c$. Thus $p \in f^{-1}(G_p) \subset f^{-1}(A^c)$ giving thereby $f^{-1}(A^c) = \bigcup_{f(p) \in G_p} f^{-1}(G_p) = \bigcup_{f(p) \in G_p} (\cup G_{p_\alpha})[G_{p_\alpha} \in S_X]$. □

Lemma 2.3. *Let $f : (X, S_X) \to (X, S_X)$, be a ps-continuous embedding. If $\vee \subseteq X$ is f_{psc}- closed, then V contains all its $f_{psc}-$ limit points.*

Proof. Let $f^{-1}(V) = \bigcap_{G_\alpha \in S_X} G_\alpha$ and let $x \in V^{/}$. Suppose if possible let $x \notin V$. Now consider $H = f(\cup G_\alpha)$ which gives $f^{-1}(H^c) = \cap G^c_\alpha$, i.e. H^c is $f_{psc}-$ closed gives that H is $f_{psc}-$ open. And $x \notin V = f(\cap G^c_\alpha)$ gives $x \notin f(G^c_\alpha), \forall \alpha$ giving thereby $x \in \cup f(G\alpha) = H$. Now $H \cap V = f(\cap G^c_\alpha) \cap f(\cup G_\alpha) = \phi$, which gives $(H \backslash \{x\} \cap V) = \phi$, a contradiction. □

Note: A is $f_{psc}-$closed implies $A^{'} \subseteq A$, this is linked with the so called continuously embedding character of the space (X, S_X), however the converse does not demand the same.

Lemma 2.4. *For any subset A of $X, \overline{A} = A \cup A^{'}$ is $f_{psc}-$ closed.*

Proof. Let $p \in f^{-1}(A \cup A^{'})^c$ which gives $f(p) \in (f \circ f^{-1})(A \cup A^{'})^c \subset (A \cup A^{'})^c$ giving thereby $f(p) \notin A, f(p) \notin A^{'}$. Now $f(p) \notin A^{'}$ gives that there exists an $f_{psc}-$open set G such that $f(p) \in G$ and $G \cap A = \phi$. Now if $g \in G$ and $G \cap A = \phi$, then $g \notin A^{'}$ gives $G \cap A^{'} = \phi$ giving thereby $G \cap (A \cup A^{'}) = \phi$. Thus $G \subset (A \cup A^{'})^c$ which gives

$p \in f^{-1}(G) \subset f^{-1}(A \cup A^{'})^c$ giving thereby $f^{-1}(A \cup A^{'})^c = \bigcup_{f(p)\in G}(f^{-1}(G)) = \bigcup_{f(p)\in G}(\bigcup G_\alpha)$, where $f^{-1}(G) = \bigcup_\alpha G_\alpha, G_\alpha \in S_X$, i.e. $(A \cup A^{'})^c$ is $f_{psc}-$ open. Thus $A \cup A^{'}$ is $f_{psc}-$ closed. □

It is yet to be verified that $\overline{A}$ may tally with the usual notion of what is known as the closure of a set in classical topology.

It is interesting to note that the notion of closeness and that of ps-continuity elegantly justifies the well behaved closure property in case of a continuous map in classical topology.

Lemma 2.5. *If $f : X \to Y$ is ps-continuous then for any subset A of $X, f(\overline{A}) \subset \overline{f(A)}$.*

Proof. Let $x \in \overline{A} = A \cup A^{'}$. If $x \in A$, then $f(x) \in f(A) \subset \overline{f(A)}$. For otherwise, we claim $f(x) \in (f(A))^{'}$. Let G be $f_{psc}-$ open in Y containing $f(x)$ such that $G \cap f(A) = \phi$. Now $f(\cup H_\alpha) \cap f(A) \subseteq G \cup f(A) = \phi$ where $f^{-1}(G) = \cup H_\alpha$ which gives $f(H_\alpha) \cap f(A) = \phi$, for each α giving thereby $f(H_\alpha \cap A) \subset f(H_\alpha) \cap f(A) = \phi$. Thus $(f^{-1} \circ f)(H_\alpha \cap A) = \phi$ gives $H_\alpha \cap A = \phi$ is f_{psc}-open. Here H_α is $f_{psc}-$ open and $x \in H_\alpha$, a contradiction as $x \in A^{'}$ which gives $(G \backslash f(x)) \cap f(A) \neq \phi$. Thus $f(x) \in (f(A))^{'}$ giving thereby $f(\overline{A}) \subset \overline{f(A)}$. □

However the converse of the above theorem does not hold. The following example justifies elegantly our claim.

Example 2.6. *Let $X = \{a, b, c\}, S_X = \{\{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Define $f : (X, S_X) \to (X, S_X)$ by $f(a) = a, f(b) = b = f(c)$. Here for every subset A of X we get $f(\overline{A}) \subset \overline{(f(A))}$. But f is not ps-continuous as $f^{-1}b = \{b, c\}$.*

Note: If S_X is the collection of all $\{x\}, x \in X$, then $f : (X, S_X) \to (X, S_X)$ is ps-continuous map.

3 Main Results

We here present semi-global aspect of Uryshon's Lemma in classical topology together with the continuity problem in broken two sheets space with some topological vigour.

3.1 Semi-global aspects of Uryshon's Lemma

The results of this section are revealed through the notion of countably f-ordered character of an $f_{psc}-$ Normal space together with its f_c-bounded (/maximal) character and an f-quasi continuous map from the same.

(X, S_X) is $f_{psc}-$ normal if for any two disjoint $f_{psc}-$ closed sets A and B, there exist $f_{psc}-$open sets U and V such that $A \subset U$ and $B \subset V$ with $U \cap V = \phi$.

The following result is a characterization of f_{psc}-normal space.

Theorem 3.1. *Let $f : (X, S_X) \to (X, S_X)$ be a ps-continuous embedding. Then the following are equivalent*

(i) (X, S_X) *is* $f_{psc}-$ *normal;*

(ii) for any $f_{psc}-$*closed set* F *and* $f_{psc}-$ *open set* G *containing* F*, there exists an* $f_{psc}-$*open set* V *such that* $F \subset V$ *and* $\overline{F} \subset G$*.*

Proof. Assume that (X, S_X) is $f_{psc}-$ normal. Let F be $f_{psc}-$ closed and G be $f_{psc}-$ open such that $F \subset G$. Now there exists $f_{psc}-$ open sets U and V such that $F \subset V$ and $G^c \subseteq U$ with $U \cap V = \phi$ which gives $V \subset U^c$ giving thereby $\overline{V} \subset \overline{U^c} = U^c$. Again $\overline{V} \subset U^c \subset G$ gives $\overline{F} \subset \overline{V} \subset G$. Conversely let L and M be two $f_{psc}-$ closed sets with $M \cap L = \phi$ which gives $L \subset M^c$. Thus $f_{psc}-$ closed set L is contained in $f_{psc}-$ open M^c. By hypothesis there exists an $f_{psc}-$ open set V such that $L \subset V$ and $\overline{V} \subset M^c$ giving thereby $M \subset \overline{V^c}$. Thus V and $\overline{V^c}$ are two $f_{psc}-$open sets with $V \cap \overline{V^c} = \phi$ such that $L \subset V$ and $M \subset (\overline{V})^c$. □

Example 3.2. *Let* $X = \{a, b, c, d, e, f\}$ *where* $S_X = \{\{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}\}$ *and we consider a map* $f : (X, S_X) \to (X, S_X)$ *defined by* $f(x) = x$*. Here* $f^{-1}(G) = G$*, for any* $G \in S_X$*. Let* $E = \{d, e, f\}$ *and* $F = \{a, b, c\}$ *be two subsets of* X*. Now* $f^{-1}(E) = \{b, c, d, e, f\} \cap \{a, c, d, e, f\} \cap \{a, b, d, e, f\}$ *which gives* E *is* $f_{psc}-$ *closed. Again* $f^{-1}(F) = \{a, b, c, e, f\} \cap \{a, c, d, e, f\} \cap \{a, b, d, e, f\}$ *gives* F *is* $f_{psc}-$ *closed and* $E \cap F = \phi$*. Now* $\{d, e, f\}$ *and* $\{a, b, c\}$ *are two* $f_{psc}-$ *open as* $f^{-1}\{d, e, f\} = \{d\} \cup \{e\} \cup \{f\}$ *and* $f^{-1}\{a, b, c\} = \{a\} \cup \{b\} \cup \{c\}$*. Here* $\{d, e, f\}$ *and* $\{a, b, c\}$ *are disjoint* $f_{psc}-$ *open sets containing* E *and* F*. Similarly, it is easy to show that for any two* $f_{psc}-$ *closed sets* E *and* F*,* $E \cap F = \phi$*, there exists* $f_{psc}-$ *open sets* G_1 *and* G_2 *such that* $E \subseteq G_1$ *and* $F \subseteq G_2$ *with* $G_1 \cap G_2 = \phi$*. Hence* (X, S_X) *is* $f_{psc}-$ *normal space. [It is easy to see that for any* $f_{psc}-$*closed set* F *contained in any* $f_{psc}-$*open set* G*, there exists an* $f_{psc}-$ *open set* V *such that* $F \subset V \subset \overline{V} \subset G$*].*

Note: Consider the mappings $f : (X, S_X) \to (Y, S_Y), g : (Y, S_Y) \to (Z, S_Z)$ and $g \circ f : (X, S_X) \to (Z, S_Z)$. If f and g are ps-continuous then so is $g \circ f$. Since for every $G \in S_Z, (g \circ f)^{-1}(G) = (f^{-1} \circ g^{-1})(G) = f^{-1}(\bigcup_i H_i) = \bigcup_i f^{-1}(H_i) = \bigcup_i(\bigcup_j M_{ij}), M_{ij} \in S_X$,where $H_i \in S_Y$.

Let $f : (X, S_X) \to (X, S_X)$ an $g : (X, S_X) \to (Y, S_Y)$ be two mappings. Then the mapping g is $f-$quasi continuous if for $G \in S_X, (f^{-1} \circ g^{-1})(G) \subseteq \cup(\bigcap_{\text{finite}} H_i), H_i \in S_Y.[f^{-1} \circ g^{-1} : (Y, S_Y) \to (X, S_X)]$.

For crafting some separation property in an $f_{psc}-$normal space the notion of f-quasi continuity plays an important role which is interestingly enough behind the important characteristic of Uryson's Lemma.

Example 3.3. *Let* $X = \{a, b, c\}$ *and* $Y = \{1, 2, 3\}$*, where* $S_X = \{\{a, b\}, \{b, c\}, \{a\}, \{b\}, \{c\}\}$ *and* $S_Y = \{\{1, 2\}, \{2\}\}$*. Consider a mapping* $f : (X, S_X) \to (Y, S_Y)$ *defined by* $f(a) = 1 = f(b), f(c) = 2$*. Then* $f^{-1}\{1, 2\} = \{a, b\} \cup \{c\}$ *and* $f^{-1}\{2\} = \{a\} \cup \{b\}$*. Here* f *is I-quasi continuous.*

Note : If $g:(X,S_X)\to(Y,S_Y)$ is ps-continuous map then g is I-guasi-continuous where $I:(X,S_X)\to(X,S_X)$ is the identity mapping . As for $G\in S_Y,(I^{-1}\circ g^{-1})(G)=I^{-1}(g^{-1}(G))=g^{-1}(G)=\bigcup_i H_i(H_i\in S_X)$.

Let $f:(X,S_X)\to(Y,S_Y)$ be ps-continuous. Then (X,S_X) is f_c- bounded (/maximal) if there exists a sub collection $C\subseteq S_X$ such that $\bigcup_{G_\alpha\in C} f(G_\alpha)$ is maximal as least upper bound (l.u.b).

(X,S_X) is countably $f-$ordered if for all rational numbers p,q with $p<q$, we get $f_{psc}-$open sets U_p and U_q such that $\overline{U_p}\subset U_q$.

Note: The trivial l.u.b. character of topological space (X,T) appears as an interesting character as what we are discussing here.

Theorem 3.4. *A $f_{psc}-$normal space with f-one-one which is f_c-bounded (/maximal) is countably f-ordered.*

Proof. Let P be the set of all rational numbers in the interval $[0,1]$. We define for each $p\in P$,an $f_{psc}-$open set U_p such that for any p,q with $\overline{U_p}\subset U_q$. Let A,B be two disjoint $f_{psc}-$ closed sets. Now by $f_{psc}-$ normality of (X,S_X), we get an $f_{psc}-$ open set U_0 such that $A\subset U_0\subset\overline{U_0}\subset V_1(V_1=B^c)$. Let P_n denote the set consists of the first n rational numbers in the sequence of all rational numbers of $[0,1]$. Suppose that U_p is defined for all rational numbers $p\in P_n$ satisfying the condition $\overline{U_p}\subset U_q$ whenever $p<q$.

Let r denote the next rational number in the sequence other than that in P_n. Consider the set $P_{n+1}=P_n\cup\{r\}$. It is a finite subset of the interval $[0,1]$and as such it has a simple ordering from the usual ordering relation '$<'$ on the real line. Here 0 is the smallest and 1 is the largest element and r is neither 0 nor 1.

So r has an immediate predecessor P in P_{n+1} an immediate successor of q in P_{n+1}. The set U_p and U_q are already defined and $\overline{U_p}\subset U_q$. Using $f_{psc}-$ normality of the space (X,S_X), we can find an $f_{psc}-$open set U_r of X such that $\overline{U_p}\subset U_r$ and $\overline{U_r}\subset U_q$. Now we extend this definition to all rational p in R defining U_p with $f^{-1}(U_p)=\bigcup_{G_\alpha\in C}G_\alpha, G_\alpha\in S_X$ if $p>1$, $f^{-1}(U_p)=\bigcup_{\alpha\in\wedge}G_\alpha, G_\alpha\in S_X,\wedge=\phi$ if $p<0$. Now, if $p>1, f^{-1}(U_p)=\bigcup_{G_\alpha\in C}G_\alpha$ gives $(f\circ f^{-1})(U_p)=\bigcup_{G_\alpha\in C}f(G_\alpha)$ giving thereby $U_p\supset(f\circ f^{-1})(U_p)=f(\bigcup_{G_\alpha\in C}G_\alpha)$. Thus $U_p=\bigcup_{G_\alpha\in C}f(G_\alpha)$. Now for $q>p>1$ we get $q>1$, and so $U_q=\bigcup_{G_\alpha\in C}f(G_\alpha)$. If $p<0, f^{-1}(U_p)=\bigcup_{\alpha\in\wedge}G_\alpha,\wedge=\phi$. Here also $\overline{U_p}\subset U_q$.

Thus in a f_c- bounded (/maximal) $f_{psc}-$normal space (X,S_X) with an injective f, for any two rationals p,q we get $U_p,U_q-f_{psc}-$open sets such that $\overline{U}_p\subset U_q$ whenever $p<q$. If we consider the collections C of all $f_{psc}-$ open sets U_p, define a relation R as U_pRU_q if an only if $\overline{U_p}\subset U_q$. Then R is a partial order in C.

(1) U_pRU_p since $\overline{U_p}$ is f_{psc}-closed, which gives $\overline{U_p}=U_p$.

(2) U_pRU_q,U_qRU_l, gives $\overline{U_p}\subset U_q,\overline{U_q}\subset U_l$ giving there by $\overline{U_p}\subset U_q,\subset\overline{U_q}\subset U_l$. Thus $\overline{U_p}\subset U_1$.

(3) U_pRU_q,U_qRU_q, gives $\overline{U_p}\subset U_q,\overline{U_q}\subset U_p$ and hence $\overline{U_p}\subset U_q,\subset\overline{U_q}\subset U_p$ giving thereby $\overline{U_p}=U_q=\overline{U_q}=U_p$ and so $U_p=U_q$.

□

Which is explained above is sufficient for motivation towards the countably ordered character of the pseudo structure.

Theorem 3.5. *Let (X, S_X) be a countably f-ordered space. Then for any two rationales $p, q \in [a, b]$ with $p < q$, we get $f_{psc}-$ open sets U_p and U_q with $\overline{U_p} \subset U_q$. Moreover there exists an f-quasi continuous map $g : (X, S_X) \rightarrow ([a, b], S_{[a,b]})$, $S_{[a,b]} = \{(p, q)\}|p, q \in$ rationals of $[a, b]\}$ such that $g^{-1}(p, q) \subseteq U_q \backslash \overline{U_p}$.*

Proof. Since (X, S_X) is a countably f-ordered space, then for each $p, q; p < q$ we get $f_{psc}-$ open sets U_p and U_q with $\overline{U_p} \subset U_q$. Now for each $x \in X$,we define $Q(x)$ to be the set of those rational numbers P such that each of the corresponding $f_{psc}-$open sets contains x, i.e. $Q(x) = \{p|x \in U_p\}$.Define g as $g(x) = \inf Q(x)$. Now we show that

i) $x \in \overline{U_r}$ gives $g(x) \leq r$.

ii) $x \notin \overline{U_r}$ gives $g(x) \geq r$.

i) If $x \in \overline{U_r}$, then $x \in U_s$ for every $s > r$. Therefore $Q(x)$ contains all rational greater than r which gives $g(x) = \inf Q(x) \geq r$.

ii)If $x \notin \overline{U_r}$, then x is not in U_s for any $s < r$. Therefore $Q(x)$ contains all rational less than r, so that $g(x) = inf Q(x) \leq r$.

Now $g(x_0) \in (p, q)$ gives $g(x_0) < q$ which gives $x_0 \in U_q$ and $g(x_0) > q$ gives $x_0 \notin \overline{U_r}$ giving thereby $x_0 \in U_q - \overline{U_p}$. Thus $g^{-1}(p) < x_0 < g^{-1}(q)$ gives $x_0 \in U_q - \overline{U_p}$. Thus $g^{-1}(p, q) \subseteq U_q - \overline{U_p}$ which gives $(f^{-1} \circ g^{-1})(p, q) \subseteq f^{-1}(U_q \cap \overline{U_p^c}) = \cup(G_\alpha \cap H_\beta)$, where $f^{-1}(U_q) = (\cup G_\alpha), G_\alpha \in S_X$ and $f^{-1}(\overline{U_p^c}) = (\cap H_\beta^c), H_\beta \in S_X$. Hence g is f-quasi continuous function. □

Now we are in a position to represent what we are intending for.

Theorem 3.6. *Let (X, S_X) be a $f_{psc}-$ normal space which is f_c-bounded (maximal) and A and B be two disjoint $f_{psc}-$closed subsets of X with f one-one. Let $[a, b]$ be a closed interval in the real line. Then there exists a f-quasi continuous map $G : X \rightarrow [a, b]$ with $g(A) = a$ and $g(B) = b$.*

Proof. Let A, B be two disjoint $f_{psc}-$closed sets. Therefore by $f_{psc}-$normality of (X, S_X), we have an $f_{psc}-$open set U_a such that $A \subset U_a \subset \overline{U_a} \subset U_b (U_b = B^c)$. Now for any two rationals $p, q \in [a, b]$, we get $U_p, U_q - f_{psc}$-Open sets such that $\overline{(U_p)} \subset U_q$ whenever $p < q$ [Theorem 3.4] . Then there exists an f-quasi continuous map $g : (X, S_X) \rightarrow ([a, b], S_{[a,b]})$, $S_{[a,b]} = \{(p, q)\}|p, q \in$ rationals of $[a, b]\}$ with $g(x) = inf.Q(x), Q(x) = \{p|x \in U_p\}$ [Theorem 3.5]. If $x \in A$, then $x \in U_p$ for all $p \geq a$ which gives $g(x) = \inf Q(x) = a$ giving thereby $g(A) = a$. If $x \in B$, then $x \in U_p$ for no $p \leq b$. Thus $Q(x)$ consists of all rationals $> b$ which gives $g(x) = \inf Q(x) = b$ giving thereby $g(B) = b$. □

Here the beauty of our result lies in the sense that the existence of such f-quasi continuous map may happen in so many ways and thus reveals the existensive dimension of the classical Uryson's Lemma.

3.2 Broken two sheets-space

In this section we deal with the case of two sided continuity and co-ordinate wise continuity separately in terms of so-called ps-continuous map in a ps-space in contrast to what has been dealt in Chowdhury et al. [1]. A mapping $f : X \times X \to X$ is ps-continuous if for $W \in S_X$, there exists collections $\{U_\alpha | U_\alpha \in S_X\}$ and $\{V_\beta | V_\beta \in S_X\}$ such that $\cup U_\alpha \times \cup V_\beta = f^{-1}(W)$. Let $W \in S_X$ be such that $f(a,b) \in W$. Then f is ps- continuous at $(a,b) \in X \times X$ if there exists $U_\alpha \in S_X$ and $V_\beta \in S_X$ with $a \in U_\alpha$ and $b \in V_\beta$ such that $(a,b) \in U_\alpha \times V_\beta \subset f^{-1}(W)$.

Now we claim what we intend to get regarding two sided and asymmetric continuity, in terms of so-called ps-continuous maps from $X \times X \to X$ and $X \to X$.

If $(G,+)$ is a topological group, A is a binary operation and T is a topology such that $A : G \times G \to G$ is continuous at (a,b),then the maps ${}_aA : G \to G$ where ${}_aA(x) = a + x$ and $A_b : G \to G$ where $A_b(x) = x + b$ for all $x \in G$ are both topologically continuous at b and a respectively. But the converse may not true. In other words if ${}_aA$ and A_b are both T-continuous then $A : G \times G \to G$ need not be continuous as it follows from what have been explained below.

It is obvious that if f is a ps-continuous map from $X \times X \to X$ at a point say $(a,b) \in X \times X$, then the mappings ${}_af : X \to X$ and $f_b : X \to X$ defined by ${}_af(x) = f(a,x)$ and $f_b(x) = f(x,b)$ are ps-continuous from $X \to X$ at b and a respectively.

At first, it is noticed that the converse of the above appears as false at least when S_X is a topology.

In each of the following examples we observe some characteristics of the binary operation of a group with respect to the respective subclasses of the power set of the group.

Example 3.7. *In the symmetric group* $(S_{S_3}, +)$ *[4, p. 339], let* $S_{S_3} = \{\{a,c\}, \{b,c\}, \{x,y\}\}$. *Here for* $\{x,y\}(\in S_{S_3})$ *containing* $y(= a+b)$ *we have* $\{b,c\}(\in S_{S_3})$ *containing* b *and* $\{a,c\}(\in S_{S_3})$ *containing* a *such that* $a + \{b,c\} = \{x,y\}$ *and* $\{a,c\} + b \subset \{x,y\}$, *but* $\{a,c\} + \{b,c\}(= \{0,x,y\} \not\subset \{x,y\}$.

Example 3.8. *In the group* $(Z_8, +)$ *[4, p. 342] consider* $S_{Z_8} = \{\{2,3\}, \{2,4\}, \{,6,7\}\}$. *Now for* $\{5,6,7\}(\in S_{Z_8})$ *containing* $7(= 3+4)$, *there exists* $\{2,4\}(\in S_{Z_8})$ *containing* 4 *such that* $3 + \{2,4\}(= \{5,7\}) \subset \{5,6,7\}$, *and also there exist* $\{2,3\}(\in S_{Z_8})$ *containing* 3 *such that* $\{2,3\} + \{2,4\}(= \{6,7\}) \subset \{5,6,7\}$ *but* $\{2,3\} + \{2,4\}(= \{4,5,6,7\}) \not\subset \{5,6,7\}$.

Theorem 3.9. *If a mapping* $f : X \times X \to X$ *satisfies the conditions*

a) *For some* $e(\in X), f(e,x) = x = f(x,e)$.

b) $f(x, f(y,z)) = f(f(x,y),z)$, *for all* $x,y,z, \in X$ *along with the following conditions at* $(a,b) \in X \times X$.

i) *For any* $V(\in S_X)$ *containing* e, *we have* ${}_af(V), f_b(V) \in S_X$.

ii) f *is ps-continuous at* (e,e).

iii) f_b *is ps-continuous at* e.

iv) ${}_af$ *is ps-continuous at* b *then* f *is ps-continuous at* (a,b).

Proof. For any $W \in S_X$ containing $f(a,b) =_a f(b)$, we get by (iv) a member $V \in S_X$ with $b \in V$ such that $b \in V \subset_a f^{-1}(W)$. Then there exists an $U \in S_X$ containing e such that $e \in U \subset f_b^{-1}(V)$ [by (a) and (iii)]. Now we get an $A \times B \in S_X \times S_X$ containing (e,e) such that $(e,e) \in A \times B \subset f^{-1}(U)$ [by (a) and (ii)] which gives that $f(e,e) \in f(A \times B) \subset f \circ f^{-1}(U)$. Finally (i) and (b) gives that $f(a,b) \in f({}_af(A) \times f_b(B)) = f(f(\{a\} \times A) \times f(B \times \{b\})) = f(\{a\} \times f(A \times f(B \times \{b\}))) = f(\{a\} \times f(f(A \times B) \times \{b\})) \subseteq f(\{a\} \times f(U \times \{b\})) = f(\{a\} \times f_b(U)) \subset f(\{a\} \times V)) =_a f(V) \subseteq W$. Therefore for $f(a,b) \in W,_a f(A) \times f_b(B) \in S_X \times S_X$ such that $(a,b) \in_a f(A) \times f_b(B) \subseteq f^{-1}(W)$. Hence f is ps-continuous at (a,b). □

Theorem 3.10. *If the two mapping* $f : X \times X \to X$ *and* $g : X \times Y \to Y$ *where* X *is equipped with a binary operation having an identity e (which is also a scalar multiplicative identity with respect to* Y*) and* $g(f(m,n),y) = g(n, g(m,y))$ *for all* $m,n \in X$ *and* $y \in Y$ *satisfying the following conditions (called a two sided S-system) at* $(x,y) \in X \times Y$.

(i) for any $V (\in S_X)$ *containing e we have* ${}_xf(V) \in S_X, g_y(V)(\in S_Y)$.

(ii) f *is ps-continuous at* (e,e).

(iii) g_y *is ps-continuous at* e.

(iv) ${}_xg$ *is ps-continuous at* y *then* g *is ps-continuous at* (x,y).

Proof. For any $W \in S_y$ containing $g(x,y) =_x g(y)$, we get by (iv) a $V \in S_Y$ with $y \in V$ such that $y \in V \subset_x g^{-1}(W)$. Again for e being a scalar multiplicative identity with respect to y, we get an U with $e \in U \subset g_y^{-1}(V)$. Thus we get $A \times B \in S_X \times S_X$ containing (e,e) such that $(e,e) \in A \times B \subset f^{-1}(U)$ [by (i) and (b)]. Finally (a) and (ii) gives that $g(x,y) \in g({}_xf(A) \times g_y(B)) = g(f(\{x\} \times A) \times g(B \times \{y\})) = g(\{x\} \times g(A \times g(B \times \{y\}))) = g(\{x\} \times g(f(A \times B) \times \{y\})) \subseteq g(\{x\} \times g(U \times \{y\})) = g(\{x\} \times g_y(U)) \subset g(\{x\} \times V) =_x g(V) \subseteq W$. Therefore for $g(x,y) \in W$ there exists ${}_xg(A) \times g_y(B) \in S_X \times S_X$ such that $(x,y) \in_x g(A) \times g_y(B) \subseteq g^{-1}(W)$. Hence g is ps-continuous at (x,y). We here present some examples in support of what has been stated above. Let $E = \{0,a,b,c\}$ be the Klein's 4- group [4, p. 339] and consider the mappings on E defined by the following table:

f_0	0	0	0	0
f_1	0	a	b	c
f_2	0	a	b	0
f_3	0	0	b	0
f_4	0	a	0	0
f_5	0	0	b	c
f_6	0	0	0	c
f_7	0	a	0	c

f_8	0	b	0	0
f_9	0	c	0	0
f_{10}	0	c	b	0
f_{11}	0	b	b	0
f_{12}	0	c	b	c
f_{13}	0	b	b	c
f_{14}	0	b	0	c
f_{15}	0	c	0	c

It follows that $N = \{f_0, f_1, \ldots, f_{15}\}$ is right near-ring with unity f_1 with respect to the following operations: $(f+g)(x) = f(x)+g(x)$ and $(f \cdot g)(x) = f(g(x))$ for all $f, g \in N, x \in E$. Here we observe the near-ring group structure of E over N with respect to the operation $N \times E \to E, (f_p, e) \to f_p(e)$. □

Example 3.11. *Consider* $S_N = \{\{f_9\}, \{f_{10}\}, \{f_9, f_{10}\}\}$ *and* $S_E = \{\{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}\}$. *Here for* $\{b,c\}(\in S_E)$ *containing* $c(= f_{10}(a))$, *we have* $\{a,b\}(\in S_E)$ *containing* $a(\in E)$ *and* $\{f_9, f_{10}\}(\in S_N)$ *containing* f_{10} *such that* $f_{10}(\{a,b\}) = \{b,c\}$ *and* $\{f_9, f_{10}\}\{a\}(= \{c\}) \subset \{b,c\}$ *but* $\{f_9, f_{10}\}\{a,b\}(= \{0,b,c\}) \not\subset \{b,c\}$.

Example 3.12. *Consider* $S_N = \{\{f_8\}, \{f_{12}\}, \{f_8, f_{12}\}\}$ *and* $S_E = \{\{a,b\}\{b,c\}\}$. *Here for* $\{b,c\}(\in S_E)$ *containing* $c(= f_{12}(a))$ *we have* $\{a,b\}(\in S_E)$ *containing* $a(\in E)$ *and* $\{f_8, f_{12}\}(\in S_N)$ *containing* $f_{12}(\in N)$ *such that* $\{f_{12}(\{a,b\}) = \{b,c\}$ *and* $\{f_8, f_{12}\}\{a\} = \{b,c\}$ *but* $\{f_8, f_{12}\}\{a,b\}(= \{0,b,c\}) \not\subset \{b,c\}$.

Example 3.13. *Consider* $S_N = \{\{f_9\}, \{f_{10}\}, \{f_{10}, f_1\}, \{f_9, f_{10}\}, \{f_1, f_4\}\}$ *and* $S_E = \{\{a,b\}, \{b,c\}\}$. *Here we note that* $f_{10}\{f_1, f_4\}(= \{f_9, f_{10}\})(\in S_N)$ *but* $\{f_1 f_4\}\{a\}(= \{a\}) \notin S_E; \{f_1, f_4\}\{f_1, f_4\} = \{f_1, f_4\}$; *for* $\{a,b\}(\in S_E)$ *containing* $a(= f_1(a))$ *there exists* $\{f_1, f_4\}(\in S_N)$ *containing* f_1 *such that* $\{f_1, f_4\}\{a\}(= \{a\} \subset \{a,b\})$, *also for* $(b,c)(\in S_E)$ *containing* $c(= f_{10}(a))$, *we get* $f_{10}, (\{a,b\})(= \{b,c\}$ *but* $\{f_9 f_{10}\}\{a,b\}(= \{0,b,c\}) \not\subset \{b,c\}$.

Example 3.14. *Consider* $S_N = \{\{f_1, f_8\}, \{f_8, f_{12}\}\}$ *and* $S_E = \{\{a,b\}, \{b,c\}$. *Here we note that* $f_{12}\{f_1, f_8\}(= \{f_8, f_{12}\})(\in S_N)$ *but* $\{f_1, f_8\}\{a\}(= \{a,b\}) \in S_E$. *Now for any* $W(\in S_N)$ *containing* $f_1(= f_1, f_1)$, *we get no* $U, V(\in S_N)$ *containing* f_1 *such that* $UV \subseteq W$. *Again for* $\{a,b\}(\in S_E)$ *containing* $a(= f_1(a))$, *there exist* $\{f_1, f_8\}(\in S_N)$ *containing* f_1 *such that* $\{f_1, f_8\}\{a\}(= \{a,b\}$. *And we have for* $\{b,c\}(\in S_E)$ *containing* $c(= f_{12}(a))$, *there exists* $\{a,b\}(\in S_E)$ *containing* a *such that* $f_{12}(\{a,b\}) = \{b,c\}$ *but* $\{f_8, f_{12}\}\{a,b\}(= \{0,b,c\}) \not\subset \{b,c\}$.

Example 3.15. $S_N = \{\{f_1, f_{12}\}, \{f_7, f_{15}\}\}$ *and* $S_E = \{\{a,c\}, \{0,a,b\}\}$, *it is seen here that* $f_7\{f_1, f_{12}\}(= \{f_7, f_{15}\}) \in S_N, \{f_1, f_{12}\}\{a\}(= \{a,c\}) \in S_E$; $\{f_1, f_{12}\}\{_1, f_{12}\} = \{f_4, f_{12}\}$; *for* $\{0,a,b\}(\in S_E)$ *containing* $a(= f_1(a))$, *there exists no* $V(\in (S_N)$ *containing* f_1 *such that* $Va \subset \{0,a,b\}$. *Again for* $\{a,c\}, \{0,a,b\}(\in S_E)$ *containing* $a(= f_7(a))$, *we get* $f_7(\{a,c\}))(= \{a,c\})$ *and* $f_7(\{0,a,b\})(= \{0,a\}) \subset \{0,a,b\}$ *but* $\{f_7, f_{15}\}\{a,c\} = \{a,c\}, \{f_7, f_{15}\}\{0,a,b\}(= \{0,a,c\}) \not\subset \{0,a,b\}$.

It can be seen in the above examples that if f is the binary operation (multiplication) on the near-ring N and g is the scalar multiplication on the N-group E defined as in Theorems 3.9 and 3.10 above, the condition (i), (iii) does not hold good in some of the examples. Thus all these justify that we have evaluated.

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