Thai Journal of Mathematics
Volume 12 (2014) Number 2 : 303–311

http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Balance and Antibalance of the Tensor Product of Two Signed Graphs[1]

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Abstract : The *tensor product* $S_1 \otimes S_2$ of two signed graphs S_1 and S_2 is a signed graph with vertex set $V(S_1) \times V(S_2)$ in which two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if u_1 is adjacent to v_1 in S_1 and u_2 is adjacent to v_2 in S_2, and the sign of the edge $(u_1, u_2)(v_1, v_2)$ is the product of the sign of the edges u_1v_1 in S_1 and u_2v_2 in S_2. A signed graph S is said to be a *tensor product signed graph* if $S = S_1 \otimes S_2$. In this paper, we establish a characterization of balanced tensor product signed graphs and antibalanced tensor product signed graphs.

Keywords : signed graph; tensor product of signed graphs; balanced signed graph; antibalanced signed graph.
2010 Mathematics Subject Classification : 05C22; 05C75.

1 Introduction

Germina et al. examined the adjacency and Laplacian matrices and their eigenvalues and energies for the general product (non-complete extended p-sum, or NEPS) of signed graphs in [1], where they expressed the adjacency matrix of the

[1]Research is supported by the Department of Science and Technology (Govt. of India), New Delhi, India under the Project SR/S4/MS: 409/06.
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product in terms of the Kronecker matrix product and the eigenvalues and energy of the product in terms of those of the factor graphs. For the cartesian product they characterized balance but for the tensor product they established only necessity. Independently, we developed a complete structural characterization of balanced and antibalanced tensor product signed graphs, which we present in this paper.

For standard terminology and notation in graph theory we refer Harary [2] and West [3] and for *signed graphs* we refer Zaslavsky [4, 5]. All graphs considered in this paper are assumed to be finite simple graphs.

A signed graph is an ordered pair $S = (S^u, \sigma)$, where S^u is a graph $G = (V, E)$, called the *underlying graph* of S and $\sigma : E \to \{+, -\}$ is a function from the edge set E of S^u into the set $\{+, -\}$, called the *signature* of S. Let $E^+(S) = \{e \in E(G) : \sigma(e) = +\}$ and $E^-(S) = \{e \in E(G) : \sigma(e) = -\}$. The elements of $E^+(S)$ and $E^-(S)$ are called *positive* and *negative* edges of S, respectively. A signed graph is said to be *homogeneous* if all its edges have the same sign and *heterogeneous* otherwise.

A *theta graph* is the union of three internally disjoint simple paths that have the same two distinct end vertices.

The *sign* of a signed subgraph S_1 in a signed graph S is the product of sign of all its edges and is denoted by $\theta(S_1)$. A cycle in a signed graph S is said to be *positive* (*negative*) if its sign is positive (negative). A given signed graph S is said be *balanced* if every cycle in S is positive [6]. A spectral characterization of balanced signed graphs was given by Acharya [7]. A *chord* of a cycle Z in S is an edge not in Z whose end vertices lie in Z. A *chordless cycle* in S is a cycle that has no chord. The following important lemma on balanced signed graph is given by Zaslavsky in [8].

Lemma 1.1 ([8]). *A signed graph in which every chordless cycle is positive, is balanced.*

The *tensor product* $G_1 \otimes G_2$ of two graphs G_1 and G_2 is a graph with vertex set $V(G_1) \times V(G_2)$, in which two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if u_1 is adjacent to v_1 in G_1 and u_2 is adjacent to v_2 in G_2. A graph G is said to be a *tensor product graph* if there exist two graphs G_1 and G_2 such that $G = G_1 \otimes G_2$. The tensor product is variously known as *Kronecker product* [9], *direct product* [10], *categorical product* [11] and *graph conjunction* [12]. Capobianco [13] used the word tensor product for it.

The tensor product of two signed graphs was defined by Mishra in [14]. Let $S = (S^u, \sigma)$ be a signed graph. S is called *tensor product signed graph* of two signed graphs $S_1 = (S_1^u, \sigma_1)$ and $S_2 = (S_2^u, \sigma_2)$, i.e. $S = S_1 \otimes S_2$ if $S^u = S_1^u \otimes S_2^u$ and for an edge $(u_1, u_2)(v_1, v_2)$ of S^u,

$$\sigma((u_1, u_2)(v_1, v_2)) = \sigma_1(u_1 v_1)\sigma_2(u_2 v_2).$$

The tensor product of two signed graphs S_1 and S_2 is shown in Figure 1.

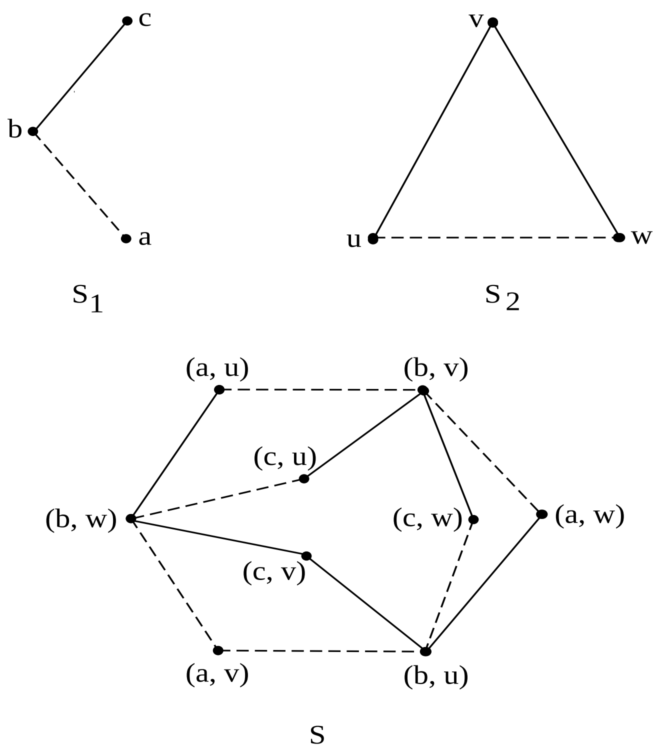

Figure 1: Showing $S = S_1 \otimes S_2$.

2 Balanced Tensor Product Signed Graphs

The following characterization of balanced signed graphs is given by Sampathkumar in [15].

Theorem 2.1 ([15])**.** *A signed graph $S = (S^u, \sigma)$ is balanced if and only if there exists a marking μ of its vertices such that each edge uv in S satisfies $\sigma(uv) = \mu(u)\mu(v)$.*

The *negation* $\eta(S)$ of a signed graph S is a signed graph obtained from S by negating the sign of every edge of S. A signed graph S is said to be *antibalanced* if $\eta(S)$ is balanced. $\sigma_\eta(uv)$ gives the sign of an edge $uv \in E(\eta(S))$. We have following lemma:

Lemma 2.2. *A signed graph $S = (S^u, \sigma)$ is antibalanced if and only if there exists a marking μ of its vertices such that each edge uv in S satisfies $\sigma(uv) = -\mu(u)\mu(v)$.*

Proof. Suppose in a signed graph S, there exists a marking μ of its vertices such that

$$\begin{aligned}&\sigma(uv) = -\mu(u)\mu(v)\ \forall uv \in E(S)\\ &\Leftrightarrow \mu(u)\mu(v) = -\sigma(uv)\ \forall uv \in E(S)\\ &\Leftrightarrow \mu(u)\mu(v) = \sigma_\eta(uv)\ \forall uv \in E(\eta(S))\\ &\Leftrightarrow \eta(S) \text{ is balanced } \Leftrightarrow S \text{ is antibalanced.}\end{aligned}$$

□

Lemma 2.3. *The tensor product signed graph $S = C_{2m} \otimes K_2$ of an even signed cycle C_{2m} and K_2 is the disjoint union of two even signed cycles C'_{2m} and C''_{2m}, and $\theta(C_{2m}) = \theta(C'_{2m}) = \theta(C''_{2m})$.*

Proof. Let $C_{2m} = u_1u_2\cdots u_{2m}u_1$ and $K_2 = (\{x,y\},\sigma_2)$. Then, $S = C_{2m} \otimes K_2$ is the disjoint union of two even cycles,

$$C'_{2m} = (u_1,x)(u_2,y)(u_3,x)(u_4,y)\ldots(u_{2m},y)(u_1,x)$$

and

$$C''_{2m} = (u_1,y)(u_2,x)(u_3,y)(u_4,x)\cdots(u_{2m},x)(u_1,y).$$

Now,

$$\theta(C'_{2m}) = \theta(C_{2m})(\sigma_2(xy))^{2m} = \theta(C_{2m}) \tag{2.1}$$

and

$$\theta(C''_{2m}) = \theta(C_{2m})(\sigma_2(xy))^{2m} = \theta(C_{2m}). \tag{2.2}$$

Thus, using equations (2.1) and (2.2), $\theta(C_{2m}) = \theta(C'_{2m}) = \theta(C''_{2m})$. □

Lemma 2.4. *The tensor product signed graph $S = C_{2m+1} \otimes C_{2n+1}$ of two odd signed cycles C_{2m+1} and C_{2n+1} contains an odd signed cycle C of length $p = lcm(2m+1, 2n+1)$ such that $\theta(C) = \theta(C_{2m+1})\theta(C_{2n+1})$.*

Proof. Let $C_{2m+1} = u_1u_2\cdots u_{2m+1}u_1$ and $C_{2n+1} = v_1v_2\cdots v_{2n+1}v_1$. Without loss of generality, suppose $m \leq n$. Then, $S = C_{2m+1} \otimes C_{2n+1}$ contains an odd cycle,

$$C = (u_1,v_1)(u_2,v_2)\cdots(u_{2m+1},v_{2m+1})(u_1,v_{2m+2})\cdots(u_{2m+1},v_{2n+1})(u_1,v_1)$$

of length $p = \mathrm{lcm}(2m+1, 2n+1)$. Now,

$$\theta(C) = (\theta(C_{2m+1}))^{p/(2m+1)}(\theta(C_{2n+1}))^{p/(2n+1)}.$$

Since $p/(2m+1)$ and $p/(2n+1)$ are odd numbers, $\theta(C) = \theta(C_{2m+1})\theta(C_{2n+1})$. □

Lemma 2.5. *If the tensor product signed graph $S = S_1 \otimes K_2$ of two connected signed graphs $S_1 = (S_1^u, \sigma_1)$ and $K_2 = (\{x,y\},\sigma_2)$ is balanced or antibalanced, then all the odd cycles in S_1 have the same sign.*

Proof. Let $C_1 = u_1u_2\cdots u_{2m+1}u_1$ and $C_2 = v_1v_2\cdots v_{2n+1}v_1$ are two arbitrary odd cycles in S_1. First, we prove that certain pairs have the same sign.

Case (i). Suppose C_1 and C_2 are disjoint cycles. Since S_1 is connected, there is a minimum connecting path $P = u_1w_1w_2\cdots w_kv_1$ between C_1 and C_2. $S = S_1 \otimes K_2$ contains a cycle,

$$C = (u_1,x)(u_2,y)\ldots(u_{2m+1},x)(u_1,y)(w_1,x)(w_2,y)\cdots(w_k,x')(v_1,y')$$
$$(v_2,x')\cdots(v_{2n+1},y')(v_1,x')(w_k,y')\ldots(w_2,x)(w_1,y)(u_1,x),$$

where $(x',y') = (x,y)$ if k is odd and $(x',y') = (y,x)$ if k is even. Now,

$$\theta(C) = \theta(C_1)(\sigma_2(xy))^{2m+1}\theta(P)(\sigma_2(xy))^{k+1}\theta(C_2)(\sigma_2(xy))^{2n+1}\theta(P)(\sigma_2(xy))^{k+1}$$
$$= \theta(C_1)\theta(C_2).$$

The length of the cycle C is even because C_1 and C_2 are odd. Since S is balanced or antibalanced, $\theta(C) = +$. Thus, $\theta(C_1) = \theta(C_2)$.

Case (ii). Suppose C_1 and C_2 are disjoint cycles except for a single common vertex. This is similar to Case (i) except that P is reduced to a single vertex $u_1 = v_1$ and $(x',y') = (x,y)$.

Case (iii). Suppose the intersection of C_1 and C_2 is a path P'. Then $C_1 \cup C_2$ is a theta graph, consisting of three internally disjoint paths, $P' = aw_1w_2\cdots w_kb$, $P_1 = au_1u_2\cdots u_mb$ and $P_2 = av_1v_2\cdots v_nb$, between two vertices a, b and we have $C_1 = P_1 \cup P'$ and $C_2 = P_2 \cup P'$. Clearly, m and n have the same parity but different parity from k. Also, $\theta(C_i) = \theta(P_i)\theta(P')$ for $i = 1, 2$. Clearly, $S = S_1 \otimes K_2$ contains a cycle

$$C' = (a,x)(u_1,y)\cdots(u_m,y')(b,x')(w_k,y')\cdots(w_1,x)(a,y)$$
$$(v_1,x)\cdots(v_n,x')(b,y')(w_k,x')\cdots(w_1,y)(a,x),$$

where $(x',y') = (x,y)$ if m, n are odd and $(x',y') = (y,x)$ if m, n are even. Now,

$$\theta(C') = \theta(P_1)(\sigma_2(xy))^{m+1}\theta(P')(\sigma_2(xy))^{k+1}\theta(P_2)(\sigma_2(xy))^{n+1}\theta(P')(\sigma_2(xy))^{k+1}$$
$$= \theta(P_1)\theta(P_2)(\sigma_2(xy))^{m+n+2k+4}$$
$$= \theta(C_1)\theta(C_2)$$

because m and n have the same parity. For the same reason the length of the cycle C' is even. Since S is balanced or antibalanced, $\theta(C') = +$. Thus, $\theta(C_1) = \theta(C_2)$.

Now, we prove that all pairs have the same sign. Cases (i) and (ii) imply that any two odd cycles in different blocks have the same sign. It follows that, if S_1 has two or more non-bipartite blocks, then every odd cycle corresponding to different blocks in S_1 has the same sign.

Assume that only one block of S_1 has an odd cycle. Let C'' and C''' be odd cycles in one block of S_1. By Tutte's Path Theorem [16], there is a chain of cycles,

$C'' = C_1, C_2, C_3, \ldots, C_n = C'''$, such that $C_i \cap C_{i+1}$ is a path of length at least one for all $i = 1, 2, \ldots, n-1$. By Case (iii),

$$\theta(C'') = \theta(C_1) = \theta(C_2) = \cdots = \theta(C_n) = \theta(C''').$$

Thus, every odd cycle in S_1 has the same sign. That concludes the proof of the lemma. □

Theorem 2.6. *Let $S_1 = (S_1^u, \sigma_1)$ and $S_2 = (S_2^u, \sigma_2)$ be two connected signed graphs of order at least 2. The tensor product signed graph $S = S_1 \otimes S_2 = (S_1^u \otimes S_2^u, \sigma)$ is balanced if and only if S_1 and S_2 are both balanced or both antibalanced.*

Proof. Necessity: Suppose $S = S_1 \otimes S_2$ is balanced. We shall show that S_1 and S_2 are both balanced or both antibalanced. If S_1 and S_2 have no cycle, then trivially they are both balanced or both antibalanced. Again, if S_1 or S_2 has at least a cycle, then we consider the following three cases:

Case (i). Let S_1^u and S_2^u be both bipartite. That means, there does not exist odd cycle in S_1 and S_2. If S_1 or S_2 contains a negative even cycle, then due to Lemma 2.3, S also contains a negative even cycle. It implies that S is not balanced, a contradiction to the assumption. Thus, all the even cycles in S_1 and S_2 are positive. Hence S_1 and S_2 are both balanced or both antibalanced.

Case (ii). Suppose S_1^u and S_2^u are both non-bipartite. That means, S_1 and S_2 both contain odd cycles. Due to Lemma 2.4, balance of S implies that every odd cycle in S_1 and S_2 has the same sign. That means, all the odd cycles in S_1 and S_2 are positive or negative and by Case (i), all the even cycles in S_1 and S_2 are positive. Thus, S_1 and S_2 are both balanced or both antibalanced.

Case (iii). Without loss of generality, suppose S_1^u is non-bipartite and S_2^u is bipartite. That means, S_1 contains odd cycles and all the cycles in S_2 are of even length. We solve this case by using $S_1 \otimes K_2$, where $K_2 \subseteq S_2$. Since $S = S_1 \otimes S_2$ is balanced, $S_1 \otimes K_2$ is also balanced. So solving $S_2 = K_2$ suffices. Due to Lemma 2.5, all the odd cycles in S_1 have the same sign. That means, all the odd cycles in S_1 are positive or negative and by Case (i), all the even cycles in S_1 and S_2 are positive. Thus, S_1 and S_2 are both balanced or both antibalanced. Hence, above three cases complete the proof of necessity.

Sufficiency: Suppose S_1 and S_2 are both balanced or both antibalanced. We shall show that $S = S_1 \otimes S_2 = (S_1^u \otimes S_2^u, \sigma)$ is balanced.

Case (i). Suppose S_1 and S_2 are both balanced. Then, all the cycles contained in S_1 and S_2 are positive. Therefore due to Theorem 2.1, there exists a marking μ_1 in S_1 such that for each edge $u_i u_j$ in S_1,

$$\sigma_1(u_i u_j) = s\mu_1(u_i)\mu_1(u_j)$$

and there exists a marking μ_2 in S_2 such that for each edge $v_k v_l$ in S_2,

$$\sigma_2(v_k v_l) = s\mu_2(v_k)\mu_2(v_l),$$

where $s = +1$. Now, we choose a marking μ in $S = S_1 \otimes S_2$ such that,

$$\mu(u_i, v_k) = \mu_1(u_i)\mu_2(v_k).$$

Suppose $(u_i, v_k)(u_j, v_l)$ is an arbitrary edge in S. Then,

$$\begin{aligned}\sigma((u_i, v_k)(u_j, v_l)) &= \sigma_1(u_i u_j)\sigma_2(v_k v_l)\\ &= (s\mu_1(u_i)\mu_1(u_j))(s\mu_2(v_k)\mu_2(v_l))\\ &= s^2(\mu_1(u_i)\mu_2(v_k))(\mu_1(u_j)\mu_2(v_l))\\ &= \mu(u_i, v_k)\mu(u_j, v_l).\end{aligned}$$

Thus, there exists a marking μ in S such that each edge $(u_i, v_k)(u_j, v_l)$ in S satisfies $\sigma((u_i, v_k)(u_j, v_l)) = \mu(u_i, v_k)\mu(u_j, v_l)$. Hence due to Theorem 2.1, $S = S_1 \otimes S_2$ is balanced.

Case (ii). Suppose S_1 and S_2 are both antibalanced. Then, using Lemma 2.2 and taking $s = -1$ in Case (i), we can prove this case in similar manner. Hence the theorem. □

Using Lemma 1.1, above theorem can be expressed equivalently as:

Corollary 2.7. *Let $S_1 = (S_1^u, \sigma_1)$ and $S_2 = (S_2^u, \sigma_2)$ be two connected signed graphs of order at least 2. The tensor product signed graph $S = S_1 \otimes S_2 = (S_1^u \otimes S_2^u, \sigma)$ is balanced if and only if the following conditions hold:*

(i) *all the chordless even cycles contained in S_1 and in S_2 are positive and*

(ii) *all the chordless odd cycles contained in S_1 and in S_2 are of the same sign.*

Corollary 2.8. *The tensor product $S_1 \otimes T$ of a connected signed graph S_1 and a signed tree T is balanced if and only if S_1 is balanced or antibalanced.*

If $\psi(G)$ denotes the set of all signed graphs whose underlying graph is G, then we have the following corollaries:

Corollary 2.9. *The tensor product $C_{2m} \otimes C_{2n}$ of two even cycles $C_{2m} \in \psi(C_{2m})$ and $C_{2n} \in \psi(C_{2n})$ is balanced if and only if C_{2m} and C_{2n} are both positive.*

Corollary 2.10. *The tensor product $C_{2m+1} \otimes C_{2n+1}$ of two odd cycles $C_{2m+1} \in \psi(C_{2m+1})$ and $C_{2n+1} \in \psi(C_{2n+1})$ is balanced if and only if C_{2m+1} and C_{2n+1} are both of the same sign.*

Corollary 2.11. *The tensor product $C_{2m} \otimes C_{2n+1}$ of an even cycle $C_{2m} \in \psi(C_{2m})$ and an odd cycle $C_{2n+1} \in \psi(C_{2n+1})$ is balanced if and only if C_{2m} is positive.*

3 Antibalanced Tensor Product Signed Graphs

Theorem 3.1. *Let $S_1 = (S_1^u, \sigma_1)$ and $S_2 = (S_2^u, \sigma_2)$ be two connected signed graphs of order at least 2. The tensor product signed graph $S = S_1 \otimes S_2 = (S_1^u \otimes S_2^u, \sigma)$ is antibalanced if and only if one of S_1 and S_2 is balanced and the other is antibalanced.*

Proof. Necessity: Suppose $S = S_1 \otimes S_2$ is antibalanced. That means, $\eta(S)$ is balanced. Clearly, $\eta(S) = \eta(S_1) \otimes S_2$. Now due to Theorem 2.6, $\eta(S_1)$ and S_2 are both balanced or both antibalanced. If $\eta(S_1)$ and S_2 are both balanced, then $\eta(\eta(S_1)) = S_1$ is antibalanced and S_2 is balanced. Again, if $\eta(S_1)$ and S_2 are both antibalanced, then $\eta(\eta(S_1)) = S_1$ is balanced and S_2 is antibalanced.

Sufficiency: Suppose one of S_1 and S_2 is balanced and the other is antibalanced. We shall show that $S = S_1 \otimes S_2$ is antibalanced. Without loss of generality, let S_1 be balanced and S_2 be antibalanced. That means, S_1 and $\eta(S_2)$ are both balanced. Now due to Theorem 2.6, $S_1 \otimes \eta(S_2) = \eta(S)$ is balanced. It follows that S is antibalanced. Hence the theorem. □

4 Conclusion

In this paper, we have established a characterization of balanced and antibalanced tensor product signed graphs. In view of main result, it become interesting to investigate a characterization of balanced and antibalanced tensor product signed graph $S = S_1 \otimes S_2 \otimes \ldots \otimes S_n$. We strongly believe that our characterization of balanced and antibalanced tensor product signed graph for $n = 2$ would work for general n, also. The problem will be taken elsewhere.

Acknowledgements : The authors express their gratitude to E. Sampathkumar and V. Mishra who in their early research have brought up the idea of tensor product of the particular graphs which on reading gave us an instant insight for further research in tensor product of two graphs and signed graphs. Also, we are deeply indebted to the referees who made extensive and constructively critical comments on the first version of the paper.

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(Received 26 June 2011)
(Accepted 22 March 2013)