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Strong Convergence Theorems by Hybrid Block Generalized f-Projection Method for Fixed Point Problems of Asymptotically Quasi-ϕ-Nonexpansive Mappings and System of Generalized Mixed Equilibrium Problems[1]

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Abstract : The purpose of this paper is to present a new hybrid block iterative scheme by the generalized f-projection method for finding a common element of the fixed point set for a countable family of uniformly asymptotically quasi-ϕ-nonexpansive mappings and the set of solutions of the system of generalized mixed equilibrium problems in a strictly convex and uniformly smooth Banach space with the Kadec-Klee property. Furthermore, we prove that our new hybrid block iterative scheme converges strongly to a common element of the afore mentioned sets. The results presented in this paper improve and extend important recent results in the literature.

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1 Introduction

Let E be a Banach space with it's dual space E^* and C be a nonempty closed convex subset of E. It is well known that the metric projection operator $P_C : E \to C$ plays an important role in nonlinear functional analysis, optimization theory, fixed point theory, nonlinear programming, game theory, variational inequality, and complementarity problems, etc. (see, for example, [1, 2] and the references therein). In 1994, Alber [3] introduced and studied the generalized projections $\pi_C : E^* \to C$ and $\Pi_E : E \to C$ from Hilbert spaces to uniformly convex and uniformly smooth Banach spaces. Moreover Alber [1] presented some applications of the generalized projections to approximately solving variational inequalities and von Neumann intersection problem in Banach spaces. In 2005, Li [2] extended the generalized projection operator from uniformly convex and uniformly smooth Banach spaces to reflexive Banach spaces and studied some properties of the generalized projection operator with applications to solving the variational inequality in Banach spaces. Later, Wu and Huang [4] introduced a new generalized f-projection operator in Banach spaces. They extended the definition of the generalized projection operators introduced by Abler [3] and proved some properties of the generalized f-projection operator. In 2009, Fan et al. [5] presented some basic results for the generalized f-projection operator, and discussed the existence of solutions and approximation of the solutions for generalized variational inequalities in noncompact subsets of Banach spaces.

In 2008, Plubtieng and Ungchittrakool [6] established strong convergence theorems of block iterative methods for a finite family of relatively nonexpansive mappings in a Banach space by using the hybrid method in mathematical programming. Block iterative method is a method which often used by many authors to solve the *convex feasibility problem* (CFP) (see, [7, 8], etc.). In 2010, Saewan and Kumam [9] introduced a new modified block hybrid projection algorithm for finding a common element of the set of solutions of the generalized equilibrium problems and the set of common fixed points of an infinite family of closed and uniformly quasi-ϕ-asymptotically nonexpansive mappings in a uniformly smooth and strictly convex Banach space E with Kadec-Klee property.

On the other hand, let $\{\theta_i\}_{i\in\Gamma} : C \times C \to \mathbb{R}$ be a bifunction, $\{\varphi_i\}_{i\in\Gamma} : C \to \mathbb{R}$ be a real-valued function, and $\{A_i\}_{i\in\Gamma} : C \to E^*$ be a monotone mapping, where Γ is an arbitrary index set. The *system of generalized mixed equilibrium problems*, is to find $x \in C$ such that

$$\theta_i(x, y) + \langle A_i x, y - x\rangle + \varphi_i(y) - \varphi_i(x) \geq 0, \quad i \in \Gamma, \quad \forall y \in C. \tag{1.1}$$

If Γ is a singleton, then problem (1.1) reduces to the *generalized mixed equilibrium* *problem*, is to find $x \in C$ such that

$$\theta(x, y) + \langle Ax, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.2}$$

The set of solutions (1.2) is denoted by $GMEP(\theta, A, \varphi)$, i.e.,

$$GMEP(\theta, A, \varphi) = \{x \in C : \theta(x, y) + \langle Ax, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C\}. \tag{1.3}$$

If $A \equiv 0$, the problem (1.2) reduces into the *mixed equilibrium problem for* θ, denoted by $MEP(\theta, \varphi)$, is to find $x \in C$ such that

$$\theta(x, y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.4}$$

If $\theta \equiv 0$, the problem (1.2) reduces into the *mixed variational inequality* of Browder type, denoted by $VI(C, A, \varphi)$, is to find $x \in C$ such that

$$\langle Ax, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.5}$$

If $A \equiv 0$ and $\varphi \equiv 0$ the problem (1.2) reduces into the *equilibrium problem for* θ, denoted by $EP(\theta)$, is to find $x \in C$ such that

$$\theta(x, y) \geq 0, \quad \forall y \in C. \tag{1.6}$$

If $\theta \equiv 0$, the problem (1.4) reduces into the *minimize problem*, denoted by $Argmin(\varphi)$, is to find $x \in C$ such that

$$\varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.7}$$

The above formulation (1.5) was shown in [10] to cover monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, optimization problems, variational inequality problems, vector equilibrium problems, Nash equilibria in noncooperative games. In other words, the $EP(\theta)$ is an unifying model for several problems arising in physics, engineering, science, optimization, economics, etc. Some solution methods have been proposed to solve the $EP(\theta)$; see, for example, [10–15] and references therein.

A point $x \in C$ is a *fixed point* of a mapping $S : C \to C$ if $Sx = x$. Denote by $F(S)$ is the set of fixed points of S; that is, $F(S) = \{x \in C : Sx = x\}$. Recall that S is said to be *nonexpansive* if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

S is said to be *quasi-nonexpansive* if $F(S) \neq \emptyset$ and

$$\|x - Sy\| \leq \|x - y\|, \quad \forall x \in F(S),\ y \in C.$$

S is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|S^n x - S^n y\| \leq k_n \|x - y\|, \quad \forall x, y \in C, \quad \forall n \geq 1.$$

S is said to be *asymptotically quasi-nonexpansive* if $F(S) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|x - S^n y\| \leq k_n \|x - y\|, \quad \forall x \in F(S), y \in C, \quad \forall n \geq 1.$$

Recall that a point p in C is said to be an *asymptotic fixed point* of S [16] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$. The set of asymptotic fixed points of S will be denoted by $\widetilde{F}(S)$.

Let E be a real Banach space with norm $\|\cdot\|$, C be a nonempty closed convex subset of E and let E^* denote the dual of E. Let $\langle \cdot, \cdot \rangle$ denote the duality pairing of E^* and E. If E is a Hilbert space $\langle \cdot, \cdot \rangle$ denotes an inner product on E. Consider the functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \text{for } x, y \in E, \tag{1.8}$$

where $J : E \to 2^{E^*}$ is the *normalized duality mapping*.

A mapping S from C into itself is said to be *relatively nonexpansive* [17–19] if $\widetilde{F}(S) = F(S) \neq \emptyset$ and

$$\phi(p, Sx) \leq \phi(p, x) \quad \forall x \in C \text{ and } p \in F(S).$$

S is said to be *relatively asymptotically nonexpansive* [20] if $\widetilde{F}(S) = F(S) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\phi(p, S^n x) \leq k_n \phi(p, x), \quad \forall x \in C,\ p \in F(S) \quad \text{and} \quad n \geq 1.$$

The asymptotic behavior of a relatively nonexpansive mapping was studied in [21–23].
S is said to be *ϕ-nonexpansive* if

$$\phi(Sx, Sy) \leq \phi(x, y) \quad \forall x, y \in C.$$

S is said to be *quasi-ϕ-nonexpansive* [14, 24, 25] if $F(S) \neq \emptyset$ and

$$\phi(p, Sx) \leq \phi(p, x) \quad \forall x \in C \text{ and } p \in F(S).$$

S is said to be *ϕ-asymptotically nonexpansive* if there exists a real sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\phi(S^n x, S^n y) \leq k_n \phi(x, y) \quad \forall x, y \in C.$$

S is said to be *quasi-ϕ-asymptotically nonexpansive* [25, 26] if $F(S) \neq \emptyset$ and there exists a real sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\phi(p, S^n x) \leq k_n \phi(p, x) \quad \forall x \in C,\ p \in F(S) \quad \text{and} \quad n \geq 1.$$

S is said to be *totally quasi-ϕ-asymptotically nonexpansive*, if $F(S) \neq \emptyset$ and there exist nonnegative real sequences ν_n, μ_n with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(0) = 0$ such that

$$\phi(p, S^n x) \leq \phi(p, x) + \nu_n \varphi(\phi(p, x)) + \mu_n, \ \forall n \geq 1, \ \forall x \in C, \ p \in F(S).$$

A mapping S is said to be *closed* if for any sequence $\{x_n\} \subset C$ with $x_n \to x$ and $Sx_n \to y$, then $Sx = y$.

Remark 1.1. *It is easy to know that each relatively nonexpansive mapping is closed. The class of quasi-ϕ-asymptotically nonexpansive mappings contains properly the class of quasi-ϕ-nonexpansive mappings as a subclass and the class of quasi-ϕ-nonexpansive mappings contains properly the class of relatively nonexpansive mappings as a subclass, but the converse may be not true (see more detail [21–23, 27]).*

As well know that if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [1] recently introduced a generalized projection Π_C from E in to C as follows:

$$\Pi_C(x) = arg \min_{y \in C} \phi(y, x), \ \ \forall x \in E. \tag{1.9}$$

It is obvious from the definition of function ϕ that

$$(\|y\| - \|x\|)^2 \leq \phi(y, x) \leq (\|y\| + \|x\|)^2, \quad \forall x, y \in E. \tag{1.10}$$

If E is a Hilbert space, then $\phi(y, x) = ||y - x\|^2$ and Π_C becomes the metric projection of E onto C. The *generalized projection* $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \inf_{y \in C} \phi(y, x). \tag{1.11}$$

The existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(y, x)$ and strict monotonicity of the mapping J (see, for example, [1, 28–31]).

Next we recall the concept of the generalized f-projection operator. Let $G : C \times E^* \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a functional defined as follows:

$$G(\xi, \varpi) = \|\xi\|^2 - 2\langle \xi, \varpi \rangle + \|\varpi\|^2 + 2\rho f(\xi), \tag{1.12}$$

where $\xi \in C$, $\varpi \in E^*$, ρ is positive number and $f : C \to \mathbb{R} \cup \{+\infty\}$ is proper, convex and lower semicontinuous. From definitions of G and f, it is easy to see the following properties:

1. $G(\xi, \varpi)$ is convex and continuous with respect to ϖ when ξ is fixed;

2. $G(\xi, \varpi)$ is convex and lower semicontinuous with respect to ξ when ϖ is fixed.

Let E be a real Banach space with its dual E^*. Let C be a nonempty closed convex subset of E. We say that $\Pi_C^f : E^* \to 2^C$ is *generalized f-projection operator* if

$$\Pi_C^f \varpi = \{u \in C : G(u, \varpi) = \inf_{\xi \in C} G(\xi, \varpi)\}, \ \forall \varpi \in E^*.$$

In 2005, Matsushita and Takahashi [27] proposed the following hybrid iteration method (it is also called the CQ method) with generalized projection for relatively nonexpansive mapping S in a Banach space E:

$$\begin{cases} x_0 \in C \ \text{ chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JSx_n), \\ C_n = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0. \end{cases} \tag{1.13}$$

They proved that $\{x_n\}$ converges strongly to $\Pi_{F(S)} x_0$, where $\Pi_{F(S)}$ is the generalized projection from C onto $F(S)$.

Recently, motivated by the results of Takahashi and Zembayashi [32], Cholumjiak and Suantai [11] proved the following strong convergence theorem by the hybrid iterative scheme for approximation of common fixed point of countable families of relatively quasi-nonexpansive mappings in a uniformly convex and uniformly smooth Banach space: $x_0 \in E$, $x_1 = \Pi_{C_1} x_0$, $C_1 = C$

$$\begin{cases} y_{n,i} = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JSx_n), \\ u_{n,i} = T^{\theta_m}_{r_{m,n}} T^{\theta_{m-1}}_{r_{m-1,n}} \cdots T^{\theta_1}_{r_{1,n}} y_{n,i}, \\ C_{n+1} = \{z \in C_n : \sup_{i>1} \phi(z, Ju_{n,i}) \leq \phi(z, Jx_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, n \geq 1. \end{cases} \tag{1.14}$$

Then, they proved that under certain appropriate conditions imposed on $\{\alpha_n\}$, and $\{r_{n,i}\}$, the sequence $\{x_n\}$ converges strongly to $\Pi_{C_{n+1}} x_0$.

Very recently, Li et al. [33] introduced the following hybrid iterative scheme for approximation of fixed point of relatively nonexpansive mapping using the properties of generalized f-projection operator in a uniformly smooth real Banach space which is also uniformly convex: $x_0 \in C$,

$$\begin{cases} y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JSx_n), \\ C_{n+1} = \{w \in C_n : G(w, Jy_n) \leq G(w, Jx_n)\}, \\ x_{n+1} = \Pi^f_{C_{n+1}} x_0, n \geq 0. \end{cases} \tag{1.15}$$

They proved a strong convergence theorem for finding an element in the fixed point set of S. We remark here that the results of Li et al. [33] extended and improved on the results of Matsushita and Takahashi [27].

In 2008, Plubtieng and Ungchittrakool [6] established strong convergence theorems of block iterative methods for a finite family of relatively nonexpansive mappings in a Banach space by using the hybrid method in mathematical programming. Block iterative method is a method which often used by many authors to solve the *convex feasibility problem* (CFP) (see, [7, 8], etc.). In 2010, Saewan and Kumam [9] introduced a new modified block hybrid projection algorithm for finding a common element of the set of solutions of the generalized equilibrium problems and the set of common fixed points of an infinite family of closed and uniformly quasi-ϕ-asymptotically nonexpansive mappings in a uniformly smooth and strictly convex Banach space E with Kadec-Klee property.

Very recently, Shehu [34], introduced a new iterative scheme by hybrid methods and prove strong convergence theorem for approximation of a common fixed point of two countable families of weak relatively nonexpansive mappings which is also a solution to a system of generalized mixed equilibrium problems in a uniformly convex real Banach space which is also uniformly smooth using the properties of the generalized f-projection operator. Chang et al. [35] used the modified block iterative method to propose an iterative algorithm for solving the convex feasibility problems for an infinite family of asymptotically quasi-ϕ-nonexpansive mappings. Kim [36] used the hybrid projection method for finding a common element in the fixed point set of an asymptotically quasi-ϕ-nonexpansive mapping and in the solution set of an equilibrium problem.

In this paper, motivated and inspired by the work mentioned above, we introduce a new hybrid block iterative scheme of the generalized f-projection operator for finding a common element of the fixed point set of uniformly quasi-ϕ-asymptotically nonexpansive mappings and the set of solutions of the system of generalized mixed equilibrium problems in a uniformly smooth and strictly convex Banach space with the Kadec- Klee property. Moreover, we prove that our new iterative scheme converges strongly to a common element of the afore mentioned sets. The results presented in this paper improve and extend the results of Shehu [34], Chang et al. [35], Li et al. [33], Takahashi and Zembayashi [32], Kim [36] and many author.

2 Preliminaries

A Banach space E is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then a Banach space E is said to be *smooth* if the limit $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in U$. It is also said to be *uniformly smooth* if the limit exists uniformly in $x, y \in U$. Let E be a Banach space. The *modulus of smoothness* of E is the function $\rho_E : [0,\infty) \to [0,\infty)$ defined by

$$\rho_E(t) = \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| \leq t\right\}.$$

The *modulus of convexity* of E is the function $\delta_E : [0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) = \inf\left\{1 - \|\frac{x+y}{2}\| : x, y \in E, \|x\| = \|y\| = 1, \|x-y\| \geq \varepsilon\right\}.$$

The *normalized duality mapping* $J : E \to 2^{E^*}$ is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2, \|x^*\| = \|x\|\}.$$

If E is a Hilbert space, then $J = I$, where I is the identity mapping.

Remark 2.1. *If E is a reflexive, strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x, y) = 0$ then $x = y$. From (1.8), we have $\|x\| = \|y\|$. This implies that $\langle x, Jy \rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, one has $Jx = Jy$. Therefore, we have $x = y$; see [29, 31] for more details.*

Recall that a Banach space E has the Kadec-Klee property [29, 31, 37], if for any sequence $\{x_n\} \subset E$ and $x \in E$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$ as $n \to \infty$. It is well known that if E is a uniformly convex Banach space, then E has the Kadec-Klee property.

Remark 2.2. *Let E be a Banach space, it is also known that*

1. *if E is an arbitrary Banach space, then J is monotone and bounded;*
2. *if E is a strictly convex, then J is strictly monotone;*
3. *if E is a smooth, then J is single valued and semi-continuous;*
4. *if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E;*
5. *if E is reflexive, smooth and strictly convex, then the normalized duality mapping $J = J_2$ is single valued, one-to-one and onto;*
6. *if E is uniformly smooth, then E is smooth and reflexive;*
7. *E is uniformly smooth if and only if E^* is uniformly convex;*

see [29] for more details.

We also need the following lemmas for the proof of our main results.

For solving the equilibrium problem for a bifunction $\theta : C \times C \to \mathbb{R}$, let us assume that θ satisfies the following conditions:

(A1) $\theta(x, x) = 0$ for all $x \in C$;

(A2) θ is monotone, i.e., $\theta(x, y) + \theta(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\lim_{t \downarrow 0} \theta(tz + (1 - t)x, y) \leq \theta(x, y);$$

(A4) for each $x \in C$, $y \mapsto \theta(x, y)$ is convex and lower semi-continuous.

For example, let A be a continuous and monotone operator of C into E^* and define

$$\theta(x,y) = \langle Ax, y-x\rangle, \forall x,y \in C.$$

Then, θ satisfies (A1)-(A4). The following result is in Blum and Oettli [10].

Motivated by Combettes and Hirstoaga[12] in a Hilbert space and Takahashi and Zembayashi [38] in a Banach space, Zhang [39] obtain the following lemma.

Lemma 2.3 (Zhang [39, Lemma 1.5], Liu et al. [40]). *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E. Assume that θ be a bifunction from $C\times C$ to $\mathbb{R}$ satisfying (A1)-(A4), $A : C \to E^*$ be a continuous and monotone mapping and $\varphi : C \to \mathbb{R}$ be a semicontinuous and convex functional. For $r > 0$ and let $x \in E$. Then, there exists $z \in C$ such that*

$$Q(z,y) + \frac{1}{r}\langle y-z, Jz-Jx\rangle \geq 0, \quad \forall y \in C,$$

where $Q(z,y) = \theta(z,y) + \langle Az, y-z\rangle + \varphi(y) - \varphi(z)$. Furthermore, define a mapping $T_r : E \to C$ as follows:

$$T_r x = \left\{ z \in C : Q(z,y) + \frac{1}{r}\langle y-z, Jz-Jx\rangle \geq 0, \quad \forall y \in C \right\}.$$

Then the following hold

1. *T_r is single-valued;*
2. *T_r is firmly nonexpansive, i.e., for all $x,y \in E$, $\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle$;*
3. *$F(T_r) = \widetilde{F}(T_r) = GMEP(\theta, A, \varphi)$;*
4. *$GMEP(\theta, A, \varphi)$ is closed and convex;*
5. *$\phi(p, T_r z) + \phi(T_r z, z) \leq \phi(p,z)$, $\forall p \in F(T_r)$ and $z \in E$.*

For the generalized f-projection operator, Wu and Hung [4] proved the following basic properties.

Lemma 2.4 (Wu and Hung [4]). *Let E be a reflexive Banach space with its dual E^* and C is a nonempty closed convex subset of E. The following statement hold:*

1. *$\Pi_C^f \varpi$ is nonempty closed convex subset of C for all $\varpi \in E^*$;*
2. *if E is smooth, then for all $\varpi \in E^*$, $x \in \Pi_C^f \varpi$ if and only if*

$$\langle x-y, \varpi - Jx\rangle + \rho f(y) - \rho f(x) \geq 0, \ \forall y \in C;$$

3. *if E is strictly convex and $f : C \to \mathbb{R} \cup \{+\infty\}$ is positive homogeneous (i.e., $f(tx) = tf(x)$ for all $t > 0$ such that $tx \in C$ where $x \in C$), then Π_C^f is single valued mapping.*

Recently, Fan et al. [5] show that the condition f is positive homogeneous which appeared in [5, Lemma 2.1 (iii)] can be removed.

Lemma 2.5 (Fan et al. [5]). *Let E be a reflexive Banach space with its dual E^* and C is a nonempty closed convex subset of E. If E is strictly convex, then $\Pi_C^f \varpi$ is single valued.*

Recall that J is single value mapping when E is a smooth Banach space. There exists a unique element $\varpi \in E^*$ such that $\varpi = Jx$ where $x \in E$. This substitution for (1.12) give

$$G(\xi, Jx) = \|\xi\|^2 - 2\langle \xi, Jx\rangle + \|x\|^2 + 2\rho f(\xi). \tag{2.1}$$

Now we consider the second generalized f projection operator in Banach spaces (see [33]).

Definition 2.6. Let E be a real smooth Banach space and C be a nonempty closed convex subset of E. We say that $\Pi_C^f : E^* \to 2^C$ is generalized f-projection operator if

$$\Pi_C^f x = \{u \in C : G(u, Jx) = \inf_{\xi \in C} G(\xi, Jx)\}, \ \forall x \in E.$$

Lemma 2.7 (Deimling [41]). *Let E be a Banach space and $f : E \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous convex functional. Then there exist $x^* \in E^*$ and $\alpha \in \mathbb{R}$ such that*

$$f(x) \geq \langle x, x^*\rangle + \alpha, \ \forall x \in E.$$

Lemma 2.8 (Li et al. [33]). *Let E be a reflexive smooth Banach space and C be a nonempty closed convex subset of E. The following statements hold*

1. *$\Pi_C^f x$ is nonempty closed convex subset of C for all $x \in E$;*
2. *for all $x \in E$, $\hat{x} \in \Pi_C^f x$ if and only if*

$$\langle \hat{x} - y, Jx - J\hat{x}\rangle + \rho f(y) - \rho f(\hat{x}) \geq 0, \ \forall y \in C;$$

3. *if E is strictly convex, then Π_C^f is single valued mapping.*

Lemma 2.9 (Li et al. [33]). *Let E be a reflexive smooth Banach space and C be a nonempty closed convex subset of E and let $x \in E$, $\hat{x} \in \Pi_C^f x$. Then*

$$\phi(y, \hat{x}) + G(\hat{x}, Jx) \leq G(y, Jx), \ \forall y \in C.$$

Lemma 2.10 (Li et al. [33]). *Let E be a Banach space and $f : E \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous mapping with convex domain $D(f)$. If $\{x_n\}$ is a sequence in $D(f)$ such that $x_n \rightharpoonup \hat{x} \in D(f)$ and $\lim_{n\to\infty} G(x_n, Jy) = G(\hat{x}, Jy)$, then $\lim_{n\to\infty} \|x_n\| = \|\hat{x}\|$.*

Remark 2.11. *Let E be a uniformly convex and uniformly smooth Banach space and $f(x) = 0$ for all $x \in E$, then Lemma 2.9 reduces to the property of the generalized projection operator considered by Alber [1].*

Lemma 2.12 (Chang et al. [35])**.** *Let E be a real uniformly smooth and strictly convex Banach space with Kadec-Klee property and C be a nonempty closed convex subset of E. Let $S : C \to C$ be a closed and quasi-ϕ-asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$. Then $F(S)$ is a closed convex subset of C.*

Lemma 2.13 (Chang et al. [35])**.** *Let E be a uniformly convex Banach space, $r > 0$ be a positive number and $B_r(0)$ be a closed ball of E. Then, for any given sequence $\{x_i\}_{i=1}^{\infty} \subset B_r(0)$ and for any given sequence $\{\lambda_i\}_{i=1}^{\infty}$ of positive number with $\sum_{n=1}^{\infty} \lambda_n = 1$, there exists a continuous, strictly increasing, and convex function $g : [0, 2r) \to [0, \infty)$ with $g(0) = 0$ such that, for any positive integer i, j with $i < j$,*

$$\left\| \sum_{n=1}^{\infty} \lambda_n x_n \right\|^2 \leq \sum_{n=1}^{\infty} \lambda_n \|x_n\|^2 - \lambda_i \lambda_j g(\|x_i - x_j\|). \tag{2.2}$$

Definition 2.14.

(1) Let $\{S_i\}_{i=1}^{\infty} : C \to C$ be a sequence of mapping. $\{S_i\}_{i=1}^{\infty}$ is said to be *a family of uniformly asymptotically quasi-ϕ-nonexpansive mappings*, if $\mathcal{F} := \cap_{i=1}^{\infty} F(S_i) \neq \emptyset$, and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that for each $i \geq 1$

$$\phi(p, S_i^n x) \leq k_n \phi(p, x), \forall p \in \mathcal{F}, x \in C, \quad \forall n \geq 1. \tag{2.3}$$

(2) A mapping $S : C \to C$ is said to be *uniformly L-Lipschitz continuous*, if there exists a constant $L > 0$ such that

$$\|S^n x - S^n y\| \leq L\|x - y\|, \quad \forall x, y \in C. \tag{2.4}$$

If $f(x) \geq 0$, it is clearly by the definition of mappings $\{S_i\}_{i=1}^{\infty}$ is a family of uniformly quasi-ϕ-asymptotically nonexpansive is equivalent to if $\cap_{i=1}^{\infty} F(S_i) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that for each $i \geq 1$,

$$G(p, S_i^n x) \leq k_n G(p, x), \ \forall p \in \cap_{i=1}^{\infty} F(S_i), \ x \in C, \quad \forall n \geq 1. \tag{2.5}$$

3 Main Results

Theorem 3.1. *Let C be a nonempty closed and convex subset of a uniformly smooth and strictly convex Banach space E with the Kadec-Klee property. Let $\{S_i\}_{i=1}^{\infty} : C \to C$ be an infinite family of closed uniformly L_i-Lipschitz continuous and uniformly quasi-ϕ-asymptotically nonexpansive mappings with a sequence*

$\{k_n\} \subset [1, \infty)$, $k_n \to 1$ and $f : E \to \mathbb{R}$ be a convex lower semicontinuous mapping with $C \subset int(D(f))$. For each $j = 1, 2, ..., m$ let θ_j be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies conditions (A1)-(A4), $A_j : C \to E^*$ be a continuous and monotone mapping and $\varphi_j : C \to \mathbb{R}$ be a lower semicontinuous and convex function. Assume that $F := (\cap_{i=1}^{\infty} F(S_i)) \cap (\cap_{j=1}^{m} GMEP(\theta_j, A_j, \varphi_j)) \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1}^f x_0$ and $C_1 = C$ we define the sequence $\{x_n\}$ as follows:

$$\begin{cases} z_n = J^{-1}(\alpha_{n,0} Jx_n + \sum_{i=1}^{\infty} \alpha_{n,i} JS_i^n x_n), \\ u_n = T_{r_{m,n}}^{Q_m} T_{r_{m-1,n}}^{Q_{m-1}} ... T_{r_{2,n}}^{Q_2} T_{r_{1,n}}^{Q_1} z_n, \\ C_{n+1} = \{z \in C_n : G(z, Ju_n) \leq G(z, Jx_n) + (k_n - 1)M_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}^f x_0, \quad \forall n \geq 1, \end{cases} \tag{3.1}$$

where $M_n = \sup_{q \in F}\{G(q, Jx_n)\}$, for $i \geq 0$, $\{\alpha_{n,i}\}$ is a sequence in $[0, 1]$ and $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ for all $n \geq 0$, satisfy the following conditions:

(i) $\{r_{j,n}\} \subset [d, \infty)$ *for some* $d > 0$*;*

(ii) $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,i} > 0$ *for all* $i \geq 1$*;*

(iii) $f(x) \geq 0$ *for all* $x \in C$ *and* $f(0) = 0$.

Then $\{x_n\}$ converges strongly to $p \in F$, where $p = \Pi_F^f x_0$.

Proof. We split the proof into six steps.

Step 1. We first show that C_{n+1} is closed and convex for each $n \geq 1$.

Clearly $C_1 = C$ is closed and convex. Suppose that C_n is closed and convex for each $n \in \mathbb{N}$. Since for any $z \in C_n$, we know that $G(z, Ju_n) \leq G(z, Jx_n) + (k_n - 1)M_n$ is equivalent to

$$2\langle z, Jx_n - Ju_n \rangle \leq \|x_n\|^2 - \|u_n\|^2 + (k_n - 1)M_n,$$

it follow that

$$C_{n+1} = \{z \in C_n : 2\langle z, Jx_n - Ju_n \rangle \leq \|x_n\|^2 - \|u_n\|^2 + (k_n - 1)M_n\}.$$

So, C_{n+1} is closed and convex. This implies that $\Pi_{C_{n+1}}^f x_0$ is well defined $\forall n \geq 1$.

Step 2. We show that $F \subset C_n$ for all $n \geq 1$.

We show by induction that $F \subset C_n$ for all $n \in \mathbb{N}$. It is obvious that $F \subset C_1 = C$. Suppose that $F \subset C_n$ for some $n \geq 1$. Let $q \in F \subset C_n$, by the convexity of $\| \cdot \|^2$, Lemma 2.13, uniformly asymptotically quasi-ϕ-nonexpansive of S_i and $u_n = \Omega_n^m y_n$, when $\Omega_n^j = T_{r_{j,n}}^{Q_j} T_{r_{j-1,n}}^{Q_{j-1}} \cdots T_{r_{2,n}}^{Q_2} T_{r_{1,n}}^{Q_1}$, $j = 1, 2, 3, ..., m$, $\Omega_n^0 = I$, we compute

$$
\begin{aligned}
G(q, Ju_n) &= G(q, J\Omega_n^m z_n) \\
&\leq G(q, Jz_n) \\
&= G\left(q, \left(\alpha_{n,0}Jx_n + \sum_{i=1}^{\infty}\alpha_{n,i}JS_i^n x_n\right)\right) \\
&= \|q\|^2 - 2\left\langle q, \alpha_{n,0}Jx_n + \sum_{i=1}^{\infty}\alpha_{n,i}JS_i^n x_n\right\rangle + \left\|\alpha_{n,0}Jx_n + \sum_{i=1}^{\infty}\alpha_{n,i}JS_i^n x_n\right\|^2 \\
&\quad + 2\rho f(q) \\
&= \|q\|^2 - 2\alpha_{n,0}\langle q, Jx_n\rangle - 2\sum_{i=1}^{\infty}\alpha_{n,i}\langle q, JS_i^n x_n\rangle \\
&\quad + \left\|\alpha_{n,0}Jx_n + \sum_{i=1}^{\infty}\alpha_{n,i}JS_i^n x_n\right\|^2 + 2\rho f(q) \\
&\leq \|q\|^2 - 2\alpha_{n,0}\langle q, Jx_n\rangle - 2\sum_{i=1}^{\infty}\alpha_{n,i}\langle q, JS_i^n x_n\rangle + \alpha_{n,0}\|Jx_n\|^2 \\
&\quad + \sum_{i=1}^{\infty}\alpha_{n,i}\|JS_i^n x_n\|^2 - \alpha_{n,0}\alpha_{n,j}g\|Jx_n - JS_j^n x_n\| + 2\rho f(q) \\
&= \|q\|^2 - 2\alpha_{n,0}\langle q, Jx_n\rangle + \alpha_{n,0}\|Jx_n\|^2 - 2\sum_{i=1}^{\infty}\alpha_{n,i}\langle q, JS_i^n x_n\rangle \\
&\quad + \sum_{i=1}^{\infty}\alpha_{n,i}\|JS_i^n x_n\|^2 - \alpha_{n,0}\alpha_{n,j}g\|Jx_n - JS_j^n x_n\| + 2\rho f(q) \\
&= \alpha_{n,0}G(q, Jx_n) + \sum_{i=1}^{\infty}\alpha_{n,i}G(q, JS_i^n x_n) - \alpha_{n,0}\alpha_{n,j}g\|Jx_n - JS_j^n x_n\| \\
&\leq \alpha_{n,0}G(q, Jx_n) + \sum_{i=1}^{\infty}\alpha_{n,i}k_n G(q, Jx_n) - \alpha_{n,0}\alpha_{n,j}g\|Jx_n - JS_j^n x_n\| \\
&= G(q, Jx_n) + (1-\alpha_{n,0})(k_n - 1)G(q, Jx_n) - \alpha_{n,0}\alpha_{n,j}g\|Jx_n - JS_j^n x_n\| \\
&\leq G(q, Jx_n) + (1-\alpha_{n,0})(k_n - 1)G(q, Jx_n) \\
&= G(q, Jx_n) + (k_n - 1)M_n. \qquad (3.2)
\end{aligned}
$$

This shows that $q \in C_{n+1}$ which implies that $F \subset C_{n+1}$ and hence, $F \subset C_n$ for all $n \geq 1$. Since F is nonempty, C_n is a nonempty closed convex subset of E and hence $\Pi_{C_n}^f$ exist for all $n \geq 0$. This implies that the sequence $\{x_n\}$ is well defined.

Step 3. We show that $\{x_n\}$ is bounded.

Since $f : E \to \mathbb{R}$ is convex and lower semicontinuous mapping, from Lemma 2.7, we known that there exist $x^* \in E^*$ and $\alpha \in \mathbb{R}$ such that

$$f(y) \geq \langle y, x^*\rangle + \alpha, \forall y \in E.$$

Since $x_n \in E$, it follows that

$$\begin{aligned} G(x_n, Jx_0) &= \|x_n\|^2 - 2\langle x_n, Jx_0\rangle + \|x_0\|^2 + 2\rho f(x_n) \\ &\geq \|x_n\|^2 - 2\langle x_n, Jx_0\rangle + \|x_0\|^2 + 2\rho\langle x_n, x^*\rangle + 2\rho\alpha \\ &= \|x_n\|^2 - 2\langle x_n, Jx_0 - \rho x^*\rangle + \|x_0\|^2 + 2\rho\alpha \\ &\geq \|x_n\|^2 - 2\|x_n\|\|Jx_0 - \rho x^*\| + \|x_0\|^2 + 2\rho\alpha \\ &= (\|x_n\| - \|Jx_0 - \rho x^*\|)^2 + \|x_0\|^2 - \|Jx_0 - \rho x^*\|^2 + 2\rho\alpha. \end{aligned} \tag{3.3}$$

For each $q \in F \subset C_n$ and by the definition of C_n that $x_n = \Pi^f_{C_n} x_0$, it follows from (3.3), that

$$G(q, Jx_0) \geq G(x_n, Jx_0) \geq (\|x_n\| - \|Jx_0 - \rho x^*\|)^2 + \|x_0\|^2 - \|Jx_0 - \rho x^*\|^2 + 2\rho\alpha.$$

This implies that $\{x_n\}$ is bounded and so are $\{G(x_n, Jx_0)\}$.

Step 4. We show that $\lim_{n\to\infty} \|u_n - x_n\| = 0$ and $\lim_{n\to\infty} \|Ju_n - Jx_n\| = 0$.

By the fact that $x_{n+1} = \Pi^f_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$ and $x_n = \Pi^f_{C_n} x_0$, it follows by Lemma 2.9, we get

$$0 \leq (\|x_{n+1} - \|x_n\|)^2 \leq \phi(x_{n+1}, x_n) \leq G(x_{n+1}, Jx_0) - G(x_n, Jx_0). \tag{3.4}$$

This implies that $\{G(x_n, Jx_0)\}$ is nondecreasing. So, we obtain that $\lim_{n\to\infty} G(x_n, Jx_0)$ exist and taking $n \to \infty$, we obtain that

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{3.5}$$

Since $\{x_n\}$ is bounded in C and E is reflexive, we can assume that $x_n \rightharpoonup p$. From the fact that $x_n = \Pi^f_{C_n} x_0$ when C_n is closed and convex for each $n \geq 1$, it is easy to see that $p \in C_n$, we get

$$G(x_n, Jx_0) \leq G(p, Jx_0), \quad \forall n \geq 1. \tag{3.6}$$

Since f is convex and lower semicontinuous, we have

$$\begin{aligned} \liminf_{n\to\infty} G(x_n, Jx_0) &= \liminf_{n\to\infty} \{\|x_n\|^2 - 2\langle x_n, Jx_0\rangle + \|x_0\|^2 + 2\rho f(x_n)\} \\ &\geq \|p\|^2 - 2\langle p, Jx_0\rangle + \|x_0\|^2 + 2\rho f(p) \\ &= G(p, Jx_0). \end{aligned} \tag{3.7}$$

By (3.6) and (3.7), we get

$$G(p, Jx_0) \leq \liminf_{n\to\infty} G(x_n, Jx_0) \leq \limsup_{n\to\infty} G(x_n, Jx_0) \leq G(p, Jx_0).$$

That is $\lim_{n\to\infty} G(x_n, Jx_0) = G(p, Jx_0)$, by Lemma 2.10, we have $\|x_n\| \to \|p\|$, from the Kadec-Klee property of E, we obtain that

$$\lim_{n\to\infty} x_n = p, \tag{3.8}$$

and we also have

$$\lim_{n\to\infty} x_{n+1} = p. \tag{3.9}$$

Since $x_{n+1} = \Pi^f_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$ and the definition of C_{n+1}, we have

$$G(x_{n+1}, Ju_n) \leq G(x_{n+1}, Jx_n) + (k_n - 1)M_n, \quad \forall n \in \mathbb{N}$$

is equivalence to

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n) + (k_n - 1)M_n, \quad \forall n \in \mathbb{N}.$$

By (3.5) and in view of $\lim_{n\to\infty}(k_n - 1)M_n = 0$, we also have

$$\lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0. \tag{3.10}$$

From (1.10), it follow that

$$(\|x_{n+1}\| - \|u_n\|)^2 \to 0.$$

Since $\|x_{n+1}\| \to \|p\|$, we also have

$$\|u_n\| \to \|p\| \;\; as\, n \to \infty. \tag{3.11}$$

It follows that

$$\|Ju_n\| \to \|Jp\| \;\; as\, n \to \infty. \tag{3.12}$$

This implies that $\{\|Ju_n\|\}$ is bounded in E^*. Note that E is reflexive and E^* is also reflexive, we can assume that $Ju_n \rightharpoonup x^* \in E^*$. In view of the reflexive of E, we see that $J(E) = E^*$. Hence there exists $x \in E$ such that $Jx = x^*$. It follows that

$$\begin{aligned}\phi(x_{n+1}, u_n) &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Ju_n\rangle + \|u_n\|^2 \\ &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Ju_n\rangle + \|Ju_n\|^2.\end{aligned}$$

Taking $\liminf_{n\to\infty}$ on the both sides of equality above and in view of the weak lower semicontinuity of norm $\|\cdot\|$, it yields that

$$\begin{aligned}0 &\geq \|p\|^2 - 2\langle p, x^*\rangle + \|x^*\|^2 \\ &= \|p\|^2 - 2\langle p, Jx\rangle + \|Jx\|^2 \\ &= \|p\|^2 - 2\langle p, Jx\rangle + \|x\|^2 \\ &= \phi(p, x).\end{aligned}$$

That is $p = x$, which implies that $x^* = Jp$. It follows that $Ju_n \rightharpoonup Jp \in E^*$. From (3.12) and the Kadec-Klee property of E^* that is $Ju_n \to Jp$ as $n \to \infty$. We known that $J^{-1} : E^* \to E$ is norm-weak*-continuous, that is $u_n \rightharpoonup p$. From (3.11) and the Kadec-Klee property of E, we have

$$\lim_{n\to\infty} u_n = p. \tag{3.13}$$

Since $\|x_n - u_n\| \le \|x_n - p\| + \|p - u_n\|$, it follows that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.14}$$

From J is uniformly norm-to-norm continuous on bounded subsets of E, we obtain

$$\lim_{n\to\infty} \|Ju_n - Jx_n\| = 0. \tag{3.15}$$

Step 5. We will show that $x_n \to p \in F := (\cap_{i=1}^{\infty} F(S_i)) \cap (\cap_{j=1}^{m} GMEP(\theta_j, A_j, \varphi_j))$.
(a) We show that $x_n \to p \in \cap_{i=1}^{\infty} F(S_i)$. For $p \in F$, we note that

$$\begin{aligned}\phi(p, x_n) - &\phi(p, u_n) + (k_n - 1)M_n \\ &= \|x_n\|^2 - \|u_n\|^2 - 2\langle q, Jx_n - Ju_n\rangle + (k_n - 1)M_n \\ &\le \|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|p\|\|Jx_n - Ju_n\| + (k_n - 1)M_n.\end{aligned}$$

It follows from $\|x_n - u_n\| \to 0$, $\|Jx_n - Ju_n\| \to 0$ and $(k_n - 1)M_n \to 0$ as $n \to \infty$, that

$$\phi(p, x_n) - \phi(p, u_n) + (k_n - 1)M_n \to 0 \quad as \quad n \to \infty. \tag{3.16}$$

For any $i \ge 1$ and any $p \in F$, it follows from (3.2) that

$$\begin{aligned}G(p, Ju_n) &\le \alpha_{n,0} G(p, Jx_n) + \sum_{i=1}^{\infty} \alpha_{n,i} k_n G(p, Jx_n) - \alpha_{n,0}\alpha_{n,j} g\|Jx_n - JS_j^n x_n\| \\ &= G(p, Jx_n) + (1 - \alpha_{n,0})(k_n - 1)G(p, Jx_n) - \alpha_{n,0}\alpha_{n,j} g\|Jx_n - JS_j^n x_n\| \\ &= G(p, Jx_n) + (k_n - 1)M_n - \alpha_{n,0}\alpha_{n,j} g\|Jx_n - JS_j^n x_n\|. \end{aligned} \tag{3.17}$$

It follows that

$$\alpha_{n,0}\alpha_{n,j} g\|Jx_n - JS_j^n x_n\| \le G(p, Jx_n) - G(p, Ju_n) + (k_n - 1)M_n \tag{3.18}$$

is equivalence to

$$\alpha_{n,0}\alpha_{n,j} g\|Jx_n - JS_j^n x_n\| \le \phi(p, x_n) - \phi(p, u_n) + (k_n - 1)M_n \tag{3.19}$$

From (3.16), $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,i} > 0$, we see that

$$g(\|Jx_n - JS_i^n x_n\|) \to 0, \ n \to \infty.$$

It follows from the property of g that

$$\lim_{n\to\infty} \|Jx_n - JS_i^n x_n\| = 0, \quad \forall i \ge 1. \tag{3.20}$$

Since $x_n \to p$ and J is uniformly continuous, it yields that $Jx_n \to Jp$. Thus from (3.20), we have

$$JS_i^n x_n \to Jp, \quad \forall i \ge 1 \tag{3.21}$$

Since $J^{-1} : E^* \to E$ is norm-weak*-continuous, we also have

$$S_i^n x_n \rightharpoonup p, \forall i \ge 1. \tag{3.22}$$

On the other hand, for each $i \geq 1$, we observe that

$$|\|S_i^n x_n\| - \|p\|| = |\|J(S_i^n x_n)\| - \|Jp\|| \leq \|J(S_i^n x_n) - Jp\|.$$

In view of (3.21), we obtain $\|S_i^n x_n\| \to \|p\|$ for each $i \geq 1$. Since E has the Kadee-Klee property, we get

$$S_i^n x_n \to p \quad \text{for each} \ \ i \geq 1 \text{ and } n \in \mathbb{N}. \tag{3.23}$$

By the assumption that for each $i \geq 1$, S_i is uniformly L_i-Lipschitz continuous, so we have

$$\begin{aligned} \|S_i^{n+1} x_n - S_i^n x_n\| &\leq \|S_i^{n+1} x_n - S_i^{n+1} x_{n+1}\| + \|S_i^{n+1} x_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &\quad + \|x_n - S_i^n x_n\| \\ &\leq (L_i + 1)\|x_{n+1} - x_n\| + \|S_i^{n+1} x_{n+1} - x_{n+1}\| + \|x_n - S_i^n x_n\|. \end{aligned} \tag{3.24}$$

By (3.8), (3.9) and (3.23), it yields that $\|S_i^{n+1} x_n - S_i^n x_n\| \to 0,\ n \to \infty,\ \forall i \geq 1$. From $S_i^n x_n \to p$, we get $S_i^{n+1} x_n \to p$, that is $S_i S_i^n x_n \to p$. In view of closeness of S_i, we have $S_i p = p$, for all $i \geq 1$. This imply that $p \in \cap_{i=1}^{\infty} F(S_i)$.

(b) We show that $x_n \to p \in F := \cap_{j=1}^{m} GMEP(\theta_j, A_j, \varphi_j)$. Since $x_{n+1} = \Pi_{C_{n+1}}^f x_0 \in C_{n+1} \subset C_n$ and from (3.2), we have

$$G(x_{n+1}, Jz_n) \leq G(x_{n+1}, Jx_n) + (k_n - 1)M_n,$$

is equivalence to

$$\phi(x_{n+1}, z_n) \leq \phi(x_{n+1}, x_n) + (k_n - 1)M_n.$$

From (3.5) and $(k_n - 1)M_n \to 0$ as $n \to \infty$, we see that

$$\lim_{n \to \infty} \phi(x_{n+1}, z_n) = 0. \tag{3.25}$$

From (1.10), it follows that

$$(\|x_{n+1}\| - \|z_n\|)^2 \to 0$$

Since $\|x_{n+1}\| \to \|p\|$, we have

$$\|z_n\| \to \|p\| \ \ as\, n \to \infty. \tag{3.26}$$

It follow that

$$\|Jz_n\| \to \|Jp\| \ \ as\, n \to \infty. \tag{3.27}$$

This implies that $\{\|Jz_n\|\}$ is bounded in E^* and E^* is reflexive, we can assume that $Jz_n \rightharpoonup z^* \in E^*$. In view of $J(E) = E^*$. Hence there exists $\hat{z} \in E$ such that $J\hat{z} = z^*$. It follows that

$$\begin{aligned} \phi(x_{n+1}, z_n) &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jz_n\rangle + \|z_n\|^2 \\ &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jz_n\rangle + \|Jz_n\|^2. \end{aligned}$$

Taking $\liminf_{n\to\infty}$ on the both sides of equality above and in view of the weak lower semicontinuous of norm $\|\cdot\|$, it yields that

$$\begin{aligned}0 &\geq \|p\|^2 - 2\langle p, z^*\rangle + \|z^*\|^2\\ &= \|p\|^2 - 2\langle p, J\hat{z}\rangle + \|J\hat{z}\|^2\\ &= \|p\|^2 - 2\langle p, J\hat{z}\rangle + \|\hat{z}\|^2\\ &= \phi(p, \hat{z}).\end{aligned}$$

That is, $p = \hat{z}$, which implies that $z^* = Jp$. It follows that $Jz_n \rightharpoonup Jp \in E^*$. From (3.27) and the Kadec-Klee property of E^* that is $Ju_n \to Jp$ as $n \to \infty$. Note that $J^{-1} : E^* \to E$ is norm-weak*-continuous, that is $z_n \rightharpoonup p$. From (3.26) and the Kadec-Klee property of E, we have

$$\lim_{n\to\infty} z_n = p. \tag{3.28}$$

For $p \in F \subset C_n$, by nonexpansiveness, we observe that

$$\begin{aligned}\phi(p, u_n) &= \phi(p, \Omega_n^m z_n)\\ &\leq \phi(p, \Omega_n^{m-1} z_n)\\ &\leq \phi(p, \Omega_n^{m-2} z_n)\\ &\vdots\\ &\leq \phi(p, \Omega_n^j z_n).\end{aligned} \tag{3.29}$$

By Lemma 2.3(5), we have for $j = 1, 2, 3, ..., m$

$$\begin{aligned}\phi(\Omega_n^j z_n, z_n) &\leq \phi(p, z_n) - \phi(p, \Omega_n^j z_n)\\ &\leq \phi(p, x_n) - \phi(p, \Omega_n^j z_n) + (k_n - 1)M_n\\ &\leq \phi(p, x_n) - \phi(p, u_n) + (k_n - 1)M_n.\end{aligned} \tag{3.30}$$

From (3.16), we get $\phi(\Omega_n^j z_n, z_n) \to 0 \quad as \quad n \to \infty$, for $j = 1, 2, 3, ..., m$. From (1.10), it follow that

$$(\|\Omega_n^j z_n\| - \|z_n\|)^2 \to 0.$$

Since $\|z_n\| \to \|p\|$, we also have

$$\|\Omega_n^j z_n\| \to \|p\| \;\; as\, n \to \infty. \tag{3.31}$$

Since $\{\Omega_n^j z_n\}$ is bounded and E is reflexive, without loss of generality we may assume that $\Omega_n^j z_n \rightharpoonup h$. From the first step, we known that C_n is closed and convex for each $n \geq 1$, it is obvious that $h \in C_n$. Again since

$$\phi(\Omega_n^j z_n, z_n) = \|\Omega_n^j z_n\|^2 - 2\langle \Omega_n^j z_n, Jz_n\rangle + \|z_n\|^2.$$

Taking $\liminf_{n\to\infty}$ on the both sides of equality above, we have

$$\begin{aligned}0 &\geq \|h\|^2 - 2\langle h, Jp\rangle + \|p\|^2\\ &= \phi(h, p).\end{aligned}$$

This implies that $h = p$, $\forall j = 1, 2, 3, ..., m$, it follow that

$$\Omega_n^j z_n \rightharpoonup p, \tag{3.32}$$

from (3.31), (3.32) and the Kadec-Klee property, we have

$$\lim_{n\to\infty} \Omega_n^j z_n = p \quad \forall j = 1, 2, 3, ..., m. \tag{3.33}$$

By using the triangle inequality, we obtain

$$\|\Omega_n^j z_n - \Omega_n^{j-1} z_n\| \leq \|\Omega_n^j z_n - p\| + \|p - \Omega_n^{j-1} z_n\|.$$

Hence, we obtain that

$$\lim_{n\to\infty} \|\Omega_n^j z_n - \Omega_n^{j-1} z_n\| = 0, \quad \forall j = 1, 2, 3, ..., m. \tag{3.34}$$

Since $\{r_{j,n}\} \subset [d, \infty)$ and J is uniformly norm-to-norm continuous on bounded subsets, so

$$\lim_{n\to\infty} \frac{\|J\Omega_n^j z_n - J\Omega_n^{j-1} z_n\|}{r_{j,n}} = 0, \quad \forall j = 1, 2, 3, ..., m. \tag{3.35}$$

From Lemma 2.3, we get for $j = 1, 2, 3, ..., m$

$$Q_j(\Omega_n^j z_n, y) + \frac{1}{r_{j,n}} \langle y - \Omega_n^j z_n, J\Omega_n^j z_n - J\Omega_n^{j-1} z_n \rangle \geq 0, \quad \forall y \in C.$$

From (A2), that

$$\frac{1}{r_{j,n}} \langle y - \Omega_n^j z_n, J\Omega_n^j z_n - J\Omega_n^{j-1} z_n \rangle \geq Q_j(y, \Omega_n^j z_n), \quad \forall y \in C, \quad \forall j = 1, 2, 3, ..., m.$$

From (3.33) and (3.35), we have

$$0 \geq Q_j(y, p), \quad \forall y \in C, \quad \forall j = 1, 2, 3, ..., m. \tag{3.36}$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)p$. Then, we get that $y_t \in C$. From (3.36), it follows that

$$Q_j(y_t, p) \leq 0, \quad \forall y \in C, \quad \forall j = 1, 2, 3, ..., m. \tag{3.37}$$

By the conditions (A1) and (A4), we have for $j = 1, 2, 3, ..., m$

$$\begin{aligned} 0 &= Q_j(y_t, y_t) \\ &\leq tQ_j(y_t, y) + (1-t)Q_j(y_t, p) \\ &\leq tQ_j(y_t, y) \\ &\leq Q_j(y_t, y). \end{aligned} \tag{3.38}$$

From (A3), we get

$$\begin{aligned} 0 &\le Q_j(y_t, y) \\ &= Q_j(ty + (1-t)p, y) \end{aligned}$$

letting $t \to 0$,

$$0 = \lim_{n\to\infty} 0 \le \lim_{n\to\infty} Q_j(ty + (1-t)p, y) \le Q_j(p, y), \quad \forall y \in C \quad \forall j = 1, 2, 3, ..., m.$$

This implies that $p \in GMEP(\theta_j, A_j, \varphi_j)$, $\forall j = 1, 2, 3, ..., m$. Therefore $p \in \cap_{j=1}^{m} GMEP(\theta_j, A_j, \varphi_j)$. Hence, from (a) and (b), we obtain $p \in F$.

Step 6. We show that $p = \Pi_F^f x_0$.

Since F is closed and convex set from Lemma 2.8, we have $\Pi_F^f x_0$ is single value, denote by v. From $x_n = \Pi_{C_n}^f x_0$ and $v \in F \subset C_n$, we also have

$$G(x_n, Jx_0) \le G(v, Jx_0), \forall n \ge 1.$$

By definition of G and f, we know that, for each given x, $G(\xi, Jx)$ is convex and lower semicontinuous with respect to ξ. So

$$G(p, Jx_0) \le \liminf_{n\to\infty} G(x_n, Jx_0) \le \limsup_{n\to\infty} G(x_n, Jx_0) \le G(v, Jx_0).$$

From definition of $\Pi_F^f x_0$ and $p \in F$, we can conclude that $v = p = \Pi_F^f x_0$ and $x_n \to p$ as $n \to \infty$. This completes the proof. □

For a special case that $i = 1, 2$, we can obtain the following results on a pair of asymptotically quasi-ϕ-nonexpansive mappings immediately from Theorem 3.1.

Corollary 3.2. *Let C be a nonempty closed and convex subset of a uniformly smooth and strictly convex Banach space E with the Kadec-Klee property and $f : E \to \mathbb{R}$ be a convex lower semicontinuous mapping with $C \subset int(D(f))$. Let $S_1, S_2 : C \to C$ be closed uniformly L_1, L_2-Lipschitz continuous and asymptotically quasi-ϕ-nonexpansive mappings with a sequence $\{k_n^1\} \subset [1, \infty)$, $k_n^1 \to 1$ and $\{k_n^2\} \subset [1, \infty)$, $k_n^2 \to 1$ respectively and $\{k_n\} = \sup\{\{k_n^1\}, \{k_n^2\}\}$. For each $j = 1, 2, ..., m$ let θ_j be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies conditions (A1)-(A4), $A_j : C \to E^*$ be a continuous and monotone mapping and $\varphi_j : C \to \mathbb{R}$ be a lower semicontinuous and convex function. Assume that $F := (F(S_1) \cap F(S_2)) \cap (\cap_{j=1}^{m} GMEP(\theta_j, A_j, \varphi_j)) \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1}^f x_0$ and $C_1 = C$, we define the sequence $\{x_n\}$ as follows:*

$$\begin{cases} z_n = J^{-1}(\alpha_{n,0} Jx_n + \alpha_{n,1} JS_1^n x_n + \alpha_{n,2} JS_2^n x_n), \\ u_n = T_{r_{m,n}}^{Q_m} T_{r_{m-1,n}}^{Q_{m-1}} \cdots T_{r_{2,n}}^{Q_2} T_{r_{1,n}}^{Q_1} z_n, \\ C_{n+1} = \{z \in C_n : G(z, Ju_n) \le G(z, Jx_n) + (k_n - 1)M_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}^f x_0, \quad \forall n \ge 1, \end{cases} \tag{3.39}$$

where $M_n = \sup_{q\in F}\{G(q, Jx_n)\}$, $\{\alpha_{n,i}\}_{i=0}^{2}$ are sequences in $[0, 1]$ such that $\sum_{i=0}^{2} \alpha_{n,i} = 1$ for all $n \ge 0$ and satisfy the following conditions:

(i) $\{r_{j,n}\} \subset [d, \infty)$ *for some* $d > 0$*;*

(ii) $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,i} > 0$ *for all* $i = 1, 2$*;*

(iii) $f(x) \geq 0$ *for all* $x \in C$ *and* $f(0) = 0$*.*

Then $\{x_n\}$ *converges strongly to* $p \in F$*, where* $p = \Pi_F^f x_0$*.*

Remark 3.3. *Corollary 3.2 improves and extends the result of Shehu [34, Theorem 3.1] following senses:*

(i) for the mappings, we extend the mappings from two closed weak relatively nonexpansive mappings (or relatively quasi-nonexpansive) mappings to a countable infinite family of closed and uniformly asymptotically quasi-ϕ-nonexpansive mappings;

(ii) for the framework of spaces, we extend the space from a uniformly smooth and uniformly convex Banach space to a uniformly smooth and strictly convex Banach space with the Kadec-Klee property;

If $S_i = S$ for each $i \in \mathbb{N}$, then Theorem 3.1 is reduced to the following corollary.

Corollary 3.4. *Let* C *be a nonempty closed and convex subset of a uniformly smooth and strictly convex Banach space* E *with the Kadec-Klee property. Let* $S : C \to C$ *be an infinite family of closed uniformly* L*-Lipschitz continuous and asymptotically quasi-ϕ-nonexpansive mappings with a sequence* $\{k_n\} \subset [1, \infty)$*,* $k_n \to 1$ *and* $f : E \to \mathbb{R}$ *be a convex lower semicontinuous mapping with* $C \subset int(D(f))$*. For each* $j = 1, 2, ..., m$ *let* θ_j *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *which satisfies conditions (A1)-(A4),* $A_j : C \to E^*$ *be a continuous and monotone mapping and* $\varphi_j : C \to \mathbb{R}$ *be a lower semicontinuous and convex function. Assume that* $F := F(S) \cap (\cap_{j=1}^m GMEP(\theta_j, A_j, \varphi_j)) \neq \emptyset$*. For an initial point* $x_0 \in E$ *with* $x_1 = \Pi_{C_1}^f x_0$ *and* $C_1 = C$*, we define the sequence* $\{x_n\}$ *as follows:*

$$\begin{cases} z_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J S^n x_n), \\ u_n = T_{r_{m,n}}^{Q_m} T_{r_{m-1,n}}^{Q_{m-1}} \cdots T_{r_{2,n}}^{Q_2} T_{r_{1,n}}^{Q_1} z_n, \\ C_{n+1} = \{z \in C_n : G(z, Ju_n) \leq G(z, Jx_n) + (k_n - 1)M_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}^f x_0, \quad \forall n \geq 1, \end{cases} \tag{3.40}$$

where $M_n = \sup_{q \in F}\{G(q, Jx_n)\}$ *and* $\{\alpha_n\}$ *is a sequence in* $[0, 1]$ *satisfy the following conditions:*

(i) $\{r_{j,n}\} \subset [d, \infty)$ *for some* $d > 0$*;*

(ii) $\liminf_{n\to\infty} \alpha_n(1 - \alpha_{n)} > 0$*;*

(iii) $f(x) \geq 0$ *for all* $x \in C$ *and* $f(0) = 0$*.*

Then $\{x_n\}$ *converges strongly to* $p \in F$*, where* $p = \Pi_F^f x_0$*.*

If taking $\theta_j \equiv 0, A_j \equiv 0, \varphi_j \equiv 0, r_{j,n} = 1$ in algorithm (3.40) in Corollary 3.4, we obtain the following corollary.

Corollary 3.5 (Li et al. [33]). *Let E be a uniformly convex and uniformly smooth Banach space, C be a nonempty closed and convex subset of E, $S : C \to C$ be a waek relative nonexpansive mapping, and $f : E \to \mathbb{R}$ be a convex lower semicontinuous mapping with $C \subset int(D(f))$. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ is sequence in $[0,1)$ such that $\limsup_{n\to\infty}(\alpha_n) < 1$. Define a sequence $\{x_n\}$ in C by the following algorithm*

$$\begin{cases} x_n = x_0 \in C, \quad C_0 = C, \\ z_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JSx_n,) \\ C_{n+1} = \{w \in C_n : G(w, Jz_n) \leq G(w, Jx_n)\}, \\ x_{n+1} = \Pi^f_{C_{n+1}} x_0, n \geq 1. \end{cases} \tag{3.41}$$

If $F(S)$ is nonempty, then $\{x_n\}$ converge to $\Pi^f_{F(S)}x$.

Taking $f(x) = 0$ for all $x \in E$ we have $G(\xi, Jx) = \phi(\xi, x)$ and $\Pi^f_C x = \Pi_C x$. By Theorem 3.1, then we obtain the following corollaries.

Corollary 3.6. *Let C be a nonempty closed and convex subset of a uniformly smooth and strictly convex Banach space E with Kadec-Klee property. Let $\{S_i\}_{i=1}^{\infty} : C \to C$ be an infinite family of closed uniformly L_i-Lipschitz continuous and uniformly asymptotically quasi-ϕ-nonexpansive mappings with a sequence $\{k_n\} \subset [1,\infty)$, $k_n \to 1$. For each $j = 1, 2, ..., m$ let θ_j be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies conditions (A1)-(A4), $A_j : C \to E^*$ be a continuous and monotone mapping and $\varphi_j : C \to \mathbb{R}$ be a lower semicontinuous and convex function. Assume that $F := (\cap_{i=1}^{\infty} F(S_i)) \cap (\cap_{j=1}^{m} GMEP(\theta_j, A_j, \varphi_j)) \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1}x_0$ and $C_1 = C$, we define the sequence $\{x_n\}$ as follows:*

$$\begin{cases} z_n = J^{-1}(\alpha_{n,0}Jx_n + \sum_{i=1}^{\infty} \alpha_{n,i} JS_i^n x_n), \\ u_n = T^{Q_m}_{r_{m,n}} T^{Q_{m-1}}_{r_{m-1,n}} \cdots T^{Q_2}_{r_{2,n}} T^{Q_1}_{r_{1,n}} z_n, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + (k_n - 1)M_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 1, \end{cases} \tag{3.42}$$

where $M_n = \sup_{q \in F} \phi(q, x_n)$, $\{\alpha_{n,i}\}$ are sequences in $[0,1]$ such that $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ for all $n \geq 0$. If $\{x_n\}$ is satisfy the following conditions:

(i) $\{r_{j,n}\} \subset [d, \infty)$ *for some* $d > 0$*;*

(ii) $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,i} > 0$ *for all* $i \geq 1$*;*

Then $\{x_n\}$ converges strongly to $p \in F$, where $p = \Pi_F x_0$.

Corollary 3.7. *Let C be a nonempty closed and convex subset of a uniformly smooth and strictly convex Banach space E with Kadec-Klee property. Let $S_1, S_2 : C \to C$ be closed uniformly L_1, L_2-Lipschitz continuous and uniformly asymptotically quasi-ϕ-nonexpansive mappings with a sequence $\{k_n^1\} \subset [1,\infty)$, $k_n^1 \to 1$ and $\{k_n^2\} \subset [1,\infty)$, $k_n^2 \to 1$, respectively such that $\{k_n\} = \sup\{\{k_n^1\}, \{k_n^2\}\}$. For each $j = 1, 2, ..., m$ let θ_j be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies* *conditions (A1)-(A4), $A_j : C \to E^*$ be a continuous and monotone mapping and $\varphi_j : C \to \mathbb{R}$ be a lower semicontinuous and convex function. Assume that $F := (F(S_1) \cap F(S_2)) \cap (\cap_{j=1}^m GMEP(\theta_j, A_j, \varphi_j)) \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, we define the sequence $\{x_n\}$ as follows:*

$$\begin{cases} z_n = J^{-1}(\alpha_{n,0} Jx_n + \alpha_{n,1} JS_1^n x_n + \alpha_{n,2} JS_2^n x_n), \\ u_n = T_{r_{m,n}}^{Q_m} T_{r_{m-1,n}}^{Q_{m-1}} \cdots T_{r_{2,n}}^{Q_2} T_{r_{1,n}}^{Q_1} z_n, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + (k_n - 1) M_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}^f x_0, \quad \forall n \geq 1, \end{cases} \tag{3.43}$$

where $M_n = \sup_{q \in F} \phi(q, x_n)$, $\{\alpha_{n,i}\}_{i=0}^2$ are sequences in $[0,1]$ such that $\sum_{i=0}^2 \alpha_{n,i} = 1$ for all $n \geq 0$ with the following conditions:

(i) $\{r_{j,n}\} \subset [d, \infty)$ for some $d > 0$;

(ii) $\liminf_{n \to \infty} \alpha_{n,0} \alpha_{n,i} > 0$ for all $i = 1, 2$;

Then $\{x_n\}$ converges strongly to $p \in F$, where $p = \Pi_F x_0$.

Remark 3.8. *Corollary 3.6 and Corollary 3.7 extend and improve the results of many authors in the literature works.*

4 An Application

Let E be a reflexive strictly convex and smooth Banach space and let A be a maximal monotone operator from E to E^*. For every $\lambda > 0$ we can defined a single valued mapping $J_\lambda : E \to D(A)$ by $J_\lambda = (J + \lambda A)^{-1} J$ and J_λ is call the *resolvent* of A. We known that $A^{-1}0 = F(J_\lambda)$ (see [31, 42]). We can define the following Theorem of zero point for maximal monotone operators.

Zhou et al. [25] showed that let E be a reflexive strictly convex and smooth Banach space and let A be a maximal monotone mapping such that the set of zero point: $A^{-1}0 \neq \emptyset$. Then the mapping J_r is closed and asymptotically quasi-ϕ-nonexpansive from E onto $D(A)$.

Theorem 4.1. *Let C be a nonempty closed and convex subset of a uniformly smooth and strictly convex Banach space E with the Kadec-Klee property. Let J_{λ_n} be the resovent of A where $\lambda_n > 0$ and $f : E \to \mathbb{R}$ be a convex lower semicontinuous mapping with $C \subset int(D(f))$. Assume that $(\cap_{j=1}^m GMEP(\theta_j, A_j, \varphi_j)) \cap A^{-1}0 \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1}^f x_0$ and $C_1 = C$ we define the sequence $\{x_n\}$ as follows:*

$$\begin{cases} z_n = J^{-1}(\alpha_{n,0} Jx_n + \sum_{i=1}^\infty \alpha_{n,i} JJ_{\lambda_i} x_n), \\ u_n = T_{r_{m,n}}^{Q_m} T_{r_{m-1,n}}^{Q_{m-1}} \cdots T_{r_{2,n}}^{Q_2} T_{r_{1,n}}^{Q_1} z_n, \\ C_{n+1} = \{z \in C_n : G(z, Ju_n) \leq G(z, Jx_n) + (k_n - 1) M_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}^f x_0, \quad \forall n \geq 1, \end{cases} \tag{4.1}$$

where $M_n = \sup_{q \in F} \{G(q, Jx_n)\}$, $\{\alpha_n\}$ is a sequence in $[0,1]$ and $\sum_{i=0}^\infty \alpha_{n,i} = 1$ for all $n \geq 0$, satisfy the following conditions:

(i) $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,i} > 0$ *for all* $i \geq 1$*;*

(ii) $f(x) \geq 0$ *for all* $x \in C$ *and* $f(0) = 0$.

Then $\{x_n\}$ *converges strongly to* $p \in F$*, where* $p = \Pi_F^f x_0$.

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