Thai Journal of Mathematics
Volume 12 (2014) Number 2 : 265–274

http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

The Generalized Stability of an n-Dimensional Jensen Type Functional Equation

J. Tipyan[1], C. Srisawat, P. Udomkavanich and P. Nakmahachalasint

Department of Mathematics, Faculty of Science, Chulalongkorn University
254 Phayathai Road, Pathumwan, Bangkok 10330, Thailand
e-mail : tipyan.j@gmail.com(J.Tipyan)
Paisan.N@chula.ac.th(P.Nakmahachalasint)

Abstract : In this paper, we will investigate the generalized Hyer-Ulam-Rassias stability of an n-dimensional functional equation,

$$\sum_{i=1}^{n} p_i f(x_i) = f\left(\sum_{i=1}^{n} p_i x_i\right),$$

where $n > 1$ is an integer, and $p_1, \ldots, p_n$ are positive real number with

$$\sum_{i=1}^{n} p_i = 1.$$

Keywords : functional equation; Jensen functional equation; stability; generalized stability
2010 Mathematics Subject Classification : 47H09; 47H10 (2010 MSC)

1 Introduction and preliminaries

In 1940 S.M. Ulam[6] proposed the famous stability problem of linear functions. In 1941 D.H. Hyers[1] considered the case of an approximately additive function $f : E \to E'$ where E and E' are Banach spaces and f satisfies the inequality

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon$$

[1]Corresponding author.

for all $x, y \in E$ and for some $\varepsilon > 0$. It was shown that the limit

$$L(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$$

exists for all $x \in E$ and that $L : E \to E'$ is the unique additive function satisfying

$$\|f(x) - L(x)\| \leq \epsilon.$$

Hyers' theorem was generalized by Aoki[7] and Bourgin[2] for additive mappings by considering bounded Cauchy differences. In 1978 Th.M. Rassias[8] considered an approximately additive function $f : E \to E'$ satisfying

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in E$ where $\theta \geq 0$ and $0 \leq p < 1$ are constants. Since then, the stability problem has been widely investigated for different types of functional equations. The Jensen functional equation given by

$$\frac{f(x)+f(y)}{2} = f\Big(\frac{x+y}{2}\Big)$$

has close connection[3, 4] with the Cauchy functional equation

$$f(x) + f(y) = f(x+y).$$

Stability of Jensen equation has been studied at first by Kominek[?]. In 1998, S.M. Jung[5] investigated the Hyers-Ulam stability for Jensen's equation on a restricted domain.

In this paper, we will extend the Jensen functional equation to an n-dimensional version,

$$\sum_{i=1}^{n} p_i f(x_i) = f\left(\sum_{i=1}^{n} p_i x_i\right), \tag{1.1}$$

where $n > 1$ is an integer, and $p_1, \ldots, p_n$ are positive rational numbers with

$$\sum_{i=1}^{n} p_i = 1, \tag{1.2}$$

will study a general solution and investigate its generalized stability. We will also discuss Hyers-Ulam stability and Hyers-Ulam-Rassias stability.

For our convenience, we let $n > 1$ be an integer, $p_1, \ldots, p_n$ be positive rational numbers with (1.2), X be a real vector space and Y be a Banach spaces.

2 Main Results

In this section, we will study the general solution and generalized stability of (1.1). The results are as follows.

2.1 General Solution

Theorem 2.1. *Let X and Y be real vector spaces. A mapping $f : X \to Y$ satisfies the functional equation (1.1) where $n > 1$ is an integer, and $p_1, \ldots p_n$ are positive rational numbers with $\sum_{i=1}^{n} p_i = 1$ for all $x_1, \ldots x_n \in X$, if and only if $f(x) = A(x) - f(0)$ for all $x \in X$ where $A : X \to Y$ is additive function and $f(0)$ is a constant.*

Proof. (Neccessity) Suppose $f : X \to Y$ satisfies the functional equation (1.1). Define a function $g : X \to y$ by

$$g(x) = f(x) - f(0)$$

for all $x \in X$. Note that $g(0) = 0$.
Consider

$$\begin{aligned} g\left(\sum_{i=1}^{n} p_i x_i\right) &= f\left(\sum_{i=1}^{n} p_i x_i\right) - f(0) = \sum_{i=1}^{n} p_i f(x_i) - f(0) \\ &= \sum_{i=1}^{n} p_i g(x_i). \end{aligned} \tag{2.1}$$

Thus g is satisfies (1.1). Let $s \in \{1, \ldots n\}$. Then set $x_s = x$ and $x_1 = \ldots = x_{s-1} = x_{s+1} = \ldots = x_n = 0$, then (2.1) becomes

$$g(p_s x) = p_s g(x) \quad \text{for all} \quad s \in \{1, \ldots n\} \quad \text{for all} \quad x \in X. \tag{2.2}$$

Next, we put $x_s = x, x_{s+1} = y$ and $x_1 = \ldots = x_{s-1} = x_{s+2} = \ldots = x_n = 0$ in (2.1) and using (2.2), we will have

$$g(p_s x + p_{s+1} y) = p_s g(x) + p_{s+1} g(y)$$

for all $x, y \in X$. Therefore g is additive function, by definition of g we get $f(x) = A(x) - f(0)$ for all $x \in X$.
(Sufficiency) Suppose $f(x) = A(x) - f(0)$ for all $x \in X$ where $A : X \to Y$ is additive function and $f(0)$ is a constant. Then,

$$\begin{aligned} f\left(\sum_{i=1}^{n} p_i x_i\right) &= A\left(\sum_{i=1}^{n} p_i x_i\right) - f(0) = A\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} p_i f(0) \\ &= \sum_{i=1}^{n} p_i (A(x_i) - f(0)) = \sum_{i=1}^{n} p_i f(x_i). \end{aligned}$$

This completes the proof. □

2.2 Generalied Stabiltiy

Theorem 2.2. *Let* $\phi : X^n \to [0,\infty)$ *be a function. For each integer* $s = 1, \ldots, n$, *let* $\phi_s : X \to [0,\infty)$ *be a function such that*

$$\phi_s(x) = \phi(\underbrace{0, \ldots, 0}_{s-1}, x, \underbrace{0, \ldots, 0}_{n-s}) \tag{2.3}$$

and $\sum_{i=0}^{\infty} p_s^{-i}\phi(p_s^i x)$ *converges and* $\lim_{m\to\infty} p_s^{-m}\phi(p_s^m x_1, \ldots, p_s^m x_n) = 0$ *for all* $x_1, \ldots, x_n \in X$. *If a function* $f : X \to Y$ *satisfies the inequality*

$$\left\| \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \right\| \le \phi(x_1, \ldots, x_n) \tag{2.4}$$

for all $x_1, \ldots, x_n \in X$, *then there exists a unique function* $L : X \to Y$ *that satisfies functional equation (1.1) and the inequality*

$$\|f(x) - L(x)\| \le \sum_{i=0}^{\infty} p_s^{-i-1}\phi_s(p_s^i x) \tag{2.5}$$

for all $x \in X$. *The function* L *is given by*

$$L(x) = f(0) + \lim_{m\to\infty} p_s^{-m}\big(f(p_s^m x) - f(0)\big) \tag{2.6}$$

for all $x \in X$.

Proof. Suppose $f : X \to Y$ satisfies the inequality (2.4). Define a function $g : X \to Y$ by

$$g(x) = f(x) - f(0) \tag{2.7}$$

for all $x_1, \ldots, x_n \in X$. It should be noted that $g(0) = 0$. By (1.2), we get

$$\left\| \sum_{i=1}^{n} p_i g(x_i) - g\left(\sum_{i=1}^{n} p_i x_i\right) \right\| \le \phi(x_1, \ldots, x_n) \tag{2.8}$$

for all $x_1, \ldots, x_n \in X$. Let $s \in \{1, \ldots, n\}$. Set $x_s = x$ and $x_1 = \cdots = x_{s-1} = x_{s+1} = \cdots = x_n = 0$, then (2.8) becomes

$$\|p_s g(x) - g(p_s x)\| \le \phi_s(x) \tag{2.9}$$

for all $x \in X$. Rewrite the above equation to

$$\|g(x) - p_s^{-1} g(p_s x)\| \le p_s^{-1}\phi_s(x) \tag{2.10}$$

for all $x \in X$. For each positive integer m and each $x \in X$, we have

$$\begin{aligned}
\left\|g(x) - p_s^{-m} g(p_s^m x)\right\| &= \left\| \sum_{i=0}^{m-1} \left(p_s^{-i} g(p_s^i x) - p_s^{-(i+1)} g(p_s^{i+1} x) \right) \right\| \\
&\le \sum_{i=0}^{m-1} \left\| p_s^{-i} g(p_s^i x) - p_s^{-(i+1)} g(p_s^{i+1} x) \right\| \\
&= \sum_{i=0}^{m-1} p_s^{-i} \left\| g(p_s^i x) - p_s^{-1} g(p_s p_s^i x) \right\| \\
&\le \sum_{i=0}^{m-1} p_s^{-i-1} \phi_s(p_s^i x). \qquad (2.11)
\end{aligned}$$

Consider the sequence $\{p_s^{-m} g(p_s^m x)\}$. For each positive integers $k < l$ and each $x \in X$,

$$\begin{aligned}
\left\|p_s^{-k} g(p_s^k x) - p_s^{-l} g(p_s^l x)\right\| &= p_s^{-k} \left\| g(p_s^k x) - p_s^{-(l-k)} g(p_s^{l-k} p_s^k x) \right\| \\
&\le p_s^{-k} \sum_{i=0}^{l-k-1} p_s^{-i-1} \phi_s(p_s^{i+k} x) \\
&\le p_s^{-k-1} \sum_{i=0}^{\infty} p_s^{-i} \phi_s(p_s^{i+k} x).
\end{aligned}$$

Since $\sum_{i=0}^{\infty} p_s^{-i} \phi(p_s^i x)$ converges, $\lim_{k\to\infty} p_s^{-k-1} \sum_{i=0}^{\infty} p_s^{-i} \phi_s(p_s^{i+k} x) = 0$; therefore,

$$L(x) = f(0) + \lim_{m\to\infty} p_s^{-m} g(p_s^m x) \qquad (2.12)$$

is well-defined in the Banach space Y. Moreover, as $m \to \infty$, (2.11) becomes

$$\left\| g(x) + f(0) - L(x) \right\| \le \sum_{i=0}^{\infty} p_s^{-i-1} \phi_s(p_s^i x).$$

Recalling the definition of $g(x)$, we see that inequality (2.5) is valid.

To show that L indeed satisfies (1.1), replace each x_i in (2.8) with $p_s^m x_i$,

$$\left\| \sum_{i=1}^{n} p_i g(p_s^m x_i) - g\left(p_s^m \sum_{i=1}^{n} p_i x_i \right) \right\| \le \phi(p_s^m x_1, \ldots, p_s^m x_n). \qquad (2.13)$$

If we multiply the above inequality by p_s^{-m} and take the limit as $m \to \infty$, then by the definition of L in (2.12) and (1.2), we obtain

$$\left\| \sum_{i=1}^{n} p_i L(x_i) - L\left(\sum_{i=1}^{n} p_i x_i \right) \right\| \le \lim_{m\to\infty} p_s^{-m} \phi(p_s^m x_1, \ldots, p_s^m x_n) = 0, \qquad (2.14)$$

which implies that

$$\sum_{i=1}^{n} p_i L(x_i) = L\left(\sum_{i=1}^{n} p_i x_i\right) \tag{2.15}$$

for all $x_1, \ldots, x_n \in X$.

To prove the uniqueness, suppose there is another function $L^{'} : X \to Y$ satisfying (1.1) and (2.5). Observe that if we replace x_s by x and put $x_1 = \cdots = x_{s-1} = x_{s+1} = \cdots = x_n = 0$ in (2.15), then

$$p_s L(x) + (1 - p_s)L(0) = L(p_s x) \tag{2.16}$$

for all $x \in X$, and

$$L(0) = f(0) + \lim_{m\to\infty} p_s^{-m} g(0) = f(0).$$

The function $L^{'}$ obviously possesses the same properties. Therefore,

$$p_s\big(L(x) - L^{'}(x)\big) = L(p_s x) - L^{'}(p_s x) \tag{2.17}$$

for all $x \in X$. We can prove by mathematical induction that for each positive integer m,

$$p_s^m\big(L(x) - L^{'}(x)\big) = L(p_s^m x) - L^{'}(p_s^m x)$$

for all $x \in X$. Therefore, for each positive integer m,

$$\begin{aligned}\big\|L(x) - L^{'}(x)\big\| &= p_s^{-m}\big\|L(p^m x) - L^{'}(p_s^m x)\big\| \\ &\le p_s^{-m}\big(\big\|L(p^m x) - f(p_s^m x)\big\| + \big\|L^{'}(p_s^m x) - f(p_s^m x)\big\|\big) \\ &\le 2p_s^{-m}\sum_{i=0}^{\infty} p_s^{-i-1}\phi_s(p_s^{i+m}x)\end{aligned}$$

for all $x \in X$. Since $\sum\limits_{i=0}^{\infty} p_s^{-i}\phi(p_s^i x)$ converges, $\lim\limits_{m\to\infty} p_s^{-m}\sum\limits_{i=0}^{\infty} p_s^{-i-1}\phi(p_s^{i+m}x) = 0$. We conclude that $L(x) = L^{'}(x)$ for all $x \in X$. □

Theorem 2.3. *Let $\phi : X^n \to [0,\infty)$ be a function. For each integer $s = 1, \ldots, n$, let $\phi_s : X \to [0,\infty)$ be a function such that (2.3) and $\sum\limits_{i=0}^{\infty} p_s^i\phi(p_s^{-i}x)$ converges and $\lim\limits_{m\to\infty} p_s^m\phi(p_s^{-m}x_1, \ldots, p_s^{-m}x_n) = 0$ for all $x_1, \ldots, x_n \in X$.*
If a function $f : X \to Y$ satisfies the inequality (2.4) then there exists a unique function $L : X \to Y$ that satisfies functional equation (1.1) and the inequality

$$\big\|f(x) - L(x)\big\| \le \sum_{i=1}^{\infty} p_s^{i-1}\phi_s(p_s^{-i}x) \tag{2.18}$$

for all $x \in X$. *The function* L *is given by*

$$L(x) = f(0) + \lim_{m\to\infty} p_s^m f(p_s^{-m}x) \tag{2.19}$$

for all $x \in X$.

Proof. Let $f : X \to Y$ satisfy the inequality (2.4). Referring the process (2.7)-(2.10), we can replace inequality (2.10) with

$$\left\|g(x) - p_s g(p_s^{-1}x)\right\| \le \phi_s(p_s^{-1}x)$$

for all $x \in X$. For each positive integer m and each $x \in X$, we get

$$\begin{aligned}\left\|g(x) - p_s^m g(p_s^{-m}x)\right\| &= \left\|\Big(\sum_{i=1}^{m} p_s^{i-1} g(p_s^{-(i-1)}x) - p_s^i g(p_s^{-i}x)\Big)\right\| \\ &\le \sum_{i=1}^{m} \left\|p_s^{i-1} g(p_s^{-(i-1)}x) - p_s^i g(p_s^{-i}x)\right\| \\ &= \sum_{i=1}^{m} p_s^{i-1}\left\|g(p_s^{-(i-1)}x) - p_s g(p_s^{-1}p_s^{-(i-1)}x)\right\| \\ &\le \sum_{i=1}^{m} p_s^{i-1}\phi_s(p_s^{-i}x). \end{aligned} \tag{2.20}$$

We investigate the sequence $\{p_s^m g(p_s^{-m}x)\}$. For each positive integer $k < l$ and each $x \in X$,

$$\begin{aligned}\left\|p_s^k g(p_s^{-k}x) - p_s^l g(p_s^{-l}x)\right\| &= p_s^k\left\|g(p_s^{-k}x) - p_s^{l-k} g(p_s^{-(l-k)}p_s^{-k}x)\right\| \\ &\le p_s^k \sum_{i=1}^{l-k} p_s^{i-1}\phi_s(p_s^{-i-k}x) \\ &\le p_s^{k-1} \sum_{i=1}^{\infty} p_s^{i}\phi_s(p_s^{-i-k}x).\end{aligned}$$

Since $\sum\limits_{i=0}^{\infty} p_s^i \phi(p_s^{-i}x)$ converges, $\lim\limits_{k\to\infty} p_s^{k-1} \sum\limits_{i=0}^{\infty} p_s^i \phi_s(p_s^{-i-k}x) = 0$. Thus,

$$L(x) = f(0) + \lim_{m\to\infty} p_s^m g(p_s^{-m}x) \tag{2.21}$$

is well-defined in the Banach space Y. Furthermore, (2.20) becomes as $m \to \infty$,

$$\left\|g(x) + f(0) - L(x)\right\| \le \sum_{i=1}^{\infty} p_s^{i-1}\phi_s(p_s^{-i}x).$$

By the definition of $g(x)$, inequality (2.18) is valid.

In order to show that L satisfies (1.1). We replace each x_i by $p_s^{-m}x_i$ in (2.8) and multiply p_s^m, then take the limit as $m \to \infty$, we have

$$\left\| \sum_{i=1}^{n} p_i L(x_i) - L\left(\sum_{i=1}^{n} p_i x_i\right)\right\| \leq \lim_{m\to\infty} p_s^m \phi(p_s^{-m}x_1, \ldots, p_s^{-m}x_n) = 0,$$

which implies (2.15).

To prove the uniqueness, suppose there is another function $L' : X \longrightarrow Y$ satisfying (1.1) and (2.5). Replacing x by $p_s^{-1}x$ and put $x_1 = \cdots = x_{s-1} = x_{s+1} = \cdots = x_n = 0$ in (2.15); consequently, (2.17) becomes

$$p_s\big(L(p_s^{-1}x) - L^{'}(p_s^{-1}x)\big) = L(x) - L^{'}(x).$$

For each positive m, we can show by mathematical induction that

$$p_s^m\big(L(p_s^{-m}x) - L^{'}(p_s^{-m}x)\big) = L(x) - L^{'}(x)$$

for all $x \in X$. Therefore, for each positive integer m,

$$\begin{aligned}\big\|L(x) - L^{'}(x)\big\| &= p_s^m\big\|L(p^{-m}x) - L^{'}(p_s^{-m}x)\big\| \\ &\leq p_s^m\big(\big\|L(p^{-m}x) - f(p_s^{-m}x)\big\| + \big\|L^{'}(p_s^{-m}x) - f(p_s^{-m}x)\big\|\big) \\ &\leq 2p_s^m \sum_{i=1}^{\infty} p_s^{i-1}\phi_s(p_s^{-i-m}x)\end{aligned}$$

for all $x \in X$. Since $\sum_{i=1}^{\infty} p_s^i \phi(p_s^{-i}x)$ converges, $\lim_{m\to\infty} p_s^m \sum_{i=0}^{\infty} p_s^{i-1}\phi(p_s^{-i-m}x) = 0$. We obtain that $L(x) = L^{'}(x)$ for all $x \in X$. □

2.3 Stability

This section will give the stability of (1.1) in various case. The following theorem proves stability of (1.1).

Theorem 2.4. *Let $\varepsilon > 0$ be a real number. If a function $f : X \to Y$ satisfies the inequality*

$$\left\| \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right)\right\| \leq \varepsilon \tag{2.22}$$

for all $x_1, \ldots, x_n \in X$, then there exists a unique function $L : X \to Y$ that satisfies (1.1) and

$$\big\|f(x) - L(x)\big\| \leq \frac{\varepsilon}{1 - p_{\min}}$$

for all $x \in X$, where $p_{\min} = \min\{p_1, \ldots, p_n\}$.

Proof. Let

$$\phi(x_1, \ldots, x_n) = \varepsilon$$

for all $x_1, \ldots, x_n \in X$ in Theorem 2.3. We can see that Theorem 2.3 holds for every $s = 1, \ldots, n$. We choose s such that $p_s = p_{\min} = \min\{p_1, \ldots, p_n\}$. Then (2.18) becomes

$$\|f(x) - L(x)\| \le \varepsilon \sum_{i=1}^{\infty} p_s^{i-1} = \frac{\varepsilon}{1 - p_s} = \frac{\varepsilon}{1 - p_{\min}}$$

for all $x \in X$ as desired. □

The following theorem proves the stability of (1.1).

Theorem 2.5. *Let $\varepsilon > 0$ and $r > 0$ be real numbers with $r \neq 1$. If a function $f : X \to Y$ satisfies the inequality*

$$\left\| \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i \right) \right\| \le \varepsilon \sum_{i=1}^{n} \|x_i\|^r \tag{2.23}$$

for all $x_1, \ldots, x_n \in X$, then there exists a unique function $L : X \to Y$ that satisfies (1.1) and

$$\|f(x) - L(x)\| \le \frac{\varepsilon}{M} \|x\|^r$$

for all $x \in X$, where $M = \max_{i=1,\ldots,n} | p_i - p_i^r |$.

Proof. In the case $0 < r < 1$, let

$$\phi(x_1, \ldots, x_n) = \varepsilon \sum_{i=1}^{n} \|x_i\|^r$$

for all $x_1, \ldots, x_n \in X$ in Theorem 2.3. Then we can see that Theorem 2.3 holds for every $s = 1, \ldots, n$. We choose s such that

$$| p_s - p_s^r | = M = \max_{i=1,\ldots,n} | p_i - p_i^r | .$$

Thus, (2.5) becomes

$$\begin{aligned} \|f(x) - L(x)\| &\le \varepsilon \sum_{i=1}^{\infty} p_s^{i-1} \|p_s^{-i} x\|^r = \varepsilon \|x\|^r p_s^{-1} \sum_{i=1}^{\infty} p_s^{i(1-r)} \\ &= \varepsilon p_s^{-1} \|x\|^r \left(\frac{p_s^{1-r}}{1 - p_s^{1-r}} \right) = \frac{\varepsilon}{p_s^r - p_s} \|x\|^r = \frac{\varepsilon}{M} \|x\|^r \end{aligned}$$

for all $x \in X$. In the case $r > 1$, let

$$\phi(x_1, \ldots, x_n) = \varepsilon \sum_{i=1}^{n} \|x_i\|^r$$

for all $x_1, \ldots, x_n \in X$ in Theorem 2.1. Since Theorem 2.1 holds for every $s = 1, \ldots, n$, (2.5) becomes

$$\begin{aligned}\big\|f(x) - L(x)\big\| &\leq \varepsilon \sum_{i=0}^{\infty} p_s^{-i-1} \|p_s^i x\|^r = \varepsilon \|x\|^r p_s^{-1} \sum_{i=0}^{\infty} p_s^{i(r-1)} \\ &= \varepsilon \|x\|^r p_s^{-1} \Big(\frac{1}{1 - p_s^{r-1}} \Big) = \frac{\varepsilon}{p_s - p_s^r} \|x\|^r = \frac{\varepsilon}{M} \|x\|^r\end{aligned}$$

for all $x \in X$. This completes the proof. □

References

[1] Acél J Dhombres J (1989) *Functional Equations in Several variable*, pp 142-145. Cambridge University Press, Cambridge, UK.

[2] Czewik S (2002) *Functional Equations and Inequalities in Several variable*, pp 10-13. World Scientific, Singapore.

[3] Hyers DH (1941) *On the Stability of the Linear Functional Equations*, Proc. Nat.. Acad. Sci. Usa 27, 222-224.

[4] Jung SM (1998) *Hyers-Ulam-Rassias stability of Jensen's equation and its application*, Proc. Amer. Math. Soc. 80, 411-416.

[5] Rassias JM (2003) *On the Ulam stability of Jensen and Jensen type mappings on restricted domains*, J. Math. Anal. Appl. 281, 516-524.

[6] Rassias JM and Rassias MJ (2005) *Asympotic behavior of alternative Jensen and Jensen type functional equation*, Bull. Sci. Math. 129, 545-558.

[7] Rassias ThM (1978) *On the Stability of the Linear Mapping in Banach Spaces*, Proc. Nat.. Acad. Sci. USA 72, 297-300.

[8] Ulam SM (1964) *Problems in Modern Mathematics*, chapter 6. Wiley, NY, USA.

(Received 13 June 2013)
(Accepted 26 October 2013)