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Some Convolution and Inclusion Properties for Subclasses of Bounded Univalent Functions of Complex Order

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Abstract : Making use of the Salagean operator, we defined two subclasses of bounded univalent functions of complex order and investigate convolution and inclusion properties of these classes.

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1 Introduction

Let $\mathcal{A}$ denote the class of analytic functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ $(0 \le \alpha < 1)$ denote the subclasses of $\mathcal{A}$ that consists, respectively, of starlike of order α and convex of order α in $\mathbb{U}$. It is well-known that $\mathcal{S}^*(\alpha) \subset \mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{K}(\alpha) \subset \mathcal{K}(0) = \mathcal{K}$.

If $f(z)$ and $g(z)$ are analytic in $\mathbb{U}$, we say that $f(z)$ is subordinate to $g(z)$, written $f(z) \prec g(z)$ if there exists a Schwarz function ω, which (by definition) is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \mathbb{U}$, such that $f(z) = g(\omega(z))$, $z \in \mathbb{U}$. Furthermore, if the function $g(z)$ is univalent in $\mathbb{U}$, then we have the following equivalence, (cf., e.g., [1–3]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

For functions $f(z)$ given by (1.1) and $g(z)$ given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \tag{1.2}$$

the Hadamard product or convolution of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.3}$$

Nasr and Aouf [4] defined and studied the class $F(b, M)(b \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}, M > \frac{1}{2})$, of bounded starlike functions of complex order, for $f(z)/z \neq 0, z \in \mathbb{U}$ and fixed M, satisfying

$$\left| \frac{b - 1 + \frac{zf'(z)}{f(z)}}{b} - M \right| < M \quad (z \in \mathbb{U}). \tag{1.4}$$

We note that

(i) $F(1,1) =$ class of functions $f(z)$ satisfying the condition

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 \quad (z \in \mathbb{U}) \text{ (see Singh [5]);}$$

(ii) $F(1, M) =$ class of functions $f(z)$ satisfying the condition

$$\left| \frac{zf'(z)}{f(z)} - M \right| < M \quad \left(M > \frac{1}{2}, z \in \mathbb{U} \right) \text{ (see Singh and Singh [6]);}$$

(iii) $F(\cos\lambda e^{-i\lambda}, M) = F_{\lambda,M}$ $(|\lambda| < \frac{\pi}{2}; M > \frac{1}{2})$ is the bounded $\lambda-$spirallike functions $f(z)$ satisfying the condition

$$\left| \frac{e^{i\lambda}\frac{zf'(z)}{f(z)} - i\sin\lambda}{\cos\lambda} - M \right| < M \quad (z \in \mathbb{U}) \text{ (see Kulshestha [7]).}$$

(iv) $F((1-\alpha)\cos\lambda e^{-i\lambda}, M) = F_{\lambda,M}(\alpha)$ $(|\lambda| < \frac{\pi}{2}; 0 \leq \alpha < 1; M > \frac{1}{2})$ is the bounded $\lambda-$spirallike functions $f(z)$ of order α satisfying the condition

$$\left| \frac{e^{i\lambda}\frac{zf'(z)}{f(z)} - \alpha\cos\lambda - i\sin\lambda}{(1-\alpha)\cos\lambda} - M \right| < M \quad (z \in \mathbb{U}) \text{ (see Aouf [8], with } p = 1).$$

Obviously $F(\cos\lambda e^{-i\lambda},\infty) = S^{\lambda}(|\lambda| < \frac{\pi}{2})$, is the class of $\lambda-$spirallike functions introduced by Špaček [9] and studied by [10, 11]. Also $F((1-\alpha)\cos\lambda e^{-i\lambda},\infty) = S^{\lambda}(\alpha)(\ |\lambda| < \frac{\pi}{2}, 0 \leq \alpha < 1)$, is the class of $\lambda-$spirallike functions of order α introduced by Libera [11]. Furthermore, $F(b,\infty) = S(b)$, the class of starlike functions of complex order introduced and studied by Nasr and Aouf [12] and $F(1-\alpha,\infty) = S^{*}(\alpha)$ $(0 \leq \alpha < 1)$ the class of starelike functions of order α (see Robertson [13]).

Also, Nasr and Aouf [14] defined and studied the class $G(b,M)(b \in \mathbb{C}^{*}, M > \frac{1}{2})$, of bounded convex functions of complex order, for $g'(z) \neq 0, z \in \mathbb{U}$ and fixed M, satisfying

$$\left|\frac{b+\frac{zg''(z)}{g'(z)}}{b} - M\right| < M \quad (z \in \mathbb{U})\,. \tag{1.5}$$

It follows from (1.4) and (1.5) that

$$g(z) \in G(b,M) \text{ if and only if } zg\prime(z) \in F(b,M) \tag{1.6}$$

We note that

(i) $G(\cos\lambda e^{-i\lambda}, M) = G_{\lambda,M}$ $(|\lambda| < \frac{\pi}{2})$ is the class of bounded Robertson functions investigated by Kulshestha [7];

(ii) $G((1-\alpha)\cos\lambda e^{-i\lambda}, M) = G_{\lambda,M}(\alpha)$ $(|\lambda| < \frac{\pi}{2}; 0 \leq \alpha < 1)$ is the class of boundsd Robertson functions of order α investigated by Aouf [8] with $p = 1$.

Obviously $G(\cos\lambda e^{-i\lambda},\infty) = C^{\lambda}(|\lambda| < \frac{\pi}{2})$, is the class of functions $f(z)$ regular in $\mathbb{U}$ and satisfying the condition that $zf'(z)$ is $\lambda-$spirallike, the class C^{λ} was introduced by Robertson [15]. Also $G((1-\alpha)\cos\lambda e^{-i\lambda},\infty) = C^{\lambda}(\alpha)(|\lambda| < \frac{\pi}{2}; 0 \leq \alpha < 1)$, is the class of functions $f(z)$ regular in $\mathbb{U}$ and satisfying the condition that $zf'(z)$ is $\lambda-$spirallike of order α was introduced by Libera and Ziegler [16] (see also [17, 18]). Furthermore, $G(b,\infty) = C(b)$, is the class of convex functions of complex order introduced and studied by Nasr and Aouf [19] and $G(1-\alpha,\infty) = C(\alpha)(0 \leq \alpha < 1)$ the class of convex functions of order α (see Robertson [13]).

In [20] Aouf et al. used Salagean operator [21] to define the class $H_n(b,M)$ $(n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1,2,\ldots\}, b \in \mathbb{C}^{*}, M > \frac{1}{2})$ of functions $f(z) \in A$ satisfying the condition

$$\left|\frac{b-1+\frac{D^{n+1}f(z)}{D^{n}f(z)}}{b} - M\right| < M \quad (z \in \mathbb{U})\,. \tag{1.7}$$

We note that $H_0(b,M) = F(b,M)$.

Also, we define $K_n(b,M)$ $(n \in \mathbb{N}_0, b \in \mathbb{C}^{*}, M > \frac{1}{2})$ as follows

$$\left|\frac{b-1+\frac{D^{n+2}f(z)}{D^{n+1}f(z)}}{b} - M\right| < M \quad (z \in \mathbb{U})\,. \tag{1.8}$$

We note that $K_0(b,M) = H_1(b,M) = G(b,M)$.

It follows from (1.7) and (1.8) that

$$g(z) \in K_n(b,M) \text{ if and only if } zg\prime(z) \in H_n(b,M). \tag{1.9}$$

Making use of the principal of subordination between analytic functions, we introduce the subclasses $F^*(b,M)$ and $G^*(b,M)$ of the class $\mathcal{A}$

$$F^*(b,M) = \left\{ f \in \mathcal{A} : \frac{zf^{'}(z)}{f(z)} \prec \frac{1+[b(1+m)-m]z}{1-mz} \right.$$
$$\left. \left(b \in \mathbb{C}^*; m = 1 - \frac{1}{M}, M > \frac{1}{2}; z \in \mathbb{U} \right) \right\}, \tag{1.1}$$

and

$$G^*(b,M) = \left\{ f \in \mathcal{A} : \frac{zg''(z)}{g'(z)} \prec \frac{b(1+m)z}{1-mz} \right.$$
$$\left. \left(b \in \mathbb{C}^*; m = 1 - \frac{1}{M}, M > \frac{1}{2}; z \in \mathbb{U} \right) \right\}. \tag{1.2}$$

In this paper, we investigate convolution properties of the classes $F^*(b,M)$ and $G^*(b,M)$ associated with the Salagean operator. Using convolution properties, we find the necessary and sufficient condition, coefficient estimate and inclusion properties for these classes.

2 Convolution properties

Unless otherwise mentioned, we assume throughout this paper that $0 \leq \theta < 2\pi, b \in \mathbb{C}^*, n \in \mathbb{N}_0$ and $m = 1 - \frac{1}{M}, M > \frac{1}{2}$.

Theorem 2.1. *The function $f(z)$ defined by* (1.1) *is in the class $F^*(b,M)$ if and only if*

$$\frac{1}{z}\left[f(z) * \frac{z - Cz^2}{(1-z)^2} \right] \neq 0 \ \ (z \in \mathbb{U}) \tag{2.1}$$

for all $C = C_\theta = \dfrac{e^{-i\theta} + [b(1+m)-m]}{b(1+m)}$, $\theta \in [0, 2\pi)$, *and also for* $C = 1$.

Proof. First suppose $f(z)$ defined by (1.1) is in the class $F^*(b,M)$, we have

$$\frac{zf^{'}(z)}{f(z)} \prec \frac{1+[b(1+m)-m]z}{1-mz} \quad (z \in \mathbb{U}) \tag{2.2}$$

since the function from the left-hand side of the subordination is analytic in $\mathbb{U}$, it follows $f(z) \neq 0, z \in \mathbb{U}^* = U \backslash \{0\}$, *i.e.*$\frac{1}{z} f(z) \neq 0, z \in \mathbb{U}$, this is equivalent to the fact that (2.1) holds for $C = 1$.

From (2.2) according to the subordination of two functions we say that there exists a function $w(z)$ analytic in U with $w(0)=0$, $\mid w(z) \mid < 1$ such that

$$\frac{zf'(z)}{f(z)} = \frac{1+[b(1+m)-m]w(z)}{1-mw(z)} \qquad (z \in \mathbb{U})$$

which is equivalent to

$$\frac{zf'(z)}{f(z)} \neq \frac{1+[b(1+m)-m]e^{i\theta}}{1-me^{i\theta}} \qquad (z \in \mathbb{U}; 0 \leq \theta < 2\pi),$$

or

$$z\{zf'(z)(1-me^{i\theta}) - f(z)(1+[b(1+m)-m]e^{i\theta})\} \neq 0. \tag{2.3}$$

Since

$$f(z) * \frac{z}{(1-z)} = f(z) \tag{2.4}$$

and

$$f(z) * \left[\frac{z}{(1-z)^2}\right] = zf'(z) \ . \tag{2.5}$$

Now from (2.3), (2.4) and (2.5), we obtain

$$= \frac{1}{z}\left[f(z) * \frac{z - \left[\frac{e^{-i\theta}+[b(1+m)-m]}{b(1+m)}\right] z^2}{(1-z)^2} . - b(1+m)e^{i\theta} \right] \neq 0 \ (z \in \mathbb{U}; 0 \leq \theta < 2\pi),$$

which leads to (2.1), which proves the necessary part of Theorem 2.1.

(ii) Reversely, because the assumption (2.1) holds for $C=1$, it follows that $\frac{1}{z}f(z) \neq 0$ for all $z \in \mathbb{U}$, hence the function $\varphi(z) = \frac{zf'(z)}{f(z)}$ is analytic in $\mathbb{U}$ (i.e. it is regular at $z_0 = 0$, with $\varphi(0) = 1$).

Since it was shown in the first part of the proof that the assumption (2.1) is equivalent to (2.3), we obtain that

$$\frac{zf'(z)}{f(z)} \neq \frac{1+[b(1+m)-m]e^{i\theta}}{1-me^{i\theta}} \qquad (z \in \mathbb{U}; \theta \in [0, 2\pi)), \tag{2.6}$$

if we denote

$$\psi(z) = \frac{1+[b(1+m)-m]z}{1-mz},$$

the relation (2.6) shows that $\varphi(\mathbb{U}) \cap \psi(\partial\mathbb{U}) = \emptyset$. Thus, the simply-connected domain $\varphi(\mathbb{U})$ is included in a connected component of $\mathbb{C} \setminus \psi(\partial\mathbb{U})$. From here, using the fact that $\varphi(0) = \psi(0)$ together with the univalence of the function ψ, it follows that $\varphi(z) \prec \psi(z)$, which represents in fact the subordination (2.2), i.e. $f \in F^*(b; M)$. $\square$

Remark 2.2.

(i) *Putting* $M = \infty$ *and* $\mathrm{e}^{-i\theta} = -\varkappa\,(0 < \theta < 2\pi)$ *in Theorem 2.1, we obtain the result obtained by Nasr and Aouf [22, Theorem 2];*

(ii) *Putting* $b = 1$, $M = \infty$ *and* $\mathrm{e}^{i\theta} = \varkappa\,(0 < \theta < 2\pi)$ *in Theorem 2.1 , we obtain the result obtained by Padmanabhan and Ganesan [23, Theorem 2, with* $B = -1$ *and* $A = 1$*];*

(iii) *Putting* $b = 1 - \alpha\ \ (0 \le \alpha < 1)$, $M = \infty$ *and* $\mathrm{e}^{-i\theta} = -\varkappa\,(0 < \theta < 2\pi)$ *in Theorem 2.1, we obtain the result obtained by Silverman et al. [24, Theorem 2];*

(iv) *Putting* $b = \cos\lambda e^{-i\lambda}\ \left(|\lambda| < \frac{\pi}{2}\right)$, $M = \infty$ *and* $\mathrm{e}^{i\theta} = \varkappa\,(0 < \theta < 2\pi)$ *in Theorem 2.1, we obtain the result obtained by Padmanabhan and Ganesan [23, Theorem 4, with* $B = -1$ *and* $A = 1$*];*

(v) *Putting* $b = \cos\lambda e^{-i\lambda}\ \left(|\lambda| < \frac{\pi}{2}\right)$, $M = \infty$ *and* $\mathrm{e}^{-i\theta} = -\varkappa\,(0 < \theta < 2\pi)$ *in Theorem 2.1, we obtain the result obtained by Silverman et al. [24, Theorem 4];*

(vi) *Putting* $b = \cos\lambda e^{-i\lambda}\ \left(|\lambda| < \frac{\pi}{2}\right)$, $M = \infty$ *and* $\mathrm{e}^{-i\theta} = -\varkappa\,(0 < \theta < 2\pi)$ *in Theorem 2.1, we obtain the result obtained by Ahuja [25, Corollary 1].*

Putting $b = (1-\alpha)\cos\lambda e^{-i\lambda}\ \left(|\lambda| < \frac{\pi}{2}; 0 \le \alpha < 1\right)$ and $M = \infty$ in Theorem 2.1, we obtain the following corollary (see Ahuja [26, Lemma 1] with $\mathrm{e}^{-i\theta} = -\varkappa$ and $\gamma = 1$).

Corollary 2.3. *The function* $f(z)$ *defined by* (1.1) *is in the class* $S^{\lambda}(\alpha)$ *if and only if*

$$\frac{1}{z}\left[f(z) * \frac{z - \frac{e^{-i\theta}+(1-\alpha)[e^{-2i\lambda}+1]-1}{(1-\alpha)[e^{-2i\lambda}+1]}z^2}{(1-z)^2}\right] \neq 0\ \ (0 \le \theta < \pi; z \in \mathbb{U}).$$

Theorem 2.4. *The function* $f(z)$ *defined by* (1.1) *is in the class* $G^*(b, M)$ *if and only if*

$$\frac{1}{z}\left[f(z) * \frac{z + (1-2C)z^2}{(1-z)^3}\right] \neq 0\ \ (z \in \mathbb{U}). \tag{2.7}$$

for all $C = C_\theta = \dfrac{e^{-i\theta} + [b(1+m) - m]}{b(1+m)}$, $\theta \in [0, 2\pi)$, *and also for* $C = 1$.

Proof. Set

$$g(z) = \frac{z - Cz^2}{(1-z)^2},$$

and we note that

$$zg'(z) = \frac{z + (1-2C)z^2}{(1-z)^3}. \tag{2.8}$$

From the identity $zf^{'}(z)*g(z)=f(z)*zg^{'}(z)\ \ (f,g\in\mathcal{A})$ and the fact that

$$f(z)\in G^{*}(b,M)\Leftrightarrow zf^{'}(z)\in F^{*}(b,M).$$

The result follows from Theorem 2.1. □

Remark 2.5.

(i) *Putting $M=\infty$ and $\mathrm{e}^{-i\theta}=-\varkappa\,(0<\theta<2\pi)$ in Theorem 2.4, we obtain the result obtained by Nasr and Aouf [22, Theorem 1];*

(ii) *Putting $b=1$, $M=\infty$ and $\mathrm{e}^{i\theta}=\varkappa\,(0<\theta<2\pi)$ in Theorem 2.4, we obtain the result obtained by Padmanabhan and Ganesan [23, Theorem 1, with $B=-1$ and $A=1$];*

(iii) *Putting $b=1-\alpha\ \ (0\le\alpha<1)$, $M=\infty$ and $\mathrm{e}^{-i\theta}=-\varkappa\,(0<\theta<2\pi)$ in Theorem 2.4, we obtain the result obtained by Silverman et al. [24, Theorem 1];*

(iv) *Putting $b=\cos\lambda e^{-i\lambda}\ \ \left(|\lambda|<\frac{\pi}{2}\right)$, $M=\infty$ and $\mathrm{e}^{i\theta}=\varkappa\,(0<\theta<2\pi)$ in Theorem 2.4, we obtain the result obtained by Padmanabhan and Ganesan [23, Theorem 3, with $B=-1$ and $A=1$];*

(v) *Putting $b=\cos\lambda e^{-i\lambda}\ \ \left(|\lambda|<\frac{\pi}{2}\right)$, $M=\infty$ and $\mathrm{e}^{-i\theta}=-\varkappa\,(0<\theta<2\pi)$ in Theorem 2.4, we obtain the result obtained by Silverman et al. [24, Theorem 3].*

Putting $b=(1-\alpha)\cos\lambda e^{-i\lambda}\ \ \left(|\lambda|<\frac{\pi}{2},0\le\alpha<1\right)$ and $M=\infty$ in Theorem 2.4, we obtain the following corollary.

Corollary 2.6. *The function $f(z)$ defined by (1.1) is in the class $C^{\lambda}(\alpha)$ if and only if*

$$\frac{1}{z}\left[f(z)*\frac{z-\frac{2e^{-i\theta}+(1-\alpha)[e^{-2i\lambda}+1]-2}{(1-\alpha)[e^{-2i\lambda}+1]}z^{2}}{(1-z)^{3}}\right]\neq0\ \ (0\le\theta<\pi;z\in\mathbb{U}).$$

Theorem 2.7. *A necessary and sufficient condition for the function $f(z)$ defined by (1.1) to be in the class $H_n(b,M)$ is that*

$$1-\sum_{k=2}^{\infty}k^{n}a_{k}z^{k-1}\neq0\ \ (z\in\mathbb{U}). \tag{2.9}$$

and

$$1-\sum_{k=2}^{\infty}k^{n}\frac{(k-1)\left[\mathrm{e}^{-i\theta}-m\right]-b(1+m)}{b(1+m)}a_{k}z^{k-1}\neq0\ \ (z\in\mathbb{U}). \tag{2.10}$$

Proof. From Theorem 2.1, we find that $f(z) \in H_n(b, M)$ if and only if

$$\frac{1}{z}\left[D^n f(z) * \frac{z - Cz^2}{(1-z)^2}\right] \neq 0 \ \ (z \in \mathbb{U})\,. \tag{2.11}$$

for all $C = C_\theta = \dfrac{e^{-i\theta} + [b(1+m) - m]}{b(1+m)}$, $\theta \in [0, 2\pi)$, and also for $C = 1$.

The left hand side of (2.11) may be written as

$$\begin{aligned}
&\frac{1}{z}\left[D^n f(z) * \left(\frac{z}{(1-z)^2} - \frac{Cz^2}{(1-z)^2}\right)\right] \\
&\qquad = \frac{1}{z}\left[D^{n+1} f(z) - C\left\{D^{n+1} f(z) - D^n f(z)\right\}\right] \\
&\qquad = 1 - \sum_{k=2}^{\infty} k^n \frac{(k-1)\left[\mathrm{e}^{-i\theta} - m\right] - b(1+m)}{b(1+m)} a_k z^{k-1}.
\end{aligned}$$

Thus, the proof of Theorem 2.7 is completed. □

Theorem 2.8. *A necessary and sufficient condition for the function $f(z)$ defined by* (1.1) *to be in the class $K_n(b, M)$ is that*

$$1 - \sum_{k=2}^{\infty} k^{n+1} a_k z^{k-1} \neq 0 \ \ (z \in \mathbb{U})\,. \tag{2.12}$$

and

$$1 - \sum_{k=2}^{\infty} k^{n+1} \frac{(k-1)\left[\mathrm{e}^{-i\theta} - m\right] - b(1+m)}{b(1+m)} a_k z^{k-1} \neq 0 \ \ (z \in \mathbb{U})\,. \tag{2.13}$$

Proof. From Theorem 2.4, we find that $f(z) \in K_n(b.M)$ if and only if

$$\frac{1}{z}\left[D^n f(z) * \frac{z + (1-2C)z^2}{(1-z)^3}\right] \neq 0 \qquad (z \in \mathbb{U})\,. \tag{2.14}$$

for all $C = C_\theta = \dfrac{e^{-i\theta} + [b(1+m) - m]}{b(1+m)}$, $\theta \in [0, 2\pi)$, and also for $C = 1$.

Using the relation

$$\frac{z}{(1-z)^2} = z + \sum_{k=2}^{\infty} k z^k \ \ (z \in U),$$

it is easy to see that (2.14) holds for $C = 1$ if and only if (2.12) satisfied.

Now from the formula

$$\frac{z}{(1-z)^3} = z + \sum_{k=2}^{\infty} \frac{k(k+1)}{2} z^{k-1} \ \ (z \in U),$$

we may easily deduce that

$$\frac{z+(1-2C)z^2}{(1-z)^3} = z + \sum_{k=2}^{\infty} k \frac{(k-1)[\mathrm{e}^{-i\theta}-m]-b(1+m)}{b(1+m)} z^k$$

$$\Leftrightarrow 1 - \sum_{k=1}^{\infty} k^{n+1} \left[\frac{(k-1)[\mathrm{e}^{-i\theta}-m]-b(1+m)}{b(1+m)} \right] a_k z^k \neq 0,$$

this proves Theorem 2.8. □

3 Coefficient estimate and inclusion property

As an applications of Theorems 2.7 and 2.8, we next determine coefficient estimate and inclusion property for a function of the form (1.1) to be in the classes $H_n(b,M)$ and $K_n(b,M)$.

Theorem 3.1. *If the function $f(z)$ defined by (1.1) and satisfy the inequalities*

$$\sum_{k=2}^{\infty} k^n \mid a_k \mid < 1 \ , \tag{3.1}$$

and

$$\sum_{k=2}^{\infty} k^n [(k-1)(1+m) + (1+m)\,|b|] \mid a_k \mid < (1+m)\,|b|\,, \tag{3.2}$$

then $f(z) \in H_n(b,M)$.

Proof. According to (3.1) a simple computation shows that

$$\begin{aligned} \left| 1 + \sum_{k=2}^{\infty} k^n a_k z^{k-1} \right| &\geq 1 - \left| \sum_{k=2}^{\infty} k^n a_k z^{k-1} \right| \\ &\geq 1 - \sum_{k=2}^{\infty} k^n \,|a_k|\, |z|^{k-1} \\ &\geq 1 - \sum_{k=1}^{\infty} k^n \,|a_k| > 0 \ (z \in U), \end{aligned}$$

hence the condition (2.9) is satisfied.

Using the inequality

$$\begin{aligned} \left| \frac{(k-1)\,[\mathrm{e}^{-i\theta}-m]-b(1+m)}{b(1+m)} \right| &= \frac{\left|(k-1)\,[\mathrm{e}^{-i\theta}-m]-b(1+m)\right|}{|b|\,(1+m)} \\ &\leq \frac{(k-1)\,[1+m]+|b|\,(1+m)}{|b|\,(1+m)} \end{aligned}$$

together with the assumption (3.2), we may easily deduce that

$$\begin{aligned}\left|1+\sum_{k=1}^{\infty}k^{n}\left(\frac{(k-1)\left[\mathrm{e}^{-i\theta}-m\right]-b(1+m)}{b(1+m)}\right)a_k z^{k-1}\right| \\ > 1-\sum_{k=1}^{\infty}k^{n}\left|\frac{(k-1)\left[\mathrm{e}^{-i\theta}-m\right]-b(1+m)}{b(1+m)}\right| \mid a_k \mid \\ \geq 1-\sum_{k=2}^{\infty}k^{n}\frac{(k-1)\left[1+m\right]+|b|\left(1+m\right)}{|b|\left(1+m\right)} \mid a_k \mid > 0 \ (z\in U)\end{aligned}$$

which show that (2.10) holds, hence the result follows from Theorem 2.7. □

Similarly, we can prove the following theorem.

Theorem 3.2. *If the function $f(z)$ defined by (1.1) and satisfy the inequalities*

$$\sum_{k=1}^{\infty}k^{n+1} \mid a_k \mid < 1, \tag{3.3}$$

and

$$\sum_{k=1}^{\infty}k^{n+1}[(k-1)(1+m)+(1+m)\,|b|] \mid a_k \mid < (1+m)\,|b|\,, \tag{3.4}$$

then $f(z)\in K_n\,(b;M)$.

Remark 3.3. *By specializing the parameters b, m and n, in Theorems 3.1 and 3.2, we obtain results corresponding to different subclasses of A defined in the introduction.*

Theorem 3.4. $H_{n+1}(b,M)\subset H_n(b,M)$.

Proof. If $f\in H_{n+1}(b,M)$, then Theorem 2.7 gives

$$1-\sum_{k=1}^{\infty}k^{n+1} \mid a_k \mid \neq 0, \tag{3.5}$$

and

$$1-\sum_{k=2}^{\infty}k^{n+1}\frac{(k-1)\left[\mathrm{e}^{-i\theta}-m\right]-b(1+m)}{b(1+m)}a_k z^{k-1}\neq 0 \quad (z\in\mathbb{U}; 0\leq\theta<2\pi)\,. \tag{3.6}$$

In general, we note that (3.6) may be written as

$$\left(1+\sum_{k=2}^{\infty}kz^{k-1}\right)*\left(1-\sum_{k=2}^{\infty}k^{n}\frac{(k-1)\left[\mathrm{e}^{-i\theta}-m\right]-b(1+m)}{b(1+m)}a_k z^{k-1}\right)\neq 0. \tag{3.7}$$

But

$$\left(1+\sum_{k=2}^{\infty} k z^{k-1}\right) * \left(1+\sum_{k=2}^{\infty} k^{-1} z^{k-1}\right) = 1+\sum_{k=2}^{\infty} z^{k-1} = \frac{1}{1-z} \quad (z \in \mathbb{U})\,. \tag{3.8}$$

Thus it follows from (3.7) that

$$1-\sum_{k=2}^{\infty} k^n \frac{(k-1)\left[\mathrm{e}^{-i\theta}-m\right]-b(1+m)}{b(1+m)} a_k z^{k-1} \neq 0 \quad (z \in \mathbb{U}; 0<\theta<2\pi)\,.$$

In view of Theorem 2.7, we conclude that $f \in H_n(b, M)$. □

Similary, we can prove Theorem 3.5.

Theorem 3.5. $\mathcal{K}_{n+1}(b, M) \subset \mathcal{K}_n(b, M)\,.$

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