Thai Journal of Mathematics
Volume 12 (2014) Number 2 : 251–264
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

The Clausen function $\mathsf{Cl}_2(x)$ and its Related Integrals

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Abstract : The Clausen function $\mathsf{Cl}_2(x)$ arises in several applications. A large number of indefinite integrals of logarithmic or trigonometric functions can be expressed in closed form in terms of $\mathsf{Cl}_2(x)$. The G-function introduced and systematically investigated by Barnes in about 1900, which was revived in the middle of the 1980s in connection with the study of the determinants of the Laplacians, also has several useful and widely-spread applications. Here, in this paper, we aim at presenting some interesting definite integral formulas by using a known relationship between the Clausen function $\mathsf{Cl}_2(x)$ and the Barnes G-function.

Keywords : Clausen function $\mathsf{Cl}_2(x)$; Barnes G-function; Catalan constant; Glaisher-Kinkelin constant; Euler-Mascheroni constant; Psi (or Digamma) function ; Polygamma functions; Generalized zeta function; Riemann zeta function; Determinants of the Laplacians; Series involving the zeta functions.

2010 Mathematics Subject Classification : Primary 33B10, 33B15; Secondary 11M06, 11M35.

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1 Introduction, Definitions and Preliminaries

Clausen's integral (or, synonymously, Clausen's function) $\mathsf{Cl}_2(x)$ is defined by

$$\mathsf{Cl}_2(x) := \sum_{k=1}^{\infty} \frac{\sin kx}{k^2} = -\int_0^x \log\left[2\sin\left(\frac{1}{2}\eta\right)\right] d\eta \qquad (x \in \mathbb{R}), \tag{1.1}$$

where $\mathbb{R}$ denotes the set of real numbers. This integral was first treated by Clausen in 1832 [1] and has since then been investigated by many authors (see, *e.g.*, [2], [3], [4], [5, Chapter 4], [6, Section 2.4], and many of the references cited therein). Some known properties and special values of the Clausen integral (or the Clausen function) include the periodic properties given by

$$\mathsf{Cl}_2(2n\pi \pm \theta) = \mathrm{Cl}_2(\pm\theta) = \pm\mathrm{Cl}_2(\theta), \tag{1.2}$$

which, for $n = 1$ and with θ replaced by $\pi + \theta$, yields

$$\mathsf{Cl}_2(\pi + \theta) = -\mathrm{Cl}_2(\pi - \theta). \tag{1.3}$$

From the series definition (1.1), it is obvious that

$$\mathrm{Cl}_2(n\pi) = 0 \qquad (n \in \mathbb{Z} := \{0, \pm 1, \pm 2, \cdots\}), \tag{1.4}$$

which, for $n = 1$, gives

$$\int_0^{\pi} \log\left(2\sin\frac{1}{2}\theta\right) d\theta = 0 \quad \text{and} \quad \int_0^{\pi/2} \log(\sin\theta)\, d\theta = -\frac{\pi}{2}\log 2. \tag{1.5}$$

Setting $\theta = \frac{1}{2}\pi$ in the series definition (1.1), and using the periodic property (1.3), we find that

$$\mathsf{Cl}_2\left(\frac{1}{2}\pi\right) = \mathsf{G} = -\mathrm{Cl}_2\left(\frac{3}{2}\pi\right), \tag{1.6}$$

where G is the Catalan constant defined by

$$\mathsf{G} := \frac{1}{2}\int_0^1 \mathbf{K}(\kappa)d\kappa = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2} \cong 0.91596\,55941\,\cdots, \tag{1.7}$$

where $\mathbf{K}(\kappa)$ is the complete elliptic integral of the first kind given by

$$\mathbf{K}(\kappa) := \int_0^{\pi/2} \frac{dt}{\sqrt{1-\kappa^2\sin^2 t}} \qquad (|\kappa| < 1). \tag{1.8}$$

From (1.2), (1.3) and (1.4), it suffices to consider $\mathsf{Cl}_2(x)$ in the interval $(0, \pi)$. The following special values of $\mathsf{Cl}_2(x)$ for arguments

$$x = \frac{p\,\pi}{q} \qquad (p,\, q \in \mathbb{N} := \{1,\, 2,\, 3,\, \ldots\})$$

were presented by Grosjean [3, p. 334, Eq. (12)] (see also Doelder [2]):

$$\mathsf{Cl}_2\left(\frac{p}{q}\pi\right) = \frac{1}{4\,q^2}\sum_{r=1}^{q-1}\left[\psi'\left(\frac{r}{2q}\right) - \psi'\left(1-\frac{r}{2q}\right)\right]\sin\left(r\frac{p}{q}\pi\right) \tag{1.9}$$
$$\big(q \in \mathbb{N};\; p \in \{1,\, 2,\, \ldots,\, 2q-1\};\; (p,q)=1\big),$$

where (and elsewhere in this paper) an empty sum is understood to be nil. For example, we have

$$\mathsf{Cl}_2\left(\frac{1}{3}\pi\right) = \frac{\sqrt{3}}{6}\left[\psi'\left(\frac{1}{3}\right) - \frac{2}{3}\,\pi^2\right] \tag{1.10}$$

and

$$\mathsf{Cl}_2\left(\frac{1}{4}\pi\right) = \frac{1}{32}\left[\sqrt{2}\,\psi'\left(\frac{1}{8}\right) - 2(\sqrt{2}+1)\,\pi^2 - 8(2\sqrt{2}-1)\,\mathsf{G}\right], \tag{1.11}$$

where G is the Catalan constant given in (1.7). Here $\psi(z)$ denotes the Psi (or Digamma) function defined by

$$\psi(z) := \frac{d}{dz}\{\log\,\Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)} \qquad \text{or} \qquad \log\,\Gamma(z) = \int_1^z \psi(t)\,dt \tag{1.12}$$

in terms of the familiar Gamma function $\Gamma(z)$. The Polygamma functions $\psi^{(n)}(z)$ $(n \in \mathbb{N})$ are defined by

$$\psi^{(n)}(z) := \frac{d^{n+1}}{dz^{n+1}}\log\Gamma(z) = \frac{d^n}{dz^n}\psi(z) \qquad (n \in \mathbb{N}_0;\; z \in \mathbb{C}\setminus\mathbb{Z}_0^-), \tag{1.13}$$

where $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$, $\mathbb{C}$ and $\mathbb{Z}_0^-$ denote the sets of nonnegative integers, complex numbers and nonpositive integers, respectively. In terms of the Hurwitz (or generalized) Zeta function $\zeta(s,a)$ defined by

$$\zeta(s,a) := \sum_{k=0}^{\infty}\frac{1}{(k+a)^s} \qquad (\Re(s) > 1;\; a \in \mathbb{C}\setminus\mathbb{Z}_0^-), \tag{1.14}$$

we can write

$$\psi^{(n)}(z) = (-1)^{n+1}\,n!\,\zeta(n+1,z) \qquad (n \in \mathbb{N};\; z \in \mathbb{C}\setminus\mathbb{Z}_0^-), \tag{1.15}$$

which may be used to deduce the properties of the Polygamma functions $\psi^{(n)}(z)$ $(n \in \mathbb{N})$ from those of $\zeta(s,z)$ $(s = n+1;\; n \in \mathbb{N})$ and vice versa. For example, in view of the relation (1.15), we find from (1.9), (1.10) and (1.11) that

$$\mathsf{Cl}_2\left(\frac{p}{q}\pi\right) = \frac{1}{4\,q^2}\sum_{r=1}^{q-1}\left[\zeta\left(2,\,\frac{r}{2q}\right) - \zeta\left(2,\,1-\frac{r}{2q}\right)\right]\sin\left(r\frac{p}{q}\pi\right) \tag{1.16}$$
$$\big(q \in \mathbb{N};\; p \in \{1,\, 2,\, \ldots,\, 2q-1\};\; (p,q)=1\big),$$

$$\mathsf{Cl}_2\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{6}\left[\zeta\left(2,\,\frac{1}{3}\right) - \frac{2\,\pi^2}{3}\right] \tag{1.17}$$

and

$$\mathsf{Cl}_2\left(\frac{\pi}{4}\right) = \frac{1}{32}\left[\sqrt{2}\,\zeta\left(2,\,\frac{1}{8}\right) - 2(\sqrt{2}+1)\,\pi^2 - 8(2\sqrt{2}-1)\,\mathsf{G}\right]. \tag{1.18}$$

The Riemann Zeta function $\zeta(s)$ is defined by

$$\zeta(s) := \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} & \big(\Re(s) > 1\big) \\ \displaystyle\frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} & \big(\Re(s) > 0;\ s \neq 1\big), \end{cases} \tag{1.19}$$

which, when compared with the definition in (1.14), yields

$$\zeta(s) = \zeta(s,1) = (2^s-1)^{-1}\,\zeta\left(s,\frac{1}{2}\right) = 1 + \zeta(s,2). \tag{1.20}$$

The Clausen function $\mathsf{Cl}_2(x)$ arises in several applications. A large number of indefinite integrals of logarithmic or trigonometric functions can be expressed in closed form in terms of $\mathsf{Cl}_2(x)$ (see, *e.g.*, [5] and [7, Section 1.6]). The G-function introduced and systematically investigated by Barnes in about 1900 (see [8, 9, 10]), which was revived in about the middle of the 1980s in connection mainly with the study of the determinants of the Laplacians, also has several useful and widely-spread applications (see, *e.g.*, [6, Chapter 5]). Here, in our present investigation, we present some interesting definite integral formulas by using a known relationship between the Clausen function $\mathsf{Cl}_2(x)$ and the Barnes G-function.

2 The Clausen function $\mathsf{Cl}_2(x)$ and the Barnes G-function

Barnes (see [8]; see also [6, Section 1.4]) defined the G-function satisfying each of the following properties:

(a) $G(z+1) = \Gamma(z)G(z) \quad (z \in \mathbb{C})$;
(b) $G(1) = 1$;
(c) Asymptotically,

$$\begin{aligned} \log G(z+n+2) = &\left(\frac{n+1+z}{2}\right)\log(2\pi) \\ &+ \left(\frac{n^2}{2} + n + \frac{5}{12} + \frac{z^2}{2} + (n+1)z\right)\log n \\ &- \frac{3n^2}{4} - n - nz - \log A + \frac{1}{12} + O\left(n^{-1}\right) \qquad (n \to \infty), \end{aligned} \tag{2.1}$$

where A is the Glaisher-Kinkelin constant defined by

$$\log A = \lim_{n\to\infty} \left\{ \sum_{k=1}^{n} k \log k - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12} \right) \log n + \frac{n^2}{4} \right\}, \tag{2.2}$$

the numerical value of A being given by

$$A \cong 1.282427130 \cdots .$$

From this definition, Barnes [8] deduced several explicit Weierstrass canonical product forms of the G-function, one of which is recalled here in the following form:

$$\begin{aligned} G(z+1) = (2\pi)^{\frac{1}{2}z} \exp\left(-\frac{1}{2}z - \frac{1}{2}(\gamma+1)z^2 \right) \\ \cdot \prod_{k=1}^{\infty} \left\{ \left(1+\frac{z}{k}\right)^k \exp\left(-z + \frac{z^2}{2k} \right) \right\}, \end{aligned} \tag{2.3}$$

where γ denotes the Euler-Mascheroni constant defined by

$$\begin{aligned} \gamma &:= \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n \right) \\ &\cong 0.57721\,56649\,01532\,86060\,65120\,90082\,40243\,1042 \cdots . \end{aligned} \tag{2.4}$$

For later use, we recall each of the following known special values of the G-function (see [8] and [6, Section 1.4]):

$$G\left(\frac{1}{2}\right) = 2^{\frac{1}{24}} \cdot \pi^{-\frac{1}{4}} \cdot e^{\frac{1}{8}} \cdot A^{-\frac{3}{2}}. \tag{2.5}$$

$$G(n+2) = 1!\,2!\,3! \cdots n! \quad \text{and} \quad G(n+1) = \frac{(n!)^n}{1 \cdot 2 \cdot 3^2 \cdot 4^3 \cdots n^{n-1}} \qquad (n \in \mathbb{N}). \tag{2.6}$$

$$G\left(\frac{1}{4}\right) = e^{\frac{3}{32} - \frac{\mathrm{G}}{4\pi}} \cdot A^{-\frac{9}{8}} \left[\Gamma\left(\frac{1}{4}\right) \right]^{-\frac{3}{4}} \cong 0.293756 \cdots \tag{2.7}$$

and

$$G\left(\frac{3}{4}\right) = 2^{-\frac{1}{8}} \cdot \pi^{-\frac{1}{4}} \cdot e^{\frac{3}{32} + \frac{\mathrm{G}}{\pi}} \cdot A^{-\frac{9}{8}} \left[\Gamma\left(\frac{1}{4}\right) \right]^{\frac{1}{4}} \cong 0.848718 \cdots , \tag{2.8}$$

where

$$\Gamma\left(\frac{1}{4}\right) \cong 3.62560\,99082\,21908 \cdots . \tag{2.9}$$

$$\frac{G\left(\frac{3}{4}\right)}{G\left(\frac{5}{4}\right)} = 2^{-\frac{1}{8}} \cdot \pi^{-\frac{1}{4}} \cdot e^{\frac{\mathrm{G}}{2\pi}}. \tag{2.10}$$

There is an interesting known relationship between the Clausen function $\mathsf{Cl}_2(x)$ and the Barnes G-function, which is asserted by the following lemma (see, *e.g.*, [6, p. 175]; see also [5, Section 1.11]).

Lemma. *The following relationship holds true*:

$$\mathsf{Cl}_2(x) = x\,\log\pi - x\,\log\left(\sin\frac{x}{2}\right) + 2\pi\,\log\left(\frac{G\left(1-\frac{x}{2\pi}\right)}{G\left(1+\frac{x}{2\pi}\right)}\right), \tag{2.11}$$

where G is the Barnes G-function.

Proof. For the sake of completeness, we simply and briefly outline the demonstration of the Lemma. Indeed, in view of Kinkelin's integral formula (see [11]):

$$\int_0^z \pi t\cot\pi t\,dt = z\,\log(2\pi) + \log\frac{G(1-z)}{G(1+z)}, \tag{2.12}$$

which Kinkelin [11] derived mainly from the formula (2.3) and the following known expansion:

$$\pi z\,\cot(\pi z) = 1 + 2\sum_{n=1}^{\infty}\frac{z^2}{z^2-n^2} \qquad (z \in \mathbb{C}\setminus\mathbb{Z}). \tag{2.13}$$

Now, by using integration by parts in (2.12), we have (see, *e.g.*, [6, p. 45, Eq. (28)])

$$\int_0^z \log\,\sin\left(\pi t\right)dt = z\log\left(\frac{\sin\,\pi z}{2\pi}\right) + \log\frac{G(1+z)}{G(1-z)}. \tag{2.14}$$

The Clausen integral in (1.1) is easily expressed as follows:

$$\mathsf{Cl}_2(x) = -x\,\log 2 - 2\int_0^{\frac{x}{2}}\log(\sin t)\,dt \quad (x\in\mathbb{R}). \tag{2.15}$$

On the other hand, we find from (2.14) that

$$\int_0^{\frac{x}{2}}\log(\sin t)\,dt = \frac{x}{2}\,\log\left(\frac{\sin\,x/2}{2\pi}\right) + \pi\,\log\frac{G\left(1+\frac{x}{2\pi}\right)}{G\left(1-\frac{x}{2\pi}\right)}. \tag{2.16}$$

Thus, by substituting from the integral (2.16) into (2.15), we are easily led to the desired relation (2.11). □

Setting $x = \frac{p}{q}\pi$ and $x = \frac{\pi}{k}$ $(k = 3,\,4)$ in (2.11), and using the formulas (1.16), (1.17) and (1.18), we obtain the following evaluations analogous to (2.10):

$$\begin{aligned}\log\left(\frac{G\left(1-\frac{p}{2q}\right)}{G\left(1+\frac{p}{2q}\right)}\right) &= \frac{p}{2q}\,\log\left[\frac{1}{\pi}\,\sin\left(\frac{p\,\pi}{2q}\right)\right]\\ &+ \frac{1}{8\pi q^2}\sum_{r=1}^{q-1}\left[\zeta\left(2,\,\frac{r}{2q}\right) - \zeta\left(2,\,1-\frac{r}{2q}\right)\right]\sin\left(\frac{rp\pi}{q}\right)\\ &\big(q\in\mathbb{N};\; p\in\{1,\,2,\,\ldots,\,2q-1\};\; (p,q)=1\big),\end{aligned} \tag{2.17}$$

$$\frac{G\left(\frac{5}{6}\right)}{G\left(\frac{7}{6}\right)} = (2\pi)^{-\frac{1}{6}} \exp\left[\frac{\sqrt{3}}{12\,\pi}\zeta\left(2, \frac{1}{3}\right) - \frac{\pi}{6\sqrt{3}}\right] \tag{2.18}$$

and

$$\begin{aligned}\frac{G\left(\frac{7}{8}\right)}{G\left(\frac{9}{8}\right)} &= \left(\frac{\sqrt{2-\sqrt{2}}}{2\pi}\right)^{\frac{1}{8}} \\ &\cdot \exp\left[\frac{\sqrt{2}}{64\,\pi}\zeta\left(2, \frac{1}{8}\right) - \frac{\pi}{32}(\sqrt{2}+1) - \frac{1}{8\pi}(2\sqrt{2}-1)\mathsf{G}\right].\end{aligned} \tag{2.19}$$

3 Integral Formulas Derivable from the Relationship (2.11)

By applying (2.11) and (2.12), we can deduce the following integral formula for $\mathsf{Cl}_2(x)$:

$$\mathsf{Cl}_2(x) = -x\,\log\left(2\,\sin\frac{x}{2}\right) + \frac{\pi^2}{2}\int_0^{\frac{x}{\pi}} t\,\cot\frac{\pi t}{2}\,dt \qquad (0 \leqslant x < 2\pi), \tag{3.1}$$

which was used by Wood [12] and Kölbig [4] to give the numerical calculation of $\mathsf{Cl}_2(x)$ with higher precision. Setting $x = \frac{\pi}{k}$ ($k = 1,\, 2,\, 3,\, 4$) in (3.1), and applying (1.5), (1.6), (1.17) and (1.18), we obtain the following integral formulas:

$$\int_0^{\frac{\pi}{2}} u\,\cot\,u\,du = \frac{\pi}{2}\,\log 2; \tag{3.2}$$

$$\int_0^{\frac{\pi}{4}} u\,\cot\,u\,du = \frac{\pi}{8}\,\log 2 + \frac{\mathsf{G}}{2}, \tag{3.3}$$

which is a known result (see, *e.g.*, [6, p. 51, Eq. (59)]);

$$\int_0^{\frac{\pi}{6}} u\,\cot\,u\,du = \frac{\sqrt{3}}{12}\,\left[\zeta\left(2, \frac{1}{3}\right) - \frac{2\,\pi^2}{3}\right]; \tag{3.4}$$

$$\begin{aligned}\int_0^{\frac{\pi}{8}} u\,\cot\,u\,du &= \frac{\pi}{16}\,\log(2-\sqrt{2}) \\ &+ \frac{1}{64}\,\left[\sqrt{2}\,\zeta\left(2, \frac{1}{8}\right) - 2(\sqrt{2}+1)\,\pi^2 - 8(2\sqrt{2}-1)\,\mathsf{G}\right].\end{aligned} \tag{3.5}$$

We now recall an integral formula [6, p. 46, Eq. (34)] in the following form:

$$\int_0^{\pi z} u\,\tan u\,du = \pi\,\log\frac{G(1-z)}{G(1+z)} - \frac{\pi}{2}\,\log\frac{G(1-2z)}{G(1+2z)}. \tag{3.6}$$

By setting $z = \frac{1}{4}$ and $z = \frac{1}{8}$ in (3.6) and using (2.10) and (2.19), we obtain the following two further integral formulas:

$$\int_0^{\frac{\pi}{4}} u \tan u \, du = -\frac{\pi}{8} \log 2 + \frac{\mathsf{G}}{2}, \tag{3.7}$$

which is a known result (see, *e.g.*, [6, p. 52, Eq. (61)]);

$$\begin{aligned} \int_0^{\frac{\pi}{8}} u \tan u \, du =& \frac{\pi}{16} \log \frac{2-\sqrt{2}}{2} \\ &+ \frac{\sqrt{2}}{64} \zeta\left(2, \frac{1}{8}\right) - \frac{\pi^2}{32} \left(\sqrt{2}+1\right) - \frac{1}{8}\left(1+2\sqrt{2}\right) \mathsf{G}. \end{aligned} \tag{3.8}$$

We next recall another integral formula [6, p. 46, Eq. (37)]:

$$\begin{aligned} &\int_0^z (\pi t \cot \pi t)^2 \, dt \\ &\quad = -\frac{\pi^2 z^3}{3} - \pi z^2 \cot \pi z + 2z \log(2\pi) + 2 \log \frac{G(1-z)}{G(1+z)}, \end{aligned} \tag{3.9}$$

which is the corrected version of [13, Eq. (2.11)]. By combining (3.9) and (2.11), we get the following integral formula for $\mathsf{Cl}_2(x)$:

$$\mathsf{Cl}_2(x) = \frac{x^3}{24} + \frac{x^2}{4} \cot \frac{x}{2} - x \log\left(2 \sin \frac{x}{2}\right) + \int_0^{\frac{x}{2}} (u \cot u)^2 \, du. \tag{3.10}$$

Setting $x = \frac{\pi}{k}$ $(k = 1, 2, 3, 4)$ in (3.9). and using (1.5), (1.6), (1.17) and (1.18), we arrive at the following integral formulas:

$$\int_0^{\frac{\pi}{2}} (u \cot u)^2 \, du = \pi \log 2 - \frac{\pi^3}{24}; \tag{3.11}$$

$$\int_0^{\frac{\pi}{4}} (u \cot u)^2 \, du = -\frac{\pi^3}{192} - \frac{\pi^2}{16} + \frac{\pi}{4} \log 2 + \mathsf{G}, \tag{3.12}$$

which is a known formula (see, *e.g.*, [6, p. 52, Eq. (66)]);

$$\int_0^{\frac{\pi}{6}} (u \cot u)^2 \, du = -\frac{\pi^3}{648} - \frac{5\sqrt{3}\,\pi^2}{36} + \frac{\sqrt{3}}{6} \zeta\left(2, \frac{1}{3}\right); \tag{3.13}$$

$$\begin{aligned} \int_0^{\frac{\pi}{8}} (u \cot u)^2 \, du =& -\frac{\pi^3}{1536} - \frac{5\,\pi^2}{64(\sqrt{2}-1)} + \frac{\pi}{8} \log(2-\sqrt{2}) \\ &+ \frac{\sqrt{2}}{32} \zeta\left(2, \frac{1}{8}\right) - \frac{1}{4}(2\sqrt{2}-1)\, \mathsf{G}. \end{aligned} \tag{3.14}$$

The integral formula in [6, p. 46, Eq. (33)] should be corrected as follows (*cf.* [13, Eq. (2.7)]):

$$\int_0^z \left(\frac{\pi t}{\sin \pi t}\right)^2 dt = -\pi z^2 \cot \pi z + 2z \log(2\pi) + 2 \log \frac{G(1-z)}{G(1+z)}. \tag{3.15}$$

By combining (2.11) and (3.15), we can deduce another integral formula for $\mathsf{Cl}_2(x)$:

$$\mathsf{Cl}_2(x) = \frac{x^2}{4} \cot \frac{x}{2} - x \log\left(2 \sin \frac{x}{2}\right) + \int_0^{\frac{x}{2}} \left(\frac{u}{\sin u}\right)^2 du. \tag{3.16}$$

Setting $x = \frac{\pi}{k}$ $(k = 1, 2, 3, 4)$ in (3.16), and using (1.5), (1.6), (1.17) and (1.18), we obtain the following integral formulas:

$$\int_0^{\frac{\pi}{2}} \left(\frac{u}{\sin u}\right)^2 du = \pi \log 2; \tag{3.17}$$

$$\int_0^{\frac{\pi}{4}} \left(\frac{u}{\sin u}\right)^2 du = -\frac{\pi^2}{16} + \frac{\pi}{4} \log 2 + \mathsf{G}, \tag{3.18}$$

which is a known formula (see, *e.g.*, [6, p. 52, Eq. (64)]);

$$\int_0^{\frac{\pi}{6}} \left(\frac{u}{\sin u}\right)^2 du = -\frac{5\sqrt{3}\,\pi^2}{36} + \frac{\sqrt{3}}{6}\,\zeta\left(2, \frac{1}{3}\right); \tag{3.19}$$

$$\begin{aligned}\int_0^{\frac{\pi}{8}} \left(\frac{u}{\sin u}\right)^2 du = &-\frac{5\,\pi^2}{64(\sqrt{2}-1)} + \frac{\pi}{8} \log(2-\sqrt{2}) \\ &+ \frac{\sqrt{2}}{32}\,\zeta\left(2, \frac{1}{8}\right) - \frac{1}{4}(2\sqrt{2}-1)\,\mathsf{G}.\end{aligned} \tag{3.20}$$

By using integration by parts in (3.16), we obtain the following integral formula for $\mathsf{Cl}_2(x)$:

$$\begin{aligned}\mathsf{Cl}_2(x) = &\frac{x^3}{24} \csc^2\left(\frac{x}{2}\right) + \frac{x^2}{4} \cot \frac{x}{2} - x \log\left(2 \sin \frac{x}{2}\right) \\ &+ \frac{2}{3} \int_0^{\frac{x}{2}} \frac{u^3 \cos u}{\sin^3 u}\, du.\end{aligned} \tag{3.21}$$

Setting $x = \frac{\pi}{k}$ $(k = 1, 2, 3, 4)$ in (3.21), and applying (1.5), (1.6), (1.17) and (1.18), we get the following integral formulas:

$$\int_0^{\frac{\pi}{2}} \frac{u^3 \cos u}{\sin^3 u}\, du = -\frac{\pi^3}{16} + \frac{3\,\pi}{2} \log 2, \tag{3.22}$$

$$\int_0^{\frac{\pi}{4}} \frac{u^3 \cos u}{\sin^3 u}\, du = -\frac{3\,\pi^3}{192} - \frac{3\,\pi^2}{32} + \frac{3\,\pi}{8} \log 2 + \frac{3}{2}\,\mathsf{G}, \tag{3.23}$$

$$\int_0^{\frac{\pi}{6}} \frac{u^3 \cos u}{\sin^3 u}\, du = -\frac{\pi^3}{108} - \frac{5\sqrt{3}\,\pi^2}{24} + \frac{\sqrt{3}}{4}\,\zeta\left(2,\, \frac{1}{3}\right) \tag{3.24}$$

and

$$\begin{aligned}\int_0^{\frac{\pi}{8}} \frac{u^3 \cos u}{\sin^3 u}\, du = &-\frac{\pi^3}{256(2-\sqrt{2})} - \frac{15\,\pi^2}{128(\sqrt{2}-1)} \\ &+ \frac{3\,\pi}{16}\,\log(2-\sqrt{2}) + \frac{3\sqrt{2}}{64}\,\zeta\left(2,\, \frac{1}{8}\right) - \frac{3}{8}(2\sqrt{2}-1)\,\mathsf{G}.\end{aligned} \tag{3.25}$$

Next, upon integration by parts in (3.21), we get the following integral formula for $\mathsf{Cl}_2(x)$:

$$\begin{aligned}\mathsf{Cl}_2(x) = &\frac{x^4 \cos\frac{x}{2}}{96\,\sin^3\left(\frac{x}{2}\right)} + \frac{x^3}{24}\,\csc^2\left(\frac{x}{2}\right) + \frac{x^2}{4}\,\cot\frac{x}{2} - x\,\log\left(2\sin\frac{x}{2}\right) \\ &- \frac{1}{6}\int_0^{\frac{x}{2}} u^4\,\left(2\,\csc^2 u - 3\,\csc^4 u\right)\, du.\end{aligned} \tag{3.26}$$

Setting $x = \frac{\pi}{k}$ $(k = 1, 2, 3, 4)$ in (3.26), and using (1.5), (1.6), (1.17) and (1.18), we find the following integral formulas:

$$\int_0^{\frac{\pi}{2}} u^4\,\left(2\,\csc^2 u - 3\,\csc^4 u\right)\, du = \frac{\pi^3}{4} - 6\pi\,\log 2, \tag{3.27}$$

$$\int_0^{\frac{\pi}{4}} u^4\,\left(2\,\csc^2 u - 3\,\csc^4 u\right)\, du = \frac{\pi^4}{128} + \frac{\pi^3}{16} + \frac{3\,\pi^2}{8} - \frac{3\pi}{2}\,\log 2 - 6\,\mathsf{G}, \tag{3.28}$$

$$\begin{aligned}&\int_0^{\frac{\pi}{6}} u^4\,\left(2\,\csc^2 u - 3\,\csc^4 u\right)\, du \\ &\qquad = \frac{\sqrt{3}\pi^4}{324} + \frac{\pi^3}{27} + \frac{5\,\sqrt{3}\,\pi^2}{6} - \sqrt{3}\,\zeta\left(2,\, \frac{1}{3}\right)\end{aligned} \tag{3.29}$$

and

$$\begin{aligned}&\int_0^{\frac{\pi}{8}} u^4\,\left(2\,\csc^2 u - 3\,\csc^4 u\right)\, du \\ &= \frac{\pi^4}{1024\,(3\sqrt{2}-4)} + \frac{\pi^3}{64(2-\sqrt{2})} + \frac{15\,\pi^2}{32(\sqrt{2}-1)} \\ &\quad - \frac{3\pi}{4}\,\log(2-\sqrt{2}) - \frac{3\sqrt{2}}{16}\,\zeta\left(2,\, \frac{1}{8}\right) + \frac{3}{2}(2\sqrt{2}-1)\mathsf{G}.\end{aligned} \tag{3.30}$$

4 Series Associated with the Zeta Functions

A classical about three-century-old theorem of Christian Goldbach (1690-1764), which was stated in a letter dated 1729 from Goldbach to Daniel Bernoulli

(1700-1782), was revived in 1986 by Shallit and Zikan [14] as the following problem:

$$\sum_{\omega\in\mathcal{S}} \frac{1}{\omega-1} = 1, \tag{4.1}$$

where $\mathcal{S}$ denotes the set of all nontrivial integer kth powers, that is,

$$\mathcal{S} := \left\{n^k : n,\, k \in \mathbb{N}\setminus\{1\}\right\}.$$

In fact, Shallit and Zikan [14] showed that Goldbach's theorem (4.1) assumes the following elegant form:

$$\sum_{\omega\in\mathcal{S}} \frac{1}{\omega-1} = \sum_{k=2}^{\infty}[\zeta(k)-1] = 1 \tag{4.2}$$

in terms of the Riemann Zeta function $\zeta(s)$ given in (1.19). An interesting historical introduction to the remarkably widely- and extensively-investigated subject of *closed-form* evaluation of series involving the Zeta functions was presented by Srivastava *et al.* (see [15] and [16]; see also [17] and [6]). The formula (4.2) is presumably the origin of this fascinating subject (see [14] and [15]; see also [17] and [18]). A considerably large number of formulas have been derived, by using various methods and techniques, in the vast literature on this subject (see, *e.g.*, [16, Chapter 3]; see also [19], [20], [15], [17], [6, Chapter 3], [18] and [21]). For a simple example, we recall here the following sum:

$$\sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{(2k+1)\,2^{2k}} = \log 2 - \gamma, \tag{4.3}$$

which, as noted by Srivastava [15], is contained in a memoir of 1781 by Leonhard Euler (1707-1783) (see also [20, p. 28, Eq. (8)]; it was rederived by Wilton [21, p. 92]). A rather extensive collection of closed-form sums of series involving the Zeta functions was presented in [16] and [6]. For a very recent development in this subject, see the work by Choi and Srivastava [22].

In this section. we present further integral formulas by using closed-form evaluations of series involving the Zeta functions. To do this, we begin by applying a well-known identity (see, *e.g.*, [6, p. 166, Eq. (18)]):

$$\zeta(2n) = (-1)^{n+1}\,\frac{(2\pi)^{2n}}{2\cdot(2n)!}B_{2n} \qquad (n\in\mathbb{N}_0), \tag{4.4}$$

B_n being the classical Bernoulli numbers (see, *e.g.*, [6, Section 1.7]), to a known indefinite integral formula [23, p. 201, Entry 9.2.3-8] in order to present the following integral formula:

$$\begin{aligned}\int_0^{\frac{x}{2}} \frac{u^{2n}}{\sin^2 u}\,du &= -\left(\frac{x}{2}\right)^{2n}\cot\frac{x}{2} + \frac{2n}{2n-1}\left(\frac{x}{2}\right)^{2n-1} \\ &- 4n\sum_{k=1}^{\infty}\frac{\zeta(2k)}{(2n+2k-1)\,\pi^{2k}}\left(\frac{x}{2}\right)^{2n+2k-1} \qquad (|x|<2\pi;\ n\in\mathbb{N}),\end{aligned} \tag{4.5}$$

which is expressed as a series involving the Riemann Zeta function. Setting $x = \pi$ in (4.5), and applying a known closed-form evaluation [6, p. 259, Eq. (71)] to the series resulting from (4.5), we obtain the following integral formula:

$$\int_0^{\frac{\pi}{2}} \frac{u^{2n}}{\sin^2 u}\, du = \frac{n\,\pi^{2n-1}}{2^{2n-2}} \cdot \left[\log 2 - \sum_{k=1}^{n-1} (-1)^k \binom{2n-1}{2k} \frac{(2k)!}{(2\pi)^{2k}} \left(1-2^{2k}\right) \zeta(2k+1)\right] \qquad (n \in \mathbb{N}). \tag{4.6}$$

The special cases of this last result (4.6) when $n = 1$ and $n = 2$ yield, respectively, the following integrals:

$$\int_0^{\frac{\pi}{2}} \frac{u^2}{\sin^2 u}\, du = 2 \int_0^{\frac{\pi}{2}} u \cot u \, du = \pi \log 2, \tag{4.7}$$

which is a known result (see, *e.g.*, [24, p. 427, Entry 3.747-7]);

$$\int_0^{\frac{\pi}{2}} \frac{u^4}{\sin^2 u}\, du = \frac{\pi^3}{2} \log 2 - \frac{9\,\pi}{4} \zeta(3). \tag{4.8}$$

By combining (4.8) and (3.27), we get the following integral formula:

$$\int_0^{\frac{\pi}{2}} \left(\frac{u}{\sin u}\right)^4 du = -\frac{\pi^3}{12} + \left(\frac{\pi^3}{3} + 2\pi\right) \log 2 - \frac{3\,\pi}{2} \zeta(3). \tag{4.9}$$

Setting $a = 1$ in a known result [6, p. 258, Eq. (65)], and using (1.20) and two other known identities (see, *e.g.*, [6, p. 165, Eq. (10) and p. 166, Eq. (13)]), we find that

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{2k+2n-1} t^{2k+2n-1} = \frac{1}{2} \sum_{k=0}^{2n-1} \binom{2n-1}{k} \cdot \left[\zeta'(-k, 1-t) + (-1)^k \zeta'(-k, 1+t)\right] t^{2n-1-k} + \frac{t^{2n-1}}{2(2n-1)} \tag{4.10}$$

$$(|t| < 1;\ n \in \mathbb{N}).$$

Finally, by applying (4.5) and (4.10), we obtain

$$\int_0^{\pi x} \frac{u^{2n}}{\sin^2 u}\, du = -(\pi x)^{2n} \cot(\pi x) - 2n\,\pi^{2n-1} \sum_{k=0}^{2n-1} \binom{2n-1}{k} \cdot \left[\zeta'(-k, 1-x) + (-1)^k \zeta'(-k, 1+x)\right] x^{2n-1-k} \qquad (|x| < 1;\ n \in \mathbb{N}), \tag{4.11}$$

where, as usual,

$$\zeta'(s, a) = \frac{\partial}{\partial s}\{\zeta(s, a)\}. \tag{4.12}$$

Acknowledgements: The first-named author was supported by the *Basic Science Research Program* through the *National Research Foundation of Korea* funded by the Ministry of Education, Science and Technology of the Republic of Korea (2010-0011005). The second-named author was supported, in part, by the *Natural Sciences and Engineering Research Council of Canada* under Grant OGP0007353.

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(Received 11 June 2013)
(Accepted 21 October 2013)