Thai Journal of Mathematics
Volume 4 (2006) Number 1 : 179–196

A Non-Uniform Bound on Poisson Approximation for Sums of Bernoulli Random Variables with Small Mean

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Abstract : In many situations, the Poisson approximation is appropriate for sums of Bernoulli random variables where its mean, λ, is small. In this paper, we give non-uniform bounds of Poisson approximation with small values of $\lambda,\ \lambda \in (0,3]$, by using the Stein-Chen method. These bounds are sharper than the bounds of Teerapabolarn and Neammanee [12].

Keywords : Bernoulli summands, non-uniform bound, Poisson approximation, Stein-Chen method.
2000 Mathematics Subject Classification : 60F05, 60G05.

1 Introduction

Consider a random variable W that can be written as a sum $\sum_{\alpha\in\Gamma} X_\alpha$ of Bernoulli random variables, where Γ is an arbitrary finite index set. The random variables X_α may be dependent, and we will be interested in the case where each of success probability $p_\alpha = P(X_\alpha = 1) = 1 - P(X_\alpha = 0)$ is small. It is then reasonable to approximate the distribution of W by Poisson distribution with mean $\lambda = E[W] = \sum_{\alpha\in\Gamma} p_\alpha$.

In the past few years, many mathematicians have been developed the method for approximating the distribution of W (for example, see Stein [10], Arratia, Goldstein and Gordon [1-2], Barbour, Holst and Janson [4], Neammanee [8] and Teerapabolarn and Neammanee [11]).

In 2006, Teerapabolarn and Neammanee [12] gave three formulas of non-uniform bounds as follows. For $w_0 \in \{0, 1, \ldots, |\Gamma|\}$, a non-uniform bound by using the coupling method is of the form

$$\left| P(W \le w_0) - \sum_{k=0}^{w_0} \frac{\lambda^k e^{-\lambda}}{k!} \right| \le \lambda^{-1}(1 - e^{-\lambda}) \min\left\{1, \frac{e^\lambda}{w_0 + 1}\right\} \sum_{\alpha\in\Gamma} p_\alpha E|W - W_\alpha^*|, \tag{1.1}$$

where W_α^* is a random variable which has the same distribution as $W - X_\alpha$ conditional on $X_\alpha = 1$, i.e. $W_\alpha^* \sim (W - X_\alpha)|X_\alpha = 1$. Suppose that for each α there is a subset $\Gamma_\alpha \subsetneq \Gamma$ such that X_α is independent of the collection $\{X_\beta : \beta \notin \Gamma_\alpha\}$, a non-uniform bound in this case is in the form of

$$\left|P(W \le w_0) - \sum_{k=0}^{w_0} \frac{\lambda^k e^{-\lambda}}{k!}\right|$$
$$\le \lambda^{-1}(1-e^{-\lambda}) \min\left\{1, \frac{e^\lambda}{w_0+1}\right\} \left\{\sum_{\alpha\in\Gamma} p_\alpha^2 + \sum_{\alpha\in\Gamma}\sum_{\beta\in\Gamma_\alpha\setminus\{\alpha\}} (p_\alpha p_\beta + E[X_\alpha X_\beta])\right\}. \tag{1.2}$$

In the case of independent summands, (1.2) becomes

$$\left|P(W \le w_0) - \sum_{k=0}^{w_0} \frac{\lambda^k e^{-\lambda}}{k!}\right| \le \lambda^{-1}(1-e^{-\lambda}) \min\left\{1, \frac{e^\lambda}{w_0+1}\right\} \sum_{\alpha\in\Gamma} p_\alpha^2. \tag{1.3}$$

We know that in many situations, the Poisson approximation is appropriate for W with small values of λ. In this paper, we improve the bounds in (1.1), (1.2) and (1.3) to be more accurately when λ is small, i.e. $\lambda \in (0,3]$.

Let

$$\triangle(\lambda, w_0) = \begin{cases} \lambda^{-2}(\lambda + e^{-\lambda} - 1), & \text{if } w_0 = 0, \\ \max\left\{\frac{1-e^{-\lambda}(1+\lambda)}{\lambda^2}, \frac{(1-e^{-\lambda})(1+\lambda)}{6\lambda}\right\}, & \text{if } w_0 = 1, \\ \max\left\{\frac{(2+\lambda)[1-e^{-\lambda}(1+\lambda+\frac{\lambda^2}{2})]}{\lambda^3}, \frac{4(1-e^{-\lambda})(1+\lambda+\frac{\lambda^2}{2})+\lambda}{60\lambda}\right\}, & \text{if } w_0 = 2, \\ \max\left\{\frac{[w_0!(w_0-\lambda)+e^{-\lambda}\lambda^{w_0+1}](w_0+2)}{w_0(w_0+2-\lambda)(w_0+1)!}, \frac{2\lambda^{-1}(1-e^{-\lambda})e^\lambda+1}{3(w_0+1)(w_0+2)}\right\}, & \text{if } w_0 \in \{3,...,|\Gamma|\}. \end{cases} \tag{1.4}$$

Then the followings are our main results.

Theorem 1.1 *Let $w_0 \in \{0,1,\dots,|\Gamma|\}$ and $\lambda \in (0,3]$. Then*

(i)
$$\left|P(W \le w_0) - \sum_{k=0}^{w_0} \frac{\lambda^k e^{-\lambda}}{k!}\right| \le \triangle(\lambda, w_0) \sum_{\alpha\in\Gamma} p_\alpha E|W - W_\alpha^*|, \tag{1.5}$$

where there exists W_α^* such that $W_\alpha^* \sim (W - X_\alpha)|X_\alpha = 1$ and

(ii)

$$\left| P(W \le w_0) - \sum_{k=0}^{w_0} \frac{\lambda^k e^{-\lambda}}{k!} \right| \le \triangle(\lambda, w_0) \left\{ \sum_{\alpha\in\Gamma} p_\alpha^2 + \sum_{\alpha\in\Gamma} \sum_{\beta\in\Gamma_\alpha\setminus\{\alpha\}} (p_\alpha p_\beta + E[X_\alpha X_\beta]) \right\}, \tag{1.6}$$

where there exists Γ_α such that X_α is independent of $\{X_\beta : \beta \notin \Gamma_\alpha\}$ for every $\alpha \in \Gamma$.

From (1.6), if $W \ge W_\alpha^*$ or $W - X_\alpha \le W_\alpha^*$ for every $\alpha \in \Gamma$, then we have the convenience forms in Theorem 1.2.

Theorem 1.2 *Let $w_0 \in \{0, 1, ..., |\Gamma|\}$ and $\lambda \in (0, 3]$. Then*

(i)

$$\left| P(W \le w_0) - \sum_{k=0}^{w_0} \frac{\lambda^k e^{-\lambda}}{k!} \right| \le \triangle(\lambda, w_0)\{\lambda - Var[W]\}, \tag{1.7}$$

where $W \ge W_\alpha^*$ a.s. for every $\alpha \in \Gamma$ and

(ii)

$$\left| P(W \le w_0) - \sum_{k=0}^{w_0} \frac{\lambda^k e^{-\lambda}}{k!} \right| \le \triangle(\lambda, w_0) \left\{ Var[W] - \lambda + 2\sum_{\alpha\in\Gamma} p_\alpha^2 \right\}, \tag{1.8}$$

where $W - X_\alpha \le W_\alpha^*$ a.s. for every $\alpha \in \Gamma$.

If $\{X_\alpha : \alpha \in \Gamma\}$ is independent, we have $W_\alpha^* = W - X_\alpha$ and $E|W - W_\alpha^*| = p_\alpha$. Then the following corollary follows immediately from (1.6).

Corollary 1.3 *Let $\lambda \in (0, 3]$ and $\{X_\alpha : \alpha \in \Gamma\}$ be independent Bernoulli random variables. Then, for $w_0 \in \{0, 1, ..., |\Gamma|\}$,*

$$\left| P(W \le w_0) - \sum_{k=0}^{w_0} \frac{\lambda^k e^{-\lambda}}{k!} \right| \le \triangle(\lambda, w_0) \sum_{\alpha\in\Gamma} p_\alpha^2. \tag{1.9}$$

Theorem 1.4 *For $w_0 \in \{0, 1, ..., |\Gamma|\}$ and $\lambda \in (0, 3]$, we have*

$$\triangle(\lambda, w_0) < \lambda^{-1}(1 - e^{-\lambda}) \min\left\{1, \frac{e^\lambda}{w_0 + 1}\right\}. \tag{1.10}$$

Remark 1.5

(i) From Theorem 1.4, we see that for $\lambda \in (0,3]$, the bounds in (1.5), (1.6) and (1.9) are sharper than the bounds in (1.1), (1.2) and (1.3), respectively.

(ii) There are many applications such that the Poisson approximation to be more accurately for $\lambda \in (0,1]$, see these applications in Arratia, Goldstein and Gordon [1-2], Barbour, Holst and Janson [4] and Lange [7]. In this case, we have

$$\triangle(\lambda, w_0) = \begin{cases} \lambda^{-2}(\lambda + e^{-\lambda} - 1), & \text{if } w_0 = 0, \\ \lambda^{-2}[1 - e^{-\lambda}(1+\lambda)], & \text{if } w_0 = 1, \\ \lambda^{-3}(2+\lambda)[1 - e^{-\lambda}(1 + \lambda + \frac{\lambda^2}{2})], & \text{if } w_0 = 2, \\ \dfrac{[w_0!(w_0 - \lambda) + e^{-\lambda}\lambda^{w_0+1}](w_0+2)}{w_0(w_0+2-\lambda)(w_0+1)!}, & \text{if } w_0 \in \{3, ..., |\Gamma|\}. \end{cases} \tag{1.11}$$

2 Proof of Main Results

We will prove our main results by using the Stein-Chen method. The method was originally formulated for normal approximation by Stein [9] in 1972, and the idea was applied to Poisson case by Chen [5] in 1975. This method started by the Stein's equation for Poisson distribution which is, given h, defined by

$$\lambda f(w+1) - wf(w) = h(w) - \mathcal{P}_\lambda(h), \tag{2.1}$$

where

$$\mathcal{P}_\lambda(h) = e^{-\lambda}\sum_{l=0}^{\infty} h(l)\frac{\lambda^l}{l!}$$

and f and h are bounded real valued functions defined on $\mathbb{N}\cup\{0\}$. For $w_0 \in \mathbb{N}\cup\{0\}$, let $C_{w_0} = \{0, 1, .., w_0\}$ and $h_{C_{w_0}} : \mathbb{N}\cup\{0\} \to \mathbb{R}$ be defined by

$$h_{C_{w_0}}(w) = \begin{cases} 1, & \text{if } \ w \in C_{w_0}, \\ 0, & \text{if } \ w \notin C_{w_0}. \end{cases} \tag{2.2}$$

Following Barbour, Holst and Janson [4] p.7, the solution $U_\lambda h_{C_{w_0}}$ of (2.1) can be expressd in the form

$$U_\lambda h_{C_{w_0}}(w) = \begin{cases} (w-1)!\lambda^{-w}e^{\lambda}[\mathcal{P}_\lambda(h_{C_{w_0}})\mathcal{P}_\lambda(1 - h_{C_{w-1}})], & \text{if } \ w_0 < w, \\ (w-1)!\lambda^{-w}e^{\lambda}[\mathcal{P}_\lambda(h_{C_{w-1}})\mathcal{P}_\lambda(1 - h_{C_{w_0}})], & \text{if } \ w_0 \geq w, \\ 0, & \text{if } \ \ w = 0. \end{cases} \tag{2.3}$$

Let $V_\lambda h_{C_{w_0}}(w) = U_\lambda h_{C_{w_0}}(w+1) - U_\lambda h_{C_{w_0}}(w)$. Then, from (2.3), we have

$$V_\lambda h_{C_{w_0}}(w)$$
$$= \begin{cases} (w-1)!\lambda^{-(w+1)}e^{\lambda}\mathcal{P}_\lambda(h_{C_{w_0}})[w\mathcal{P}_\lambda(1-h_{C_w}) - \lambda\mathcal{P}_\lambda(1-h_{C_{w-1}})], & \text{if } w \geq w_0+1, \\ (w-1)!\lambda^{-(w+1)}e^{\lambda}\mathcal{P}_\lambda(1-h_{C_{w_0}})[w\mathcal{P}_\lambda(h_{C_w}) - \lambda\mathcal{P}_\lambda(h_{C_{w-1}})], & \text{if } 1 \leq w \leq w_0. \end{cases}$$
$$= \begin{cases} (w-1)!\lambda^{-(w+1)}e^{-\lambda}\sum_{j=0}^{w_0}\frac{\lambda^j}{j!}\sum_{k=w+1}^{\infty}(w-k)\frac{\lambda^k}{k!}, & \text{if } w \geq w_0+1, \\ (w-1)!\lambda^{-(w+1)}e^{-\lambda}\sum_{j=w_0+1}^{\infty}\frac{\lambda^j}{j!}\sum_{k=0}^{w}(w-k)\frac{\lambda^k}{k!}, & \text{if } 1 \leq w \leq w_0. \end{cases} \tag{2.4}$$

Hence, by(2.4),

$$V_\lambda h_{C_{w_0}}(w) \begin{cases} < 0, & \text{if } w \geq w_0+1, \\ > 0, & \text{if } 1 \leq w \leq w_0. \end{cases}$$

The following lemmas are the properties of $V_\lambda h_{Cw_0}$ which are used in the main theorem.

Lemma 2.1 $V_\lambda h_{Cw_0}$ *is increasing in* w *for* $w \in \{1, \ldots, w_0\}$.

Proof. We shall show that $0 < V_\lambda h_{Cw_0}(w+1) - V_\lambda h_{Cw_0}(w)$ for $1 \leq w \leq w_0 - 1$. Note that, from (2.4),

$$V_\lambda h_{Cw_0}(w+1) - V_\lambda h_{Cw_0}(w) = (w-1)!\lambda^{-(w+2)}\mathcal{P}_\lambda(1-h_{C_{w_0}})$$
$$\times \left\{ w\sum_{k=0}^{w+1}(w+1-k)\frac{\lambda^k}{k!} - \lambda\sum_{k=0}^{w}(w-k)\frac{\lambda^k}{k!} \right\}$$

and

$$\lambda\sum_{k=0}^{w}(w-k)\frac{\lambda^k}{k!} = \sum_{k=0}^{w}(w+1-(k+1))\frac{\lambda^{k+1}}{(k+1)!}(k+1)$$
$$= \sum_{k=0}^{w+1}(w+1-k)k\frac{\lambda^k}{k!}.$$

So, we have

$$\begin{aligned} V_\lambda h_{Cw_0}(w+1) - V_\lambda h_{Cw_0}(w) &= (w-1)!\lambda^{-(w+2)}\mathcal{P}_\lambda(1-h_{C_{w_0}}) \\ &\quad \times \left\{ \sum_{k=0}^{w+1}(w-k)(w+1-k)\frac{\lambda^k}{k!} \right\} \\ &> 0. \end{aligned}$$

□

Lemma 2.2 *Let* $w_0 \in \mathbb{N} \cup \{0\}$ *and* $w \geq 1$. *Then*

$$|V_\lambda h_{C_{w_0}}(w)| \leq \triangle(\lambda, w_0). \tag{2.5}$$

Proof. **Case 1.** $w_0 = 0$.
Follows from Teerapabolarn, Neammanee and Chongcharoen[13] p.14.
Case 2. $w_0 = 1$.
Note that, from (2.4),

$$V_\lambda h_{C_1}(w) = \begin{cases} e^{-\lambda}(1+\lambda)(w-1)! \sum_{k=w+1}^{\infty} (w-k)\dfrac{\lambda^{k-(w+1)}}{k!}, & \text{if } w \geq 2, \\ \lambda^{-2}[1 - e^{-\lambda}(1+\lambda)], & \text{if } w \leq 1. \end{cases}$$

Hence, for $w \geq 2$, we have

$$\begin{aligned} 0 &< -V_\lambda h_{C_1}(w) \\ &= e^{-\lambda}(1+\lambda)(w-1)! \left\{ \frac{1}{(w+1)!} + \frac{2\lambda}{(w+2)!} + \frac{3\lambda^2}{(w+3)!} + \cdots \right\} \\ &= \frac{e^{-\lambda}(1+\lambda)(w-1)!}{(w+1)!} \left\{ 1 + \frac{2\lambda}{w+2} + \frac{3\lambda^2}{(w+2)(w+3)} + \cdots \right\} \\ &\leq \frac{e^{-\lambda}(1+\lambda)}{6} \left\{ 1 + \frac{2\lambda}{4} + \frac{3\lambda^2}{20} + \cdots \right\} \\ &\leq \frac{\lambda^{-1}(1-e^{-\lambda})(1+\lambda)}{6}, \end{aligned}$$

which implies that

$$|V_\lambda h_{C_1}(w)| \leq \max\left\{ \lambda^{-2}[1 - e^{-\lambda}(1+\lambda)], \frac{\lambda^{-1}(1-e^{-\lambda})(1+\lambda)}{6} \right\}.$$

Case 3. $w_0 = 2$.
Since $V_\lambda h_{C_{w_0}}$ is positive for $1 \leq w \leq w_0$, by (2.4) and lemma 2.1, we have

$$V_\lambda h_{C_2}(w) \leq \begin{cases} e^{-\lambda}(1+\lambda+\frac{\lambda^2}{2})(w-1)! \sum_{k=w+1}^{\infty} (w-k)\dfrac{\lambda^{k-(w+1)}}{k!}, & \text{if } w \geq 3, \\ \lambda^{-3}(2+\lambda)[1 - e^{-\lambda}(1+\lambda+\frac{\lambda^2}{2})], & \text{if } 1 \leq w \leq 2. \end{cases}$$

Hence, for $w \geq 3$,

$$\begin{aligned}
0 &< -V_\lambda h_{C_2}(w+1, w) \\
&= \frac{e^{-\lambda}(1+\lambda+\frac{\lambda^2}{2})(w-1)!}{(w+1)!}\left\{1+\frac{2\lambda}{w+2}+\frac{3\lambda^2}{(w+2)(w+3)}+\cdots\right\} \\
&\leq \frac{e^{-\lambda}(1+\lambda+\frac{\lambda^2}{2})}{12}\left\{1+\frac{2\lambda}{5}+\frac{3\lambda^2}{30}+\cdots\right\} \\
&\leq \frac{e^{-\lambda}(1+\lambda+\frac{\lambda^2}{2})}{12}\left\{1+\frac{4\lambda^{-1}}{5}\left[\frac{\lambda^2}{2!}+\frac{\lambda^3}{3!}+\cdots\right]\right\} \\
&= \frac{e^{-\lambda}(1+\lambda+\frac{\lambda^2}{2})}{12}\left\{\frac{5+4\lambda^{-1}(e^\lambda-1-\lambda)}{5}\right\} \\
&= \frac{4\lambda^{-1}(1-e^{-\lambda})(1+\lambda+\frac{\lambda^2}{2})+e^{-\lambda}(1+\lambda+\frac{\lambda^2}{2})}{60} \\
&\leq \frac{4\lambda^{-1}(1-e^{-\lambda})(1+\lambda+\frac{\lambda^2}{2})+1}{60}.
\end{aligned}$$

So,

$$\begin{aligned}
&|V_\lambda h_{C_2}(w)| \\
&\quad \leq \max\left\{\lambda^{-3}(2+\lambda)\left[1-e^{-\lambda}\left(1+\lambda+\frac{\lambda^2}{2}\right)\right], \frac{4\lambda^{-1}(1-e^{-\lambda})(1+\lambda+\frac{\lambda^2}{2})+1}{60}\right\}.
\end{aligned}$$

Case 4. $w_0 \geq 3$.
Note that for $1 \leq w \leq w_0$, we have, by (2.4) and lemma 2.1,

$$\begin{aligned}
V_\lambda h_{C_{w_0}}(w) &\leq (w_0-1)!\lambda^{-(w_0+1)}\left\{e^{-\lambda}\sum_{k=w_0+1}^{\infty}\frac{\lambda^k}{k!}\right\}\left\{(w_0-\lambda)\sum_{k=0}^{w_0}\frac{\lambda^k}{k!}+\frac{\lambda^{w_0+1}}{w_0!}\right\} \\
&\leq \frac{w_0!(w_0-\lambda)+e^{-\lambda}\lambda^{w_0+1}}{w_0}\sum_{k=w_0+1}^{\infty}\frac{\lambda^{k-(w_0+1)}}{k!} \\
&= \frac{w_0!(w_0-\lambda)+e^{-\lambda}\lambda^{w_0+1}}{w_0}\left\{\frac{1}{(w_0+1)!}+\frac{\lambda}{(w_0+2)!}+\frac{\lambda^2}{(w_0+3)!}+\cdots\right\} \\
&= \frac{w_0!(w_0-\lambda)+e^{-\lambda}\lambda^{w_0+1}}{w_0(w_0+1)!}\left\{1+\frac{\lambda}{w_0+2}+\frac{\lambda^2}{(w_0+2)(w_0+3)}+\cdots\right\} \\
&\leq \frac{w_0!(w_0-\lambda)+e^{-\lambda}\lambda^{w_0+1}}{w_0(w_0+1)!}\left\{1+\frac{\lambda}{w_0+2}+\frac{\lambda^2}{(w_0+2)^2}+\cdots\right\} \\
&= \frac{[w_0!(w_0-\lambda)+e^{-\lambda}\lambda^{w_0+1}](w_0+2)}{w_0(w_0+2-\lambda)(w_0+1)!}
\end{aligned}$$

and for $w \geq w_0 + 1$,

$$\begin{aligned}
0 < -V_\lambda h_{C_{w_0}}(w) &\leq \frac{(w-1)!}{(w+1)!}\left\{1 + \frac{2\lambda}{w+2} + \frac{3\lambda^2}{(w+2)(w+3)} + \cdots\right\} \\
&= \frac{1}{(w_0+1)(w_0+2)}\left\{1 + \frac{2\lambda}{(w_0+3)} + \frac{3\lambda^2}{(w_0+3)(w_0+4)} + \cdots\right\} \\
&\leq \frac{1}{(w_0+1)(w_0+2)}\left\{1 + \frac{\lambda}{3} + \frac{\lambda^2}{14} + \cdots\right\} \\
&\leq \frac{1}{(w_0+1)(w_0+2)}\left\{1 + \frac{2\lambda^{-1}}{3}\left[\frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots\right]\right\} \\
&\leq \frac{1}{(w_0+1)(w_0+2)}\left\{\frac{3 + 2\lambda^{-1}(e^\lambda - 1 - \lambda)}{3}\right\} \\
&= \frac{2\lambda^{-1}(1-e^{-\lambda})e^\lambda + 1}{3(w_0+1)(w_0+2)}.
\end{aligned}$$

So, we have

$$|V_\lambda h_{C_{w_0}}(w)| \leq \max\left\{\frac{[w_0!(w_0-\lambda) + e^{-\lambda}\lambda^{w_0+1}](w_0+2)}{w_0(w_2+2-\lambda)(w_0+1)!}, \frac{2\lambda^{-1}(1-e^{-\lambda})e^\lambda + 1}{3(w_0+1)(w_0+2)}\right\}.$$

Hence, from case 1 to case 4, we have (2.5). □

Proof of Theorem 1.1.
(i) Teerapabolarn and Neammanee showed in [12] that

$$\left|P(W \leq w_0) - \sum_{k=0}^{w_0}\frac{\lambda^k e^{-\lambda}}{k!}\right| \leq \sup_{w\geq 1}|V_\lambda h_{C_{w_0}}(w)|\sum_{\alpha\in\Gamma}p_\alpha E|W - W_\alpha^*|.$$

Hence, by lemma 2.2, (1.5) holds.
(ii) Since

$$\begin{aligned}
\sum_{\alpha\in\Gamma}p_\alpha E|W - W_\alpha^*| &= \sum_{\alpha\in\Gamma}p_\alpha E\left|\sum_{\beta\in\Gamma}X_\beta - \sum_{\beta\in\Gamma\setminus\{\alpha\}}X_\beta|X_\alpha = 1\right| \\
&= \sum_{\alpha\in\Gamma}p_\alpha E\left|X_\alpha + \sum_{\beta\in\Gamma_\alpha\setminus\{\alpha\}}X_\beta - \sum_{\beta\in\Gamma_\alpha\setminus\{\alpha\}}X_\beta|X_\alpha = 1\right| \\
&\leq \sum_{\alpha\in\Gamma}p_\alpha^2 + \sum_{\alpha\in\Gamma}\sum_{\beta\in\Gamma_\alpha\setminus\{\alpha\}}\{p_\alpha p_\beta + P(X_\alpha = 1)E[X_\beta|X_\alpha = 1]\} \\
&= \sum_{\alpha\in\Gamma}p_\alpha^2 + \sum_{\alpha\in\Gamma}\sum_{\beta\in\Gamma_\alpha\setminus\{\alpha\}}\{p_\alpha p_\beta + E[E[X_\alpha X_\beta|X_\alpha]]\} \\
&= \sum_{\alpha\in\Gamma}p_\alpha^2 + \sum_{\alpha\in\Gamma}\sum_{\beta\in\Gamma_\alpha\setminus\{\alpha\}}(p_\alpha p_\beta + E[X_\alpha X_\beta]), \qquad (2.6)
\end{aligned}$$

so, by (1.5) and (2.6), (1.6) holds. □

Proof of Theorem 1.2.
From (1.5),

$$\left| P(W \leq w_0) - \sum_{k=0}^{w_0} \frac{\lambda^k e^{-\lambda}}{k!} \right| \leq \triangle(\lambda, w_0) \sum_{\alpha\in\Gamma} p_\alpha E|W - W_\alpha^*|,$$

it suffices to show that $\sum_{\alpha\in\Gamma} p_\alpha E|W - W_\alpha^*| = \lambda - Var[W]$ where $W \geq W_\alpha^*$ a.s. for every $\alpha \in \Gamma$ and $\sum_{\alpha\in\Gamma} p_\alpha E|W - W_\alpha^*| = Var[W] - \lambda + 2\sum_{\alpha\in\Gamma} p_\alpha^2$ where $W - X_\alpha \leq W_\alpha^*$ a.s. for every $\alpha \in \Gamma$.

(i) If $W \geq W_\alpha^*$, then

$$\begin{aligned}
\sum_{\alpha\in\Gamma} p_\alpha E|W - W_\alpha^*| &= \sum_{\alpha\in\Gamma} p_\alpha E[(W+1) - (W_\alpha^* + 1)] \\
&= \lambda^2 + \lambda - \sum_{\alpha\in\Gamma} p_\alpha E[(W - X_\alpha + 1)|X_\alpha = 1] \\
&= \lambda^2 + \lambda - \sum_{\alpha\in\Gamma} E[E[X_\alpha W|X_\alpha]] \\
&= \lambda^2 + \lambda - \sum_{\alpha\in\Gamma} E[X_\alpha W] \\
&= \lambda^2 + \lambda - E[W^2] \\
&= \lambda - Var[W].
\end{aligned}$$

(ii) If $W - X_\alpha \leq W_\alpha^*$, then

$$\begin{aligned}
\sum_{\alpha\in\Gamma} p_\alpha E|W - W_\alpha^*| &= \sum_{\alpha\in\Gamma} p_\alpha E|X_\alpha + (W - X_\alpha) - W_\alpha^*| \\
&= \sum_{\alpha\in\Gamma} p_\alpha \{E[X_\alpha] + E|(W_\alpha^* + 1) - (W - X_\alpha + 1)|\} \\
&= \sum_{\alpha\in\Gamma} p_\alpha \{E[W_\alpha^* + 1] - E[W + 1] + 2E[X_\alpha]\} \\
&= \sum_{\alpha\in\Gamma} p_\alpha E[(W - X_\alpha + 1)|X_\alpha = 1] - \lambda^2 - \lambda + 2\sum_{\alpha\in\Gamma} p_\alpha^2 \\
&= E[W^2] - \lambda^2 - \lambda + 2\sum_{\alpha\in\Gamma} p_\alpha^2 \\
&= Var[W] - \lambda + 2\sum_{\alpha\in\Gamma} p_\alpha^2.
\end{aligned}$$

□

Proof of Theorem 1.4. We shall show that

$$\triangle(\lambda, w_0) < \lambda^{-1}(1-e^{-\lambda}) \min\left\{1, \frac{e^{\lambda}}{w_0+1}\right\}.$$

Case 1. $w_0 = 0$.
Since $0 < \lambda^{-2}e^{-\lambda}(e^{\lambda}-1-\lambda)$, we have

$$\begin{aligned}\lambda^{-2}(\lambda+e^{-\lambda}-1) &= \lambda^{-1}(1-e^{-\lambda}) - \lambda^{-2}e^{-\lambda}(e^{\lambda}-1-\lambda)\\ &< \lambda^{-1}(1-e^{-\lambda})\\ &= \lambda^{-1}(1-e^{-\lambda})\min\{1, e^{\lambda}\}.\end{aligned}$$

Case 2. $w_0 = 1$.

$$\begin{aligned}\frac{(1-e^{-\lambda})(1+\lambda)}{6\lambda} &= \frac{\lambda^{-1}(1-e^{-\lambda})(1+\lambda)}{6}\\ &< \lambda^{-1}(1-e^{-\lambda})\min\left\{1, \frac{e^{\lambda}}{2}\right\}\end{aligned}$$

and

$$\begin{aligned}\frac{1-e^{-\lambda}(1+\lambda)}{\lambda^2} &= \frac{e^{-\lambda}(e^{\lambda}-1-\lambda)}{\lambda^2}\\ &= \lambda^{-1}e^{-\lambda}\sum_{k=2}^{\infty}\frac{\lambda^{k-1}}{k!}\\ &= \frac{\lambda^{-1}e^{-\lambda}}{2}\left\{\lambda + \frac{\lambda^2}{3} + \frac{\lambda^3}{12} + \cdots\right\}\\ &\le \frac{\lambda^{-1}(1-e^{-\lambda})}{2}\\ &< \lambda^{-1}(1-e^{-\lambda})\min\left\{1, \frac{e^{\lambda}}{2}\right\}.\end{aligned}$$

So, we have

$$\max\left\{\frac{1-e^{-\lambda}(1+\lambda)}{\lambda^2}, \frac{(1-e^{-\lambda})(1+\lambda)}{6\lambda}\right\} < \lambda^{-1}(1-e^{-\lambda})\min\left\{1, \frac{e^{\lambda}}{2}\right\}.$$

Case 3. $w_0 = 2$.
we observe that, for $\lambda \in (0, 3]$,

$$\frac{1}{4} < \lambda^{-1}(1-e^{-\lambda}) \tag{2.7}$$

thus

$$\begin{aligned}\frac{4(1-e^{-\lambda})(1+\lambda+\frac{\lambda^2}{2})+\lambda}{60\lambda} &= \frac{\lambda^{-1}(1-e^{-\lambda})(1+\lambda+\frac{\lambda^2}{2})}{15}+\frac{1}{60}\\ &< \frac{\lambda^{-1}(1-e^{-\lambda})(2+\lambda+\frac{\lambda^2}{2})}{15}\\ &< \lambda^{-1}(1-e^{-\lambda})\min\left\{1,\frac{e^{\lambda}}{3}\right\}\end{aligned}$$

and

$$\begin{aligned}\frac{(2+\lambda)[1-e^{-\lambda}(1+\lambda+\frac{\lambda^2}{2})]}{\lambda^3} &= \frac{(2+\lambda)e^{-\lambda}(e^{\lambda}-1-\lambda-\frac{\lambda^2}{2})}{\lambda^3}\\ &= (2+\lambda)\lambda^{-1}e^{-\lambda}\sum_{k=3}^{\infty}\frac{\lambda^{k-2}}{k!}\\ &= \frac{(2+\lambda)\lambda^{-1}e^{-\lambda}}{3!}\left\{\lambda+\frac{\lambda^2}{4}+\frac{\lambda^3}{4\cdot 5}+\cdots\right\}\\ &< \frac{\lambda^{-1}(1-e^{-\lambda})(2+\lambda)}{6}\\ &\le \lambda^{-1}(1-e^{-\lambda})\min\left\{1,\frac{e^{\lambda}}{3}\right\}.\end{aligned}$$

Hence

$$\max\left\{\frac{(2+\lambda)[1-e^{-\lambda}(1+\lambda+\frac{\lambda^2}{2})]}{\lambda^3},\frac{4(1-e^{-\lambda})(1+\lambda+\frac{\lambda^2}{2})+\lambda}{60\lambda}\right\} < \lambda^{-1}(1-e^{-\lambda})\min\left\{1,\frac{e^{\lambda}}{3}\right\}.$$

Case 4. $w_0 \ge 3$.
By (2.7),

$$\begin{aligned}\frac{2\lambda^{-1}(1-e^{-\lambda})e^{\lambda}+1}{3(w_0+1)(w_0+2)} &= \frac{2\lambda^{-1}(1-e^{-\lambda})e^{\lambda}+4(\frac{1}{4})}{3(w_0+1)(w_0+2)}\\ &< \frac{2\lambda^{-1}(1-e^{-\lambda})(e^{\lambda}+2)}{3(w_0+1)(w_0+2)}\\ &\le \frac{2\lambda^{-1}(1-e^{-\lambda})(e^{\lambda}+2)}{15(w_0+1)}\\ &< \lambda^{-1}(1-e^{-\lambda})\min\left\{1,\frac{e^{\lambda}}{w_0+1}\right\}.\end{aligned} \quad (2.8)$$

By the fact that $\dfrac{w_0+2}{w_0+2-\lambda} = 1 + \dfrac{\lambda}{w_0+2} + \dfrac{\lambda^2}{(w_0+2)^2} + \cdots$, we have

$$\begin{aligned}
&\frac{[w_0!(w_0-\lambda)+e^{-\lambda}\lambda^{w_0+1}](w_0+2)}{w_0(w_0+2-\lambda)(w_0+1)!}\\
&\qquad = \frac{w_0!(w_0-\lambda)+e^{-\lambda}\lambda^{w_0+1}}{w_0(w_0+1)!}\left\{1+\frac{\lambda}{w_0+2}+\frac{\lambda^2}{(w_0+2)^2}+\cdots\right\}\\
&\qquad \le \frac{w_0!(w_0-\lambda)e^{\lambda}e^{-\lambda}+e^{-\lambda}\lambda^{w_0+1}}{w_0(w_0+1)!}\lambda^{-1}\left\{\lambda+\frac{\lambda^2}{5}+\frac{\lambda^3}{25}+\cdots\right\}\\
&\qquad < \frac{w_0!(w_0-\lambda)e^{\lambda}+\lambda^{w_0+1}}{w_0(w_0+1)!}\lambda^{-1}e^{-\lambda}(e^{\lambda}-1)\\
&\qquad = \frac{w_0!(w_0-\lambda)e^{\lambda}+\lambda w_0!\frac{\lambda^{w_0}}{w_0!}}{w_0(w_0+1)!}\lambda^{-1}(1-e^{-\lambda})\\
&\qquad = \lambda^{-1}(1-e^{-\lambda})\frac{w_0e^{\lambda}-\lambda(e^{\lambda}-\frac{\lambda^{w_0}}{w_0!})}{w_0(w_0+1)}\\
&\qquad < \frac{\lambda^{-1}(1-e^{-\lambda})e^{\lambda}}{w_0+1}. \qquad (2.9)
\end{aligned}$$

For $0 < \lambda \le 1$, it follows from (2.9) that

$$\frac{[w_0!(w_0-\lambda)+e^{-\lambda}\lambda^{w_0+1}](w_0+2)}{w_0(w_0+2-\lambda)(w_0+1)!} < \lambda^{-1}(1-e^{-\lambda}), \qquad (2.10)$$

and in the case of $1 < \lambda \le 3$, we observe that $\dfrac{e^{-\lambda}\lambda^{w_0+1}}{w_0!}$ is decreasing for $w_0 \ge 3$ and $f(t) = e^{-t}t^2$ has maximum at $t = 2$ for $t \in [1,3]$. Thus

$$\frac{e^{-\lambda}\lambda^{w_0+1}}{w_0!} \le \frac{e^{-\lambda}\lambda^4}{3!} \le \frac{3^2e^{-\lambda}\lambda^2}{3!} = 1.5e^{-\lambda}\lambda^2 \le 6e^{-2} < 1.$$

Hence, for $1 < \lambda \le 3$, we have

$$\begin{aligned}
\frac{[w_0!(w_0-\lambda)+e^{-\lambda}\lambda^{w_0+1}](w_0+2)}{w_0(w_0+2-\lambda)(w_0+1)!} &= \frac{[w_0-\lambda+\frac{e^{-\lambda}\lambda^{w_0+1}}{w_0!}](w_0+2)}{w_0(w_0+1)(w_0+2-\lambda)}\\
&< \frac{(w_0+2)(w_0+1-\lambda)}{w_0(w_0+1)(w_0+2-\lambda)}\\
&= \frac{1}{w_0} - \frac{\lambda}{w_0(w_0+1)(w_0+2-\lambda)}. \qquad (2.11)
\end{aligned}$$

If $w_0 \ge 4$, by (2.7) and (2.11), we have

$$\begin{aligned}
\frac{[w_0!(w_0-\lambda)+e^{-\lambda}\lambda^{w_0+1}](w_0+2)}{w_0(w_0+2-\lambda)(w_0+1)!} &< \frac{1}{4}\\
&< \lambda^{-1}(1-e^{-\lambda}). \qquad (2.12)
\end{aligned}$$

If $w_0 = 3$, then by (2.11),

$$\begin{aligned}\frac{[w_0!(w_0-\lambda)+e^{-\lambda}\lambda^{w_0+1}](w_0+2)}{w_0(w_0+2-\lambda)(w_0+1)!} &< \frac{1}{3} - \frac{\lambda}{3(4)(5-\lambda)} \\ &\leq \frac{1}{3} - \frac{1}{3(4)(4)} \\ &< 3^{-1}(1-e^{-3}) \\ &\leq \lambda^{-1}(1-e^{-\lambda}). \qquad (2.13)\end{aligned}$$

By (2.8), (2.9), (2.10), (2.12) and (2.13), we have

$$\max\left\{\frac{[w_0!(w_0-\lambda)+e^{-\lambda}\lambda^{w_0+1}](w_0+2)}{w_0(w_0+2-\lambda)(w_0+1)!}, \frac{2\lambda^{-1}(1-e^{-\lambda})e^{\lambda}+1}{3(w_0+1)(w_0+2)}\right\} < \lambda^{-1}(1-e^{-\lambda})\min\left\{1, \frac{e^{\lambda}}{w_0+1}\right\}.$$

Hence, from case 1 to 4, the theorem is proved. □

3 Applications

In this section, we apply the results in (1.5) and (1.6) of Theorem 1.1 and in (1.7) and (1.8) of Theorem 1.2 to some related problems.

Example 3.1 (The number of triangles in a random graph problem)

Consider a graph with n nodes which is created by randomly connecting some pairs of nodes by edges. If the connection probability per pair is p, then all pairs from a triple of nodes are connected with probability p^3. Let Γ be the set of all triple of nodes in the random graph, and let W be the number of such triangles in the random graph. So $W = \sum_{\alpha\in\Gamma} X_\alpha$ where $X_\alpha = 1$ if triple of nodes α is connected to be the triangle and $X_\alpha = 0$ otherwise. We then have $p_\alpha = P(X_\alpha = 1) = p^3$ and $\lambda = |\Gamma|p^3 = \binom{n}{3}p^3$. If p is small, W is approximately Poisson with mean λ.

We apply Theorem 1.1 (2) to bound the error of this approximation by taking $\Gamma_\alpha = \{\beta : |\alpha \cap \beta| \geq 2\}$, and observe that X_α and X_β are independent for $\beta \notin \Gamma_\alpha$. For $\alpha \neq \beta$, $E[X_\alpha X_\beta] = P(X_\alpha = 1, X_\beta = 1) = p^5$ and $|\Gamma_\alpha| = 3(n-3)+1$. Hence, by (1.6), we have

$$\left|P(W \leq w_0) - \sum_{k=0}^{w_0}\frac{\lambda^k e^{-\lambda}}{k!}\right| \leq \triangle(\lambda, w_0)\left\{\binom{n}{3}p^5\{(3n-8)p+3(n-3)\}\right\}, \qquad (3.1)$$

where $w_0 \in \{0, 1, ..., \binom{n}{3}\}$. For $n = 10$ and $p = 0.1$, we can show some Poisson estimate $P(W \leq w_0)$ and the bound of (3.1) in Table 3.1.

Table 3.1 Poisson estimate of $P(W \le w_0)$ for $n = 10$ and $p = 0.1$

w_0	Estimate	Uniform bound	Non-uniform bound (1.2)	Non-uniform bound (1.6)
0	0.66697681	0.18208043	0.18208043	0.09716869
1	0.93710242	0.18208043	0.13649682	0.08491173
2	0.99180285	0.18208043	0.09099788	0.06571378
3	0.99918741	0.18208043	0.06824841	0.04289475
4	0.99993510	0.18208043	0.05459873	0.03510103
5	0.99999566	0.18208043	0.04549894	0.02960146
6	0.99999975	0.18208043	0.03899909	0.02554915
7	0.99999999	0.18208043	0.03412421	0.02245364
8	1.00000000	0.18208043	0.03033263	0.02001767

Example 3.2 (The number of isolated vertices in a random graph problem)

Let $G(n,p)$ be a graph on n labeled vertices $\{1, 2, \dots, n\}$, where each possible edge is present randomly and independently with probability p. Let $X_\alpha = 1$ if vertex α is an isolated vertex in $G(n,p)$ and $X_\alpha = 0$ otherwise. Then $W = \sum_{\alpha=1}^{n} X_\alpha$ is the number of isolated vertices in $G(n,p)$, and $p_\alpha = P(X_\alpha = 1) = (1-p)^{n-1}$, $\lambda = E[W] = n(1-p)^{n-1}$ and $Var[W] = \lambda + n(n-1)(1-p)^{2n-3} - \lambda^2$. In constructing W_α^*, Barbour [3] setting W_α^* is the number of isolated vertices obtained from the graph $G(n,p)$, by dropping vertex α and deleting all the edges $\{\alpha, \beta\}$ for $1 \le \beta \le n$ and $\beta \ne \alpha$, and showed that $W_\alpha^* \ge W - X_\alpha$. By applying (1.8), a non-uniform bound of the error in Poisson approximation to the distribution of W is of the form

$$\left| P(W \le w_0) - \sum_{k=0}^{w_0} \frac{\lambda^k e^{-\lambda}}{k!} \right| \le \triangle(\lambda, w_0)\lambda^2 \left[\frac{(n-2)p+1}{n(1-p)} \right],$$

where $w_0 \in \{0, ..., n\}$. Table 3.2 shows some Poisson estimate $P(W \le w_0)$ for $n = 100$ and $p = 0.036$.

Example 3.3 (The ménage problem)

The classical ménage problem of combinatorics is defined as follows: if n married couples are seated around a circle table with men and women alternating, but husbands and wives are randomly scrambled, then the number of married couples W seated next to each other is approximately Poisson distributed. We number the places around the table from 1 to $2n$, so W can be represented as $W = \sum_{\alpha=1}^{2n} X_\alpha$ where $X_{2n+1} = X_1$, and $X_\alpha = 1$ if a couple occupies seats α and $\alpha + 1$ and $X_\alpha = 0$ otherwise. We then have, by symmetry, $p_\alpha = P(X_\alpha = 1) = 1/n$ and $\lambda = E[W] = 2$.

To construct the coupled random variable W_α^*, Janson [6] constructed it by exchange the person in seat $\alpha + 1$ with the spouse of the person in seat α and then count the number of adjacent spouse pairs, excluding the pair now occupying seats α and $\alpha+1$. Since it does not easily to calculate $E|W-W_{\alpha}^{*}|$, Lange [7] p.251 bounded it by $6/n$, i.e., $E|W-W_{\alpha}^{*}|\leq 6/n$. Hence, by (1.5), we have

$$\left|P(W\leq w_0)-\sum_{k=0}^{w_0}\frac{\lambda^k e^{-\lambda}}{k!}\right|\leq\frac{12\triangle(\lambda,w_0)}{n}, \tag{3.2}$$

where $w_0\in\{0,1,...,2n\}$. Table 3.3 shows some representative Poisson estimate of $P(W\leq w_0)$ for $n=100$.

Table 3.2 Poisson estimate of $P(W\leq w_0)$ for $n=100$ and $p=0.036$

w_0	Estimate	Uniform bound	Non-uniform bound (1.1)	Non-uniform bound (1.8)
0	0.07048730	0.11580067	0.11580067	0.08092203
1	0.25744238	0.11580067	0.11580067	0.07049023
2	0.50537497	0.11580067	0.11580067	0.06085784
3	0.72457405	0.11580067	0.11580067	0.06026915
4	0.86992072	0.11580067	0.11580067	0.05132209
5	0.94702197	0.11580067	0.11580067	0.04525973
6	0.98110487	0.11580067	0.11580067	0.04046440
7	0.99401899	0.11580067	0.11580067	0.03665955
8	0.99830055	0.11580067	0.11580067	0.03347233
9	0.99956233	0.11580067	0.11580067	0.03072631
10	0.99989700	0.11580067	0.11580067	0.02833802
11	0.99997769	0.11580067	0.11580067	0.02625339
12	0.99999553	0.11580067	0.11580067	0.02442781
13	0.99999916	0.11580067	0.11580067	0.02282233
14	0.99999985	0.11580067	0.10952390	0.02140347
15	0.99999998	0.11580067	0.10267866	0.02014297

Table 3.3 Poisson estimate of $P(W\leq w_0)$ for $n=100$

w_0	Estimate	Uniform bound	Non-uniform bound (1.1)	Non-uniform bound (1.5)
0	0.13533528	0.05187988	0.05187988	0.03406006
1	0.40600585	0.05187988	0.05187988	0.02593994
2	0.67667642	0.05187988	0.05187988	0.01939942
3	0.85712346	0.05187988	0.05187988	0.02268157
4	0.94734698	0.05187988	0.05187988	0.01962402
5	0.98343639	0.05187988	0.05187988	0.01720420
6	0.99546619	0.05187988	0.05187988	0.01532975
7	0.99890328	0.05187988	0.04791792	0.01379445
8	0.99976255	0.05187988	0.04259371	0.01250358
9	0.99995350	0.05187988	0.03833434	0.01140803
10	0.99999169	0.05187988	0.03484940	0.01047283
11	0.99999864	0.05187988	0.03194528	0.00966944

Example 3.4 (Pólya's urn model)

The urn contains N balls of n different colors in proportions $\pi_1, \pi_2, ..., \pi_n$. Balls are drawn at random from the urn in such away that at each draw the probability of obtaining a specific ball is the same for all balls. When a ball is sampled, it is put back into urn with another ball of the same color. Let $X_\alpha = 1$ if no balls of color α are drawn in the first r samplings and $X_\alpha = 0$ otherwise. So $W = \sum_{\alpha=1}^n X_\alpha$ is the number of colors which have not appeared in the first r samplings. Then

$$\begin{aligned}
p_\alpha &= P(X_\alpha = 1) \\
&= P(\text{ No balls of color } \alpha \text{ are drawn in the first } r \text{ samplings }) \\
&= \left(\frac{N - N\pi_\alpha}{N}\right)\left(\frac{N+1-N\pi_\alpha}{N+1}\right)\cdots\left(\frac{N+r-1-N\pi_\alpha}{N+r-1}\right) \\
&= \exp\left\{\sum_{k=0}^{r-1}\log\left(1 - \frac{N\pi_\alpha}{N+k}\right)\right\}, \\
p_{\alpha\beta} &= E[X_\alpha X_\beta] \\
&= P(\text{ No balls of colors } \alpha \text{ and } \beta \text{ are drawn in the first } r \text{ samplings }) \\
&= \left[\frac{N - N(\pi_\alpha + \pi_\beta)}{N}\right]\left[\frac{N+1-N(\pi_\alpha + \pi_\beta)}{N+1}\right]\cdots\left[\frac{N+r-1-N(\pi_\alpha + \pi_\beta)}{N+r-1}\right] \\
&= \exp\left\{\sum_{k=0}^{r-1}\log\left(1 - \frac{N(\pi_\alpha + \pi_\beta)}{N+k}\right)\right\}.
\end{aligned}$$

For constructing W_α^* such that $W_\alpha^* \sim (W - X_\alpha)|X_\alpha = 1$, Barbour, Holst and Janson [4] letting W_α^* is the number of colors which have not appeared in the first r samplings given that no balls of color α are drawn in the first r samplings, and showed that $W \geq W_\alpha^*$. So, by (1.7), a non-uniform bound of the error of this approximation is in the form of

$$\left|P(W \leq w_0) - \sum_{k=0}^{w_0}\frac{\lambda^k e^{-\lambda}}{k!}\right| \leq \triangle(\lambda, w_0)\{\lambda - Var[W]\}, \tag{3.3}$$

where $w_0 \in \{0, ..., n\}$ and

$$\begin{aligned}
Var[W] &= E[W^2] - \lambda^2 \\
&= \sum_{\alpha=1}^n E[X_\alpha] + \sum_{\alpha=1}^n \sum_{\beta=1,\beta\neq\alpha}^n E[X_\alpha X_\beta] - \sum_{\alpha=1}^n p_\alpha^2 - \sum_{\alpha=1}^n \sum_{\beta=1,\beta\neq\alpha}^n p_\alpha p_\beta \\
&= \sum_{\alpha=1}^n p_\alpha(1 - p_\alpha) + \sum_{\alpha=1}^n \sum_{\beta=1,\beta\neq\alpha}^n (p_{\alpha\beta} - p_\alpha p_\beta).
\end{aligned}$$

In the case of $\pi_1 = \pi_2 = \cdots = \pi_n = \pi$, we have, for all $\alpha, \beta \in \{1, ..., n\}$,

$$p_\alpha = \exp\left\{\sum_{k=0}^{r-1} \log\left(1 - \frac{N\pi}{N+k}\right)\right\},\ p_{\alpha\beta} = \exp\left\{\sum_{k=0}^{r-1} \log\left(1 - \frac{2N\pi}{N+k}\right)\right\},$$

$$\lambda = np_\alpha \text{ and } Var[W] = \lambda - \lambda^2 + n(n-1)p_{\alpha\beta}.$$

So, by (3.3),

$$\left| P(W \leq w_0) - \sum_{k=0}^{w_0} \frac{\lambda^k e^{-\lambda}}{k!} \right| \leq \triangle(\lambda, w_0)\{\lambda^2 - n(n-1)p_{\alpha\beta}\},$$

where $w_0 \in \{0, ..., n\}$. Table 3.3 shows all Poisson estimate $P(W \leq w_0)$ for $N = 100$, $n = 10$, $r = 25$ and $\pi_1 = \cdots = \pi_{10} = 0.1$.

Table 3.4 Poisson estimate of $P(W \leq w_0)$ for $N = 100,\ n = 10$ and $r = 25$

w_0	Estimate	Uniform bound	Non-uniform bound (1.1)	Non-uniform bound (1.7)
0	0.38491106	0.17364302	0.17364302	0.10043145
1	0.75240220	0.17364302	0.17364302	0.07321157
2	0.92783199	0.17364302	0.15037501	0.06604059
3	0.98366211	0.17364302	0.11278126	0.05825994
4	0.99698796	0.17364302	0.09022501	0.04900929
5	0.99953251	0.17364302	0.07518751	0.04210898
6	0.99993741	0.17364302	0.06444643	0.03676769
7	0.99999264	0.17364302	0.05639063	0.03254910
8	0.99999923	0.17364302	0.05012500	0.02915746
9	0.99999993	0.17364302	0.04511250	0.02638366
10	0.99999999	0.17364302	0.04101137	0.02407906

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(Received 5 June 2005)

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