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Iterative Solution of Fixed Points Problem, System of Generalized Mixed Equilibrium Problems and Variational Inclusion Problems[1]

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Abstract : In this paper, we study an algorithm for finding a common element of the set of common fixed points of an infinite family of asymptotically nonexpansive mappings, the set of common solutions to a system of generalized mixed equilibrium problems and the set of solutions to variational inclusion in a real Hilbert space. We prove that the scheme converges strongly to a common element of the three afore mentioned sets. Finally, we give applications of our results to Optimization problems in a real Hilbert space.

Keywords : strong convergence; asymptotically nonexpansive mapping; generalized mixed equilibrium problem; variational inclusion; Hilbert spaces.

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1 Introduction

Throughout this paper, $\mathbb{R}$ denotes the set of real numbers. We shall assume that H is a real Hilbert space with inner product $\langle , \rangle$ and norm $||.||$, while K will stand for a nonempty, closed and convex subset of H.

A mapping $A : K \to H$ is called $\alpha-$ *inverse-strongly monotone* (see, for example, [1, 2]) if and only if there exists $\alpha > 0$ such that $\langle Ax - Ay, x - y\rangle \geq \alpha||Ax - Ay||^2 \ \ \forall \ \ x, y \in K$.

Let $A : H \to H$ be a single-valued nonlinear mapping and let $M : H \to 2^H$ be a set-valued mapping. The variational inclusion is to find $u \in H$ such that

$$\theta \in A(u) + M(u), \tag{1.1}$$

where θ is a zero vector in H. The set of solutions to the variational inclusion (1.1) is denoted by $I(A, M)$. When $A \equiv 0$, (1.1) becomes the inclusion problem introduced by Rockafellar [3].

A set-valued mapping $M : H \to 2^H$ is called monotone if and only if for all $x, y \in H, f \in M(x)$ and $g \in M(y)$ we have that $\langle x - y, f - g\rangle \geq 0$. A monotone mapping M is said to be maximal if and only if the graph $G(M)$ is not properly contained in the graph of any other monotone map, where $G(M) := \{(x, y) \in H \times H : y \in M(x)\}$. Equivalently, M is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(M)$ implies that $f \in M(x)$. The resolvent operator $J_{M,\lambda}$ associated with M and λ is the mapping $J_{M,\lambda} : H \to H$ defined by

$$J_{M,\lambda}(u) = (I + \lambda M)^{-1}(u), \quad u \in H, \quad \lambda > 0. \tag{1.2}$$

It is known that the resolvent operator $J_{M,\lambda}$ is single-valued, nonexpansive and $1-$inverse-strongly monotone (see, for example, [4]) and that a solution of (1.1) is a fixed point of $J_{M,\lambda}(I - \lambda A), \forall \ \lambda > 0$ (see, for example, [5]). If $0 < \lambda \leq 2\alpha$, it is easy to see that $J_{M,\lambda}(I - \lambda A)$ is nonexpansive and $I(A, M)$ is closed and convex.

Let $\varphi : K \to \mathbb{R}$ be a real-valued function and $A : K \to H$ be a nonlinear mapping. Suppose $F : K \times K \to \mathbb{R}$ is an *equilibrium bi-function*, that is, $F(u, u) = 0, \ \ \forall \ \ u \in K$. The generalized mixed equilibrium problem is to find $x \in K$ (see e.g., [6–8]) such that

$$F(x, y) + \varphi(y) - \varphi(x) + \langle Ax, y - x\rangle \geq 0, \tag{1.3}$$

for all $y \in K$. We shall denote the set of solutions of the generalized mixed equilibrium problem by Ω. Thus

$$\Omega := \{x^* \in K : F(x^*, y) + \varphi(y) - \varphi(x^*) + \langle Ax^*, y - x^*\rangle \geq 0 \ \ \forall y \in K\}.$$

If $\varphi = 0, \ \ A = 0$, then problem (1.3) reduces to equilibrium problem studied by many authors (see e.g., [9–15]) which is to find $x^* \in K$ such that

$$F(x^*, y) \geq 0, \tag{1.4}$$

for all $y \in K$. The set of solutions of (1.4) is denoted by $EP(F)$.

If $\varphi = 0$, then problem (1.3) reduces to generalized equilibrium problem studied by many authors (see e.g., [16–18]) which is to find $x^* \in K$ such that

$$F(x^*, y) + \langle Ax^*, y - x^*\rangle \geq 0 \tag{1.5}$$

for all $y \in K$. The set of solutions of (1.5) is denoted by $GEP(F, A)$.

If $A = 0$, then problem (1.3) reduces to mixed equilibrium problem considered by many authors (see, for example, [19–22]) which is to find $x^* \in K$ such that

$$F(x^*, y) + \varphi(y) - \varphi(x^*) \geq 0, \tag{1.6}$$

for all $y \in K$. The set of solutions of (1.6) is denoted by $MEP(F, \varphi)$.

The generalized mixed equilibrium problem includes fixed point problems, optimization problems, variational inequality problems, Nash equilibrium problems and equilibrium problems as special cases (see e.g., [23]). Numerous problems in Physics, optimization and economics reduce to find a solution of problem (1.3). Several methods have been proposed to solve the fixed point problems, variational inequality problems and generalized mixed equilibrium problems in the literature. See e.g., [18, 21, 24, 25].

A mapping $T : K \to K$ is said to be *nonexpansive* if

$$||Tx - Ty|| \leq ||x - y||, \tag{1.7}$$

for all $x, y \in K$. A point $x \in K$ is called *a fixed point* of T if $Tx = x$. The set of fixed points of T is the set $F(T) := \{x \in K : Tx = x\}$.

Finding a common element of the set of fixed points of nonexpansive mappings and the set of solutions of variational inclusions and equilibrium problems has been studied by many researchers (see e.g., [18, 26–28] and the references contained therein).

Recently, Takahashi and Takahashi [18] introduced an iterative scheme for approximating the common element of the set of fixed points of a nonexpansive mapping and the set of solutions to a generalized equilibrium problem in a real Hilbert space. In particular, they proved the following theorem.

Theorem 1.1 (Takahashi and Takahashi [18]). *Let K be a nonempty, closed and convex subset of a real Hilbert space H. Let F be a bi-function from $K \times K$ satisfying $(A1) - (A4)$, ψ be an $\mu-$ inverse-strongly monotone mapping of K into H and let T be a nonexpansive mapping of K into itself. Suppose $F(T) \cap EP \neq \emptyset$ and $u \in K$. Let $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ be generated by $x_1 \in K$,*

$$\begin{cases} F(z_n, y) + \langle \psi x_n, y - z_n\rangle + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq 0 \quad \forall y \in K \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)T[\alpha_n u + (1 - \alpha_n)z_n], n \geq 1; \end{cases} \tag{1.8}$$

where $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty} \subset [0, 1)$ and $\{r_n\}_{n=1}^{\infty} \subset [0, 2\mu]$. If $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ are chosen so that $\{r_n\}_{n=1}^{\infty} \subset [a, b]$ for some a, b with $0 < a < b < 2\mu$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} |r_{n+1} - r_n| = 0, 0 < c \leq \beta_n \leq d < 1$ then, $\{x_n\}_{n=1}^{\infty}$ converges strongly to $z_0 = P_{F(T)\cap EP}u$.

Most recently, Shehu [28] modified the algorithm (1.9) and obtained strong convergence of the scheme to an element common to the set of fixed points of nonexpansive maps, set of solution of generalized equilibrium problems and the set of solution of variational inclusion. He proved the following result.

Theorem 1.2 (Shehu [28]). *Let K be a nonempty, closed and convex subset of a real Hilbert space H. Let F be a bi-function from $K\times K \to \mathbb{R}$ satisfying $(A1)-(A4)$, ψ be a $\mu-$ inverse-strongly monotone mapping of K into H A an $\alpha-$ inverse-strongly monotone mapping of K into H and $M : H \to 2^H$ a maximal monotone mapping. Let $T : H \to H$ be a nonexpansive mapping such that $\Omega := F(T) \cap I(A,M) \cap EP \neq \emptyset$ and suppose $f : H \to H$ is a contraction map with constant $\gamma \in (0,1)$. Let $\{x_n\}_{n=1}^\infty$ and $\{z_n\}_{n=1}^\infty$ be generated by $x_1 \in K$,*

$$\begin{cases} F(z_n, y) + \langle \psi x_n, y - z_n\rangle + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq 0 \quad \forall y \in K \\ x_{n+1} = \beta_n x_n + (1-\beta_n)T[\alpha_n f(x_n) + (1-\alpha_n)J_{M,\lambda}(u_n - \lambda A u_n)], n \geq 1; \end{cases} \tag{1.9}$$

where $\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty \subset [0,1]$ and $\{r_n\}_{n=1}^\infty \subset [0,\infty)$ satisfying (i) $0 < c \leq \beta_n \leq d < 1$, (ii) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$, (iii) $\lambda \in (0, 2\alpha]$, (iv) $0 < a < r_n < b < 2\mu$, $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$.
Then, $\{x_n\}_{n=1}^\infty$ converges strongly to $z_0 = P_{F(T)\cap EP}u$.

Let K be a nonempty subset of a real normed linear space E. A mapping $T : K \to K$ is called *asymptotically nonexpansive* (see e.g., Goebel and Kirk [29]) if there exists a sequence $\{k_n\}, k_n \geq 1$, such that $\lim_{n\to\infty} k_n = 1$, and

$$||T^n x - T^n y|| \leq k_n ||x - y||$$

holds for each $x, y \in K$ and for each integer $n \geq 1$. Many authors have studied the approximation of fixed points of asymptotically nonexpansive maps (see e.g., [30–34] and the references contained therein).

Motivated by [18, 28], we introduce an iterative scheme by using the so-called hybrid method, and prove that the scheme strongly converges to an element common to the set of solutions of a system of generalized mixed equilibrium problem, the set of fixed points of infinite family of asymptotically nonexpansive mappings and the set of solutions to a variational inclusion in a real Hilbert space. Finally, we give some applications of our results to Optimization problems in a real Hilbert space.

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle.,.\rangle$ and norm $||.||$ and let K be a nonempty closed and convex subset of H. In what follows, we shall write $x_n \to x$ as $n \to \infty$ to mean that $\{x_n\}_{n=1}^\infty$ converges strongly to x.

For any point $u \in H$,there exists a unique point $P_K u \in K$ such that

$$||u - P_K u|| \leq ||u - y||, \quad \forall y \in K. \tag{2.1}$$

P_K is called the *metric projection* of H onto K. We know that P_K is a nonexpansive mapping of H onto K. It is also known that P_K satisfies

$$\langle x-y, P_Kx-P_Ky\rangle \geq ||P_Kx-P_Ky||^2, \tag{2.2}$$

for all $x, y \in H$. Furthermore, P_Kx is characterized by the properties $P_Kx \in K$ and

$$\langle x-P_Kx, P_Kx-y\rangle \geq 0, \tag{2.3}$$

for all $y \in K$ and

$$||x-P_Kx||^2 \leq ||x-y||^2 - ||y-P_Kx||^2 \quad \forall\, x \in H, \quad y \in K. \tag{2.4}$$

If A is an α-inverse-strongly monotone mapping of K into H, then it is obvious that A is $\frac{1}{\alpha}$-Lipschitz continuous. We also have that for all $x, y \in K$ and $r > 0$,

$$\begin{aligned} ||(I-rA)x-(I-rA)y||^2 &= ||x-y-r(Ax-Ay)||^2 \\ &= ||x-y||^2 - 2r\langle Ax-Ay, x-y\rangle + r^2||Ax-Ay||^2 \\ &\leq ||x-y||^2 + r(r-2\alpha)||Ax-Ay||^2. \end{aligned} \tag{2.5}$$

So, if $r \leq 2\alpha$, then $I-rA$ is a nonexpansive mapping of K into H.

For solving the generalized mixed equilibrium problem for a bifunction $F : K \times K \to \mathbb{R}$, let us assume that F, φ and K satisfy the following conditions:

(A1) $F(x, x) = 0$ for all $x \in K$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y, \in K$;

(A3) for each $x, y, z \in K$, $\lim_{t\to 0} F(tz + (1-t)x, y) \leq F(x, y)$;

(A4) for each $x \in K$, $y \mapsto F(x, y)$ is convex and lower semicontinuous;

(B1) for each $x \in H$ and $r > 0$ there exist a bounded subset $D_x \subseteq K$ and $y_x \in K$ such that for any $z \in K \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0; \tag{2.6}$$

(B2) K is a bounded set.

Then, we have the following lemma.

Lemma 2.1 (Wangkeeree and Wangkeeree [35])**.** *Assume that $F : K \times K \to \mathbb{R}$ satisfies (A1)-(A4) and let $\varphi : K \to \mathbb{R}$ be a proper lower semicontinuous and convex function. Assume that either $(B1)$ or $(B2)$ holds. For $r > 0$ and $x \in H$, define a mapping $T_r^{(F,\varphi)} : H \to K$ as follows:*

$$T_r^{(F,\varphi)}(x) = \left\{z \in K : F(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \forall y \in K\right\}$$

for all $z \in H$. Then, the following hold:

1. *for each* $x \in H$, $T_r^{(F,\varphi)}(x) \neq \emptyset$;
2. $T_r^{(F,\varphi)}$ *is single-valued;*
3. $T_r^{(F,\varphi)}$ *is firmly nonexpansive, i.e., for any* $x, y \in H$,

$$||T_r^{(F,\varphi)}x - T_r^{(F,\varphi)}y||^2 \leq \langle T_r^{(F,\varphi)}x - T_r^{(F,\varphi)}y, x - y\rangle;$$

4. $F(T_r^{(F,\varphi)}) = GMEP(F)$;
5. $GMEP(F)$ *is closed and convex.*

We shall also use the following lemma in our results

Lemma 2.2 (Baillon and Haddad [36]). *Let E be a Banach space, let f be a continuously Fréchet differentiable convex functional on E and let ∇f be the gradient of f. If ∇f is $\frac{1}{\alpha}$-Lipschitz continuous, then ∇f is α-inverse-strongly monotone.*

Lemma 2.3. *Let H be a real Hilbert space, and K a nonempty closed convex subset of H. Then for all $x, y, z \in H$ and a real number $a \in \mathbb{R}$, the set*

$$\{v \in K : ||y - v||^2 \leq ||x - v||^2 + \langle z, v\rangle + a\}$$

is closed and convex.

Lemma 2.4 (Goebel and Kirk [29]). *Let K be a nonempty, closed and convex and bounded subset of a uniformly convex Banach space X, and let $T : K \to K$ be asymptotically nonexpansive. Then T has a fixed point.*

Lemma 2.5 (Lemaire [5]). *Let $M : H \to 2^H$ be a maximal monotone mapping and $A : H \to H$ be a Lipschitz continuous mapping. Then the mapping $S = M + A : H \to 2^H$ is a maximal monotone mapping.*

3 Main Results

We now prove our main theorems.

Lemma 3.1 (Goebel and Kirk [29]). *Let K be a nonempty closed and convex subset of a real Hilbert space H and let $T : K \to K$ be assymptotically nonexpansive. Then the set of fixed points of T, $F(T)$ is closed and convex.*

Lemma 3.2. *Let K be a nonempty, closed and convex subset of a real Hilbert space H. For each $m = 1, 2$, let F_m be a bi-function from $K \times K \to \mathbb{R}$ satisfying $(A1)-(A4)$, $\varphi_m : K \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function with assumption $(B1)$ or $(B2)$, A be an α-inverse-strongly monotone mapping of K into H, B be a β-inverse-strongly monotone mapping of K into H and for each $i = 1, 2, \ldots$, let $T_i : K \to K$ be an asymptotically nonexpansive mapping. Let D be a γ-inverse-strongly monotone mapping of K into H. Suppose $F :=$*

$\cap_{i=1}^{\infty} F(T_i) \cap GMEP(F_1, A, \varphi_1) \cap GMEP(F_2, B, \varphi_2) \cap I(D, M) \neq \emptyset$ *and bounded*. *Let* $\{z_n\}_{n=1}^{\infty}$, $\{u_n\}_{n=1}^{\infty}$, $\{w_n\}_{n=1}^{\infty}$, $\{y_{n,i}\}_{n=1}^{\infty}$ $(i = 1, 2, \ldots)$ *and* $\{x_n\}_{n=0}^{\infty}$ *be generated by* $x_0 \in K$, $C_{1,i} = K$, $C_1 = \cap_{i=1}^{\infty} C_{1,i}$, $x_1 = P_{C_1} x_0$

$$\begin{cases} z_n = T_{r_n}^{(F_1,\varphi_1)}(x_n - r_n A x_n) \\ u_n = T_{\lambda_n}^{(F_2,\varphi_2)}(z_n - \lambda_n B z_n) \\ w_n = J_{M,s_n}(u_n - s_n D u_n) \\ y_{n,i} = \alpha_n x_n + (1-\alpha_n) T_i^n w_n \\ C_{n+1,i} = \{z \in C_{n,i} : ||y_{n,i} - z||^2 \leq ||x_n - z||^2 \\ \qquad -\alpha_n(1-\alpha_n)||x_n - T_i^n y_{n,i}||^2 + \theta_{n,i}\} \\ C_{n+1} = \cap_{i=1}^{\infty} C_{n+1,i} \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \geq 1, \end{cases} \tag{3.1}$$

where $\theta_{n,i} = (1-\alpha_n)(k_{n,i}^2 - 1)(\sup_{x^* \in F}\{||x_n - x^*||^2\}), i = 1, 2, \ldots$. *Assume that* $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$, $\{r_n\}_{n=1}^{\infty} \subset [0, 2\alpha]$ *and* $\{\lambda_n\}_{n=1}^{\infty} \subset [0, 2\beta]$ *satisfy* $(i) 0 < a \leq r_n \leq b < 2\alpha$, $(ii) 0 < c \leq \lambda_n \leq f < 2\beta$, $(iii) \lim_{n\to\infty} \alpha_n = 0$, (iv) $0 < h \leq s_n \leq j < 2\gamma$.
Then for each $n \geq 0$, *the following hold:*

1. C_n *is closed and convex,*
2. $F \subset C_n$,
3. $\{x_n\}$ *is well defined.*

Proof. Observe that Lemma 2.3 implies that $C_{n,i}$ is closed and convex for each $n \geq 1$ and for each $i = 1, 2, \ldots$. This implies that C_n is closed and convex for $n \geq 1$, establishing (1). For $n = 1$, $F \subset K = C_{1,i}$. For $n \geq 2$, let $x^* \in F$. We have

$$\begin{aligned} ||y_{n,i} - x^*||^2 &= ||\alpha_n(x_n - x^*) + (1-\alpha_n)(T_i w_n - x^*)||^2 \\ &= \alpha_n ||x_n - x^*||^2 + (1-\alpha_n)||T_i w_n - x^*||^2 - \alpha_n(1-\alpha_n)||x_n - T_i w_n||^2 \\ &\leq \alpha_n ||x_n - x^*||^2 + (1-\alpha_n) k_{n,i}^2 ||w_n - x^*||^2 - \alpha_n(1-\alpha_n)||x_n - T_i w_n||^2 \\ &\leq \alpha_n ||x_n - x^*||^2 + (1-\alpha_n) k_{n,i}^2 ||x_n - x^*||^2 - \alpha_n(1-\alpha_n)||x_n - T_i w_n||^2 \\ &= [1 + (1-\alpha_n)(k_{n,i}^2 - 1)]||x_n - x^*||^2 - \alpha_n(1-\alpha_n)||x_n - T_i w_n||^2 \\ &\leq ||x_n - x^*||^2 - \alpha_n(1-\alpha_n)||x_n - T_i w_n||^2 + \theta_{n,i} \end{aligned}$$

which shows that $x^* \in C_{n,i}$, $\forall$ $n \geq 2$, $\forall$ $i = 1, 2, \ldots$. Thus $F \subset C_{n,i}$ $\forall$ $n \geq 1$, $\forall$ $i = 1, 2, \ldots$. Hence $F \subset C_n$ $\forall$ $n \geq 1$, establishing (2). Therefore $\{x_n\}$ is well defined, completing the proof. □

Lemma 3.3. *Let* K *be a nonempty, closed and convex subset of a real Hilbert space* H. *For each* $m = 1, 2$, *let* F_m *be a bi-function from* $K \times K \to \mathbb{R}$ *satisfying (A1) − (A4),* $\varphi_m : K \to \mathbb{R} \cup \{+\infty\}$ *be a proper lower semicontinuous and convex function with assumption (B1) or (B2),* A *be an* α*-inverse-strongly monotone mapping of* K *into* H, B *be a* β*-inverse-strongly monotone mapping of* K *into* H *and for each* $i = 1, 2, \ldots$, *let* $T_i : K \to K$ *be an asymptotically* *nonexpansive mapping such that* $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$. *Let* D *be a* γ*-inverse-strongly monotone mapping of* K *into* H. *Suppose* $F := \cap_{i=1}^{\infty} F(T_i) \cap GMEP(F_1, A, \varphi_1) \cap GMEP(F_2, B, \varphi_2) \cap I(D, M) \neq \emptyset$ *and bounded . Let* $\{z_n\}_{n=1}^{\infty}$, $\{u_n\}_{n=1}^{\infty}$, $\{w_n\}_{n=1}^{\infty}$, $\{y_{n,i}\}_{n=1}^{\infty}$ $(i = 1, 2, \ldots)$ *and* $\{x_n\}_{n=0}^{\infty}$ *be as defined in Lemma 3.2, then the sequences* $\{z_n\}_{n=1}^{\infty}$, $\{u_n\}_{n=1}^{\infty}$, $\{w_n\}_{n=1}^{\infty}$, $\{y_{n,i}\}_{n=1}^{\infty}$ $(i = 1, 2, \ldots)$, $\{x_n\}_{n=0}^{\infty}$ *are bounded and* $||x_{n+1} - x_n|| \to 0, n \to \infty$.

Proof. Since $x_n = P_{C_n} x_0 \ \ \forall n \geq 1$ and $x_{n+1} \in C_{n+1} \subset C_n \ \ \forall n \geq 1$, we have

$$||x_n - x_0|| \leq ||x_{n+1} - x_0|| \ \ \forall n \geq 0. \tag{3.2}$$

Again, from $F \subset C_n$ and using inequality (2.1), we obtain

$$||x_n - x_0|| \leq ||z - x_0|| \ \ z \in F \ \ \forall n \geq 0. \tag{3.3}$$

From inequalities (3.2) and (3.3), we have that $\lim_{n\to\infty} ||x_n - x_0||$ exists. Hence $\{x_n\}_{n=0}^{\infty}$ is bounded and so are $\{z_n\}_{n=0}^{\infty}$, $\{Ax_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$, $\{Du_n\}_{n=0}^{\infty}$, $\{Bz_n\}_{n=0}^{\infty}$, $\{w_n\}_{n=0}^{\infty}$, $\{T_i^n w_n\}_{n=0}^{\infty}$ and $\{y_{n,i}\}_{n=0}^{\infty}$ $i = 1, 2, \ldots$. For $m > n \geq 1$, we have that $x_m = P_{C_m} x_0 \in C_m \subset C_n$. By inequality (2.4), we obtain

$$||x_m - x_n||^2 \leq ||x_n - x_0||^2 - ||x_m - x_0||^2. \tag{3.4}$$

Letting $m, n \to \infty$ in inequality (3.4), we obtain $||x_m - x_n|| \to 0$. In particular $\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0$, completing the proof. □

Lemma 3.4. *Let* K *be a nonempty, closed and convex subset of a real Hilbert space* H. *For each* $m = 1, 2$, *let* F_m *be a bi-function from* $K \times K \to \mathbb{R}$ *satisfying* $(A1)-(A4)$, $\varphi_m : K \to \mathbb{R} \cup \{+\infty\}$ *be a proper lower semicontinuous and convex function with assumption* $(B1)$ *or* $(B2)$, A *be an* α*-inverse-strongly monotone mapping of* K *into* H, B *be a* β*-inverse-strongly monotone mapping of* K *into* H *and for each* $i = 1, 2, \ldots$, *let* $T_i : K \to K$ *be an asymptotically nonexpansive mapping such that* $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$. *Let* D *be a* γ*-inverse-strongly monotone mapping of* K *into* H. *Suppose* $F := \cap_{i=1}^{\infty} F(T_i) \cap GMEP(F_1, A, \varphi_1) \cap GMEP(F_2, B, \varphi_2) \cap I(D, M) \neq \emptyset$ *and bounded . Let* $\{z_n\}_{n=1}^{\infty}$, $\{u_n\}_{n=1}^{\infty}$, $\{w_n\}_{n=1}^{\infty}$, $\{y_{n,i}\}_{n=1}^{\infty}$ $(i = 1, 2, \ldots)$ *and* $\{x_n\}_{n=0}^{\infty}$ *be as defined in Lemma 3.2, then* $\lim_{n\to\infty} ||z_n - u_n|| = \lim_{n\to\infty} ||w_n - x_n|| = 0$. *In addition* $\lim_{n\to\infty} ||y_{n,i} - x_n|| = \lim_{n\to\infty} ||w_n - T_i w_n|| = 0 \ \ (i = 1, 2, \ldots)$.

Proof. By hypothesis $F \neq \emptyset$. Let $x^* \in F$, then using the fact that $J_{M,\lambda}(I - s_n D)$ is nonexpansive for all $n \in \mathbb{N}$, we have

$$\begin{aligned} ||w_n - x^*||^2 &= ||J_{M,\lambda}(u_n - s_n Du_n) - J_{M,\lambda}(x^* - s_n Dx^*)||^2 \\ &\leq ||u_n - x^*||^2. \end{aligned}$$

Since both $I - r_n A$ and $I - \lambda_n B$ are nonexpansive for each $n \geq 1$, using inequality (2.5) and the fact that $x^* = T_{r_n}^{(F_1,\varphi_1)}(x^* - r_n Ax^*)$, $x^* = T_{\lambda_n}^{(F_2,\varphi_2)}(x^* - \lambda_n Bx^*)$, we obtain

$$\begin{aligned}||u_n - x^*||^2 &= ||T_{\lambda_n}^{(F_2,\varphi_2)}(z_n - \lambda_n B z_n) - T_{\lambda_n}^{(F_2,\varphi_2)}(x^* - \lambda_n B x^*)||^2 \\ &\leq ||z_n - x^*||^2\end{aligned}$$

and

$$\begin{aligned}||z_n - x^*||^2 &= ||T_{r_n}^{(F_1,\varphi_1)}(x_n - r_n A x_n) - T_{r_n}^{(F_1,\varphi_1)}(x^* - r_n A x^*)||^2 \\ &\leq ||x_n - x^*||^2.\end{aligned}$$

Therefore, $||u_n - x^*|| \leq ||x_n - x^*||$. Since $x_{n+1} = P_{C_{n+1}} x_0 \in C_{n+1}$, then for each $i = 1, 2, \ldots,$

$$||y_{n,i} - x_{n+1}||^2 \leq ||x_n - x_{n+1}||^2 - \alpha_n(1-\alpha_n)||x_n - T_i w_n||^2 + \theta_{n,i} \to 0.$$

Using the fact that

$$||y_{n,i} - x_n|| \leq ||y_{n,i} - x_{n+1}|| + ||x_n - x_{n+1}||,$$

we obtain that $\lim_{n\to\infty} ||y_{n,i} - x_n|| = 0, \quad i = 1, 2, \ldots$. Furthermore, for each $i = 1, 2, \ldots,$

$$\begin{aligned}&||y_{n,i} - x^*||^2 \\ &\leq \alpha_n ||x_n - x^*||^2 + (1-\alpha_n)||T_i^n w_n - x^*||^2 \\ &\leq \alpha_n ||x_n - x^*||^2 + (1-\alpha_n)k_{n,i}^2 ||u_n - x^*||^2 \\ &\leq \alpha_n ||x_n - x^*||^2 + (1-\alpha_n)k_{n,i}^2 ||T_{\lambda_n}^{(F_2,\varphi_2)}(z_n - \lambda_n B z_n) - T_{\lambda_n}^{(F_2,\varphi_2)}(x^* - \lambda_n B x^*)||^2 \\ &\leq \alpha_n ||x_n - x^*||^2 + (1-\alpha_n)k_{n,i}^2 ||(z_n - \lambda_n B z_n) - (x^* - \lambda_n B x^*)||^2 \\ &\leq \alpha_n ||x_n - x^*||^2 + (1-\alpha_n)k_{n,i}^2 \Big[||z_n - x^*||^2 + \lambda_n(\lambda_n - 2\beta)||B z_n - B x^*||^2\Big] \\ &\leq \alpha_n ||x_n - x^*||^2 + (1-\alpha_n)k_{n,i}^2 \Big[||x_n - x^*||^2 + \lambda_n(\lambda_n - 2\beta)||B z_n - B x^*||^2\Big] \\ &\leq ||x_n - x^*||^2 + (1-\alpha_n)(k_{n,i}^2 - 1)||x_n - x^*||^2 \\ &\qquad + (1-\alpha_n)k_{n,i}^2 \lambda_n(\lambda_n - 2\beta)||B z_n - B x^*||^2 \\ &\leq ||x_n - x^*||^2 + (1-\alpha_n)k_{n,i}^2 \lambda_n(\lambda_n - 2\beta)||B z_n - B x^*||^2 + \theta_{n,i}.\end{aligned}$$

Since $0 < c \leq \lambda_n \leq f < 2\beta$, we have for each $i = 1, 2, \ldots,$

$$\begin{aligned}c(2\beta - f)(1-\alpha_n)k_{n,i}^2 ||B z_n - B x^*||^2 &\leq ||x_n - x^*||^2 - ||y_{n,i} - x^*||^2 + \theta_{n,i} \\ &\leq ||y_{n,i} - x_n||(||x_n - x^*|| + ||y_{n,i} - x^*||) + \theta_{n,i}.\end{aligned}$$

So,

$$\begin{aligned}||B z_n - B x^*||^2 &\leq \frac{1}{(1-\alpha_n)k_{n,i}^2 c(2\beta - f)}||y_{n,i} - x_n||(||x_n - x^*|| + ||y_{n,i} - x^*||) \\ &\quad + \frac{1}{k_{n,i}^2 c(2\beta - f)}\theta_{n,i}.\end{aligned}$$

Hence, $\lim_{n\to\infty}||Bz_n - Bx^*|| = 0$. From the recursion formula (3.1), we have

$$\begin{aligned}||y_{n,i} - x^*||^2 &\le \alpha_n||x_n - x^*||^2 + (1-\alpha_n)||T_i^n w_n - x^*||^2 \\ &\le \alpha_n||x_n - x^*||^2 + (1-\alpha_n)k_{n,i}^2||u_n - x^*||^2. \qquad (3.5)\end{aligned}$$

On the other hand,

$$\begin{aligned}||u_n - x^*||^2 &\le ||T_{\lambda_n}^{(F_2,\varphi_2)}(z_n - \lambda_n Bz_n) - T_{\lambda_n}^{(F_2,\varphi_2)}(x^* - \lambda_n Bx^*)||^2 \\ &\le \langle (z_n - \lambda_n Bz_n) - (x^* - \lambda_n Bx^*), u_n - x^*\rangle \\ &= \frac{1}{2}\Big[||(z_n - \lambda_n Bz_n) - (x^* - \lambda_n Bx^*)||^2 + ||u_n - x^*||^2 \\ &\quad - ||(z_n - \lambda_n Bz_n) - (x^* - \lambda_n Bx^*) - (u_n - x^*)||^2\Big] \\ &\le \frac{1}{2}\Big[||z_n - x^*||^2 + ||u_n - x^*||^2 \\ &\quad - ||(z_n - \lambda_n Bz_n) - (x^* - \lambda_n Bx^*) - (u_n - x^*)||^2\Big] \\ &= \frac{1}{2}\Big[||z_n - x^*||^2 + ||u_n - x^*||^2 - ||u_n - z_n||^2 \\ &\quad + 2\lambda_n\langle z_n - u_n, Bz_n - Bx^*\rangle - \lambda_n^2||Bz_n - Bx^*||^2\Big]\end{aligned}$$

and hence

$$\begin{aligned}||u_n - x^*||^2 &\le ||z_n - x^*||^2 - ||u_n - z_n||^2 + 2\lambda_n\langle z_n - u_n, Bz_n - Bx^*\rangle \\ &\quad - ||Bz_n - Bx^*||^2 \\ &\le ||z_n - x^*||^2 - ||u_n - z_n||^2 + 2\lambda_n||z_n - u_n||||Bz_n - Bx^*|| \\ &\le ||x_n - x^*||^2 - ||u_n - z_n||^2 + 2\lambda_n||z_n - u_n||||Bz_n - Bx^*||. \qquad (3.6)\end{aligned}$$

Putting inequality (3.6) into inequality (3.5), we have

$$\begin{aligned}||y_{n,i} - x^*||^2 &\le \alpha_n||x_n - x^*||^2 + (1-\alpha_n)k_{n,i}^2||x_n - x^*||^2 - (1-\alpha_n)k_{n,i}^2||u_n - z_n||^2 \\ &\quad + 2(1-\alpha_n)k_{n,i}^2\lambda_n||z_n - u_n||||Bz_n - Bx^*|| \\ &= ||x_n - x^*||^2 - (1-\alpha_n)k_{n,i}^2||u_n - z_n||^2 \\ &\quad + 2(1-\alpha_n)k_{n,i}^2\lambda_n||z_n - u_n||||Bz_n - Bx^*|| + \theta_{n,i}.\end{aligned}$$

It follows that

$$\begin{aligned}(1-\alpha_n)k_{n,i}^2||z_n - u_n||^2 &\le ||x_n - x^*||^2 - ||y_{n,i} - x^*||^2 \\ &\quad + 2(1-\alpha_n)k_{n,i}^2\lambda_n||z_n - u_n||||Bz_n - Bx^*|| + \theta_{n,i} \\ &\le ||y_{n,i} - x_n||(||x_n - x^*|| + ||y_{n,i} - x^*||) \\ &\quad + 2(1-\alpha_n)k_{n,i}^2\lambda_n||z_n - u_n||||Bz_n - Bx^*|| + \theta_{n,i}.\end{aligned}$$

Consequently,

$$||z_n - u_n||^2 \le \frac{1}{(1-\alpha_n)k_{n,i}^2}||y_{n,i} - x_n||(||x_n - x^*|| + ||y_{n,i} - x^*||) + 2\lambda_n||z_n - u_n||||Bz_n - Bx^*|| + \frac{1}{(1-\alpha_n)k_{n,i}^2}\theta_{n,i}.$$

Therefore $\lim_{n\to\infty}||z_n - u_n|| = 0$. Furthermore,

$$\begin{aligned}
||y_{n,i} - x^*||^2 &\le \alpha_n||x_n - x^*||^2 + (1-\alpha_n)||T_i^n w_n - x^*||^2 \\
&\le \alpha_n||x_n - x^*||^2 + (1-\alpha_n)k_{n,i}^2||u_n - x^*||^2 \\
&\le \alpha_n||x_n - x^*||^2 + (1-\alpha_n)k_{n,i}^2||z_n - x^*||^2 \\
&\le \alpha_n||x_n - x^*||^2 \\
&\quad + (1-\alpha_n)k_{n,i}^2||T_{r_n}^{(F_1,\varphi_1)}(x_n - r_nAx_n) - T_{r_n}^{(F_1,\varphi_1)}(x^* - r_nAx^*)||^2 \\
&\le \alpha_n||x_n - x^*||^2 + (1-\alpha_n)k_{n,i}^2||(x_n - r_nAx_n) - (x^* - r_nAx^*)||^2 \\
&\le \alpha_n||x_n - x^*||^2 \\
&\quad + (1-\alpha_n)k_{n,i}^2\Big[||x_n - x^*||^2 + r_n(r_n - 2\alpha)||Ax_n - Ax^*||^2\Big] \\
&\le ||x_n - x^*||^2 + (1-\alpha_n)(k_{n,i}^2 - 1)||x_n - x^*||^2 \\
&\quad + (1-\alpha_n)k_{n,i}^2 r_n(r_n - 2\alpha)||Ax_n - Ax^*||^2 \\
&\le ||x_n - x^*||^2 + (1-\alpha_n)k_{n,i}^2 r_n(r_n - 2\alpha)||Ax_n - Ax^*||^2 + \theta_{n,i}.
\end{aligned}$$

Since $0 < a \le r_n \le b < 2\alpha$, we have

$$\begin{aligned}
(1-\alpha_n)k_{n,i}^2 a(2\alpha - b)||Ax_n - Ax^*||^2 &\le ||x_n - x^*||^2 - ||y_{n,i} - x^*||^2 + \theta_{n,i} \\
&\le ||y_{n,i} - x_n||(||x_n - x^*|| + ||y_{n,i} - x^*||) + \theta_{n,i}.
\end{aligned}$$

So,

$$||Ax_n - Ax^*||^2 \le \frac{1}{(1-\alpha_n)k_{n,i}^2 a(2\alpha - b)}||y_{n,i} - x_n||(||x_n - x^*|| + ||y_{n,i} - x^*||) + \frac{1}{(1-\alpha_n)k_{n,i}^2 a(2\alpha - b)}\theta_{n,i}.$$

Hence $\lim_{n\to\infty}||Ax_n - Ax^*|| = 0$. From (3.1), we have

$$\begin{aligned}
||y_{n,i} - x^*||^2 &\le \alpha_n||x_n - x^*||^2 + (1-\alpha_n)||T_i^n w_n - x^*||^2 \\
&= \alpha_n||x_n - x^*||^2 + (1-\alpha_n)k_{n,i}^2||z_n - x^*||^2 \\
&\le \alpha_n||x_n - x^*||^2 + (1-\alpha_n)k_{n,i}^2||z_n - x^*||^2. \qquad (3.7)
\end{aligned}$$

Moreover,

$$\begin{aligned}||z_n - x^*||^2 &\leq ||T_{r_n}^{(F_1,\varphi_1)}(x_n - r_nAz_n) - T_{r_n}^{(F_1,\varphi_1)}(x^* - r_nAx^*)||^2\\
&\leq \langle (x_n - r_nAx_n) - (x^* - r_nAx^*), x_n - x^*\rangle\\
&= \frac{1}{2}\Big[||(x_n - r_nAx_n) - (x^* - r_nAx^*)||^2 + ||z_n - x^*||^2\\
&\quad - ||(x_n - r_nAx_n) - (x^* - r_nAx^*) - (z_n - x^*)||^2\Big]\\
&\leq \frac{1}{2}\Big[||x_n - x^*||^2 + ||z_n - x^*||^2\\
&\quad - ||(x_n - r_nAx_n) - (x^* - r_nAx^*) - (x_n - x^*)||^2\Big]\\
&= \frac{1}{2}\Big[||x_n - x^*||^2 + ||z_n - x^*||^2 - ||x_n - z_n||^2\\
&\quad + 2r_n\langle x_n - z_n, Ax_n - Ax^*\rangle - r_n^2||Ax_n - Ax^*||^2\Big]\end{aligned}$$

and hence

$$\begin{aligned}||z_n - x^*||^2 &\leq ||x_n - x^*||^2 - ||x_n - z_n||^2 + 2r_n\langle x_n - z_n, Ax_n - Ax^*\rangle\\
&\quad - ||Ax_n - Ax^*||^2\\
&\leq ||x_n - x^*||^2 - ||z_n - z_n||^2 + 2r_n||z_n - z_n||||Ax_n - Ax^*||\\
&\leq ||x_n - x^*||^2 - ||x_n - z_n||^2 + 2r_n||x_n - z_n||||Ax_n - Ax^*||. \qquad (3.8)\end{aligned}$$

Putting (3.8) into (3.7), we have for each $i = 1, 2, \ldots,$

$$\begin{aligned}||y_{n,i} - x^*||^2 &\leq \alpha_n||x_n - x^*||^2 + (1-\alpha_n)k_{n,i}^2||x_n - x^*||^2 - (1-\alpha_n)k_{n,i}^2||x_n - z_n||^2\\
&\quad + 2(1-\alpha_n)k_{n,i}^2r_n||x_n - z_n||||Ax_n - Ax^*||\\
&= ||x_n - x^*||^2 + (1-\alpha_n)(k_{n,i}^2 - 1)||x_n - x^*||^2 - (1-\alpha_n)k_{n,i}^2||x_n - z_n||^2\\
&\quad + 2(1-\alpha_n)k_{n,i}^2r_n||x_n - z_n||||Ax_n - Ax^*||\\
&\leq ||x_n - x^*||^2 - (1-\alpha_n)k_{n,i}^2||x_n - z_n||^2\\
&\quad + 2(1-\alpha_n)k_{n,i}^2r_n||x_n - z_n||||Ax_n - Ax^*|| + \theta_{n,i}.\end{aligned}$$

It follows that for each $i = 1, 2, \ldots,$

$$\begin{aligned}(1-\alpha_n)k_{n,i}^2||x_n - z_n||^2 &\leq ||x_n - x^*||^2 - ||y_{n,i} - x^*||^2\\
&\quad + 2(1-\alpha_n)k_{n,i}^2r_n||x_n - u_n||||Ax_n - Ax^*|| + \theta_{n,i}\\
&\leq ||y_{n,i} - x_n||(||x_n - x^*|| + ||y_{n,i} - x^*||)\\
&\quad + 2(1-\alpha_n)k_{n,i}^2r_n||x_n - z_n||||Ax_n - Ax^*|| + \theta_{n,i}.\end{aligned}$$

Consequently,

$$||x_n - z_n||^2 \leq \frac{1}{(1-\alpha_n)k_{n,i}^2}||y_{n,i} - x_n||(||x_n - x^*|| + ||y_{n,i} - x^*||) + 2r_n||x_n - z_n||||Az_n - Ax^*|| + \frac{1}{(1-\alpha_n)k_{n,i}^2}\theta_{n,i}, \quad i = 1, 2, \ldots.$$

Therefore, $\lim_{n\to\infty} ||x_n - z_n|| = 0$. But $y_{n,i} = \alpha_n x_n + (1-\alpha_n)T_i^n w_n$ for each $i = 1, 2, \ldots$, implies that

$$||y_{n,i} - T_i^n w_n|| = \alpha_n ||x_n - T_i^n w_n|| \to 0. \tag{3.9}$$

Consequently, we have

$$||x_n - T_i^n w_n|| \leq ||y_{n,i} - T_i^n w_n|| + ||y_{n,i} - x_n|| \to 0, \quad i = 1, 2, \ldots.$$

Furthermore, for each $i = 1, 2, \ldots$,

$$\begin{aligned}
||y_{n,i} - x^*||^2 &\leq \alpha_n ||x_n - x^*||^2 + (1-\alpha_n)||T_i^n w_n - x^*||^2 \\
&\leq \alpha_n ||x_n - x^*||^2 + (1-\alpha_n)k_{n,i}^2 ||w_n - x^*||^2 \\
&\leq \alpha_n ||x_n - x^*||^2 \\
&\quad + (1-\alpha_n)k_{n,i}^2 ||J_{M,\lambda}(u_n - s_n Du_n) - J_{M,\lambda}(x^* - s_n Dx^*)||^2 \\
&\leq \alpha_n ||x_n - x^*||^2 + (1-\alpha_n)k_{n,i}^2 ||(u_n - s_n Du_n) - (x^* - s_n Dx^*)||^2 \\
&\leq \alpha_n ||x_n - x^*||^2 \\
&\quad + (1-\alpha_n)k_{n,i}^2 [||u_n - x^*||^2 + s_n(s_n - 2\gamma)||Du_n - Dx^*||^2] \\
&\leq \alpha_n ||x_n - x^*||^2 + (1-\alpha_n)k_{n,i}^2 ||x_n - x^*||^2 \\
&\quad + (1-\alpha_n)k_{n,i}^2 s_n(s_n - 2\gamma)||Du_n - Dx^*||^2 \\
&\leq ||x_n - x^*||^2 + (1-\alpha_n)(k_{n,i}^2 - 1)||x_n - x^*||^2 \\
&\quad + (1-\alpha_n)k_{n,i}^2 s_n(s_n - 2\gamma)||Du_n - Dx^*||^2 \\
&\leq ||x_n - x^*||^2 + (1-\alpha_n)k_{n,i}^2 s_n(s_n - 2\gamma)||Du_n - Dx^*||^2 + \theta_{n,i}.
\end{aligned}$$

Thus,

$$\begin{aligned}
(1-\alpha_n)k_{n,i}^2 h(2\gamma - j)||Du_n - Dx^*||^2 &\leq ||x_n - x^*||^2 - ||y_{n,i} - x^*||^2 + \theta_{n,i} \\
&\leq ||y_{n,i} - x_n||(||x_n - x^*|| + ||y_{n,i} - x^*||) + \theta_{n,i}.
\end{aligned}$$

So,

$$||Du_n - Dx^*||^2 \leq \frac{1}{(1-\alpha_n)k_{n,i}^2 h(2\gamma - j)}||y_{n,i} - x_n||(||x_n - x^*|| + ||y_{n,i} - x^*||) + \frac{1}{(1-\alpha_n)k_{n,i}^2 h(2\gamma - j)}\theta_{n,i}, \quad i = 1, 2, \ldots.$$

Since $0 < h \le s_n \le j < 2\gamma$, condition (iii) and $||y_{n,i} - x_n|| \to 0$ as $n \to \infty$, we have that $\lim_{n\to\infty} ||Du_n - Dx^*|| = 0$. Using inequality (2.2), we obtain

$$\begin{aligned}
||w_n - x^*||^2 &\le ||J_{M,\lambda}(u_n - s_n Du_n) - J_{M,\lambda}(x^* - s_n Dx^*)||^2 \\
&\le \langle (u_n - s_n Du_n) - (x^* - s_n Dx^*), w_n - x^* \rangle \\
&= \frac{1}{2}\Big[||(u_n - s_n Du_n) - (x^* - s_n Dx^*)||^2 + ||w_n - x^*||^2 \\
&\quad - ||(u_n - s_n Du_n) - (x^* - s_n Dx^*) - (w_n - x^*)||^2 \Big] \\
&\le \frac{1}{2}\Big[||u_n - x^*||^2 \\
&\quad + ||w_n - x^*||^2 - ||(u_n - s_n Du_n) - (x^* - s_n Dx^*) - (w_n - x^*)||^2 \Big] \\
&= \frac{1}{2}\Big[||x_n - x^*||^2 + ||w_n - x^*||^2 - ||w_n - u_n||^2 \\
&\quad + 2s_n \langle u_n - w_n, Du_n - Dx^* \rangle - s_n^2 ||Du_n - Dx^*||^2 \Big].
\end{aligned}$$

Thus,

$$||w_n - x^*||^2 \le ||x_n - x^*||^2 - ||w_n - u_n||^2 + 2s_n ||w_n - u_n|| ||Du_n - Dx^*||.$$

Using this last inequality, we obtain from the recursion formula (3.1) that

$$\begin{aligned}
||y_{n,i} - x^*||^2 &\le \alpha_n ||x_n - x^*||^2 + (1-\alpha_n) k_{n,i}^2 ||T_i^n w_n - x^*||^2 \\
&\le \alpha_n ||x_n - x^*||^2 + (1-\alpha_n) k_{n,i}^2 ||w_n - x^*||^2 \\
&\le \alpha_n ||x_n - x^*||^2 + (1-\alpha_n) k_{n,i}^2 \Big[||x_n - x^*||^2 - ||w_n - u_n||^2 \\
&\quad + 2s_n ||w_n - u_n|| ||Du_n - Dx^*|| \Big] \\
&= ||x_n - x^*||^2 + (1-\alpha_n)(k_{n,i}^2 - 1) ||x_n - x^*||^2 \\
&\quad - (1-\alpha_n) k_{n,i}^2 ||w_n - u_n||^2 \\
&\quad + 2s_n (1-\alpha_n) ||w_n - u_n|| ||Du_n - Dx^*|| \\
&\le ||x_n - x^*||^2 - (1-\alpha_n) k_{n,i}^2 ||w_n - u_n||^2 \\
&\quad + 2s_n (1-\alpha_n) ||w_n - u_n|| ||Du_n - Dx^*|| + \theta_{n,i}, i = 1, 2, \ldots.
\end{aligned}$$

This implies that for each $i = 1, 2, \ldots$,

$$\begin{aligned}
(1-\alpha_n) k_{n,i}^2 ||w_n - u_n||^2 &\le ||x_n - x^*||^2 - ||y_{n,i} - x^*||^2 \\
&\quad + 2s_n (1-\alpha_n) ||w_n - u_n|| ||Du_n - Dx^*|| + \theta_{n,i}. \\
||w_n - u_n||^2 &\le \frac{1}{(1-\alpha_n) k_{n,i}^2} ||y_{n,i} - x_n|| \big(||x_n - x^*|| + ||y_{n,i} - x^*|| \big) \\
&\quad + \frac{1}{(1-\alpha_n) k_{n,i}^2} \theta_{n,i} + \frac{2j}{k_{n,i}^2} ||w_n - u_n|| ||Du_n - Dx^*||.
\end{aligned}$$

Since for each $i = 1, 2, \ldots$, $\lim_{n\to\infty} \alpha_n = 0$, $||y_{n,i} - x_n|| \to 0$ as $n \to \infty$ and $||Du_n - Dx^*|| \to 0$ as $n \to \infty$, we have $\lim_{n\to\infty} ||w_n - u_n|| = 0$. Hence $||w_n - x_n|| = ||w_n - u_n + u_n - x_n|| \leq ||w_n - u_n|| + ||u_n - x_n|| \to$ *as* $n \to \infty$. Also

$$\begin{aligned} ||w_{n+1} - w_n|| &= ||w_{n+1} - x_n + x_n - w_n|| \leq ||w_{n+1} - x_n|| + ||x_n - w_n|| \\ &\leq ||w_{n+1} - x_{n+1}|| + ||x_{n+1} - x_n|| + ||x_n - w_n||. \end{aligned}$$

Thus $\lim_{n\to\infty} ||w_{n+1} - w_n|| = 0$. Now $||w_n - T_i^n w_n|| \leq ||x_n - T_i^n w_n|| + ||w_n - x_n||$. Therefore $\lim_{n\to\infty} ||w_n - T_i^n w_n|| = 0$, for each $i = 1, 2, \ldots$.

$$\begin{aligned} ||w_{n+1} - T_i w_{n+1}|| &\leq ||w_{n+1} - T_i^{n+1} w_{n+1}|| + ||T_i^{n+1} w_{n+1} - T_i^{n+1} w_n|| \\ &\quad + ||T_i^{n+1} w_n - T_i w_{n+1}|| \\ &\leq ||w_{n+1} - T_i^{n+1} w_{n+1}|| + k_{n+1,i} ||w_n - w_{n+1}|| \\ &\quad + k_{1,i} ||T_i^n w_n - w_{n+1}|| \\ &\leq ||w_{n+1} - T_i^{n+1} w_{n+1}|| + \big(k_{n+1,i} + k_{1,i}\big) ||w_n - w_{n+1}|| \\ &\quad + k_{1,i} ||T_i^n w_n - w_n||. \end{aligned}$$

Thus $\lim_{n\to\infty} ||w_n - T_i w_n|| = 0$, $i = 1, 2, \ldots$, completing the proof. □

Theorem 3.5. *Let K be a nonempty, closed and convex subset of a real Hilbert space H. For each $m = 1, 2$, let F_m be a bi-function from $K \times K \to \mathbb{R}$ satisfying $(A1) - (A4)$, $\varphi_m : K \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function with assumption $(B1)$ or $(B2)$, A be an α-inverse-strongly monotone mapping of K into H, B be a β-inverse-strongly monotone mapping of K into H and for each $i = 1, 2, \ldots$, let $T_i : K \to K$ be an asymptotically nonexpansive mapping such that $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let D be a γ-inverse-strongly monotone mapping of K into H. Suppose $F := \cap_{i=1}^{\infty} F(T_i) \cap GMEP(F_1, A, \varphi_1) \cap GMEP(F_2, B, \varphi_2) \cap I(D, M) \neq \emptyset$ and bounded . Let $\{z_n\}_{n=1}^{\infty}$, $\{u_n\}_{n=1}^{\infty}$, $\{w_n\}_{n=1}^{\infty}$, $\{y_{n,i}\}_{n=1}^{\infty}$ $(i = 1, 2, \ldots)$ and $\{x_n\}_{n=0}^{\infty}$ be as defined in Lemma 3.2, then $\{x_n\}$ converges strongly to $P_F x_0$.*

Proof. Observe that in the proof of Lemma 3.3, we obtained that $\lim ||x_m - x_n|| \to 0$ as $m, n \to \infty$. That is $\{x_n\}$ is a Cauchy sequence. Therefore $x_n \to z, n \to \infty$ for some $z \in K$. Since $\lim_{n\to\infty} ||w_n - x_n|| = 0$ and $\lim_{n\to\infty} ||x_n - z|| = 0$, we have that $\lim_{n\to\infty} ||w_n - z|| = 0$. Using the fact that $\lim_{n\to\infty} ||w_n - z|| = 0$ and $\lim_{n\to\infty} ||w_n - T_i w_n|| = 0$, $i = 1, 2, \ldots$, we have that $z \in \cap_{i=1}^{\infty} F(T_i)$.

We show that $z \in I(D, M)$. Since $\{w_n\}_{n=0}^{\infty}$ is bounded, there exists a subsequence $\{w_{n_j}\}_{j=1}^{\infty}$ of $\{w_n\}_{n=0}^{\infty}$ that converges weakly to z. From the fact that D is a $\frac{1}{\gamma}-$ Lipschitz continuous mapping and $\mathcal{D}(D) = H$, we obtain from Lemma 2.5 that $M + D$ is maximal monotone. Let $(v, g) \in G(M + A)$, that is, $g - Av \in M(v)$. Since $w_{n_j} = J_{M,s_n}(I - s_n D)u_{n_j}$, we get $(I - s_n D)u_{n_j} \in (I + s_n M)w_{n_j}$, that is, $\frac{1}{s_n}(u_{n_j} - s_n Du_{n_j} - w_{n_j}) \in M(w_{n_j})$. Using the maximal monotonicity of $M + D$, we obtain

$$\left\langle v - w_{n_j}, g - Dv + \frac{1}{s_n}(u_{n_j} - s_n D u_{n_j} - w_{n_j})\right\rangle \geq 0,$$

$$\begin{aligned}\left\langle v - w_{n_j}, g\right\rangle &\geq \left\langle v - w_{n_j}, Dv + \frac{1}{s_n}(u_{n_j} - s_n D u_{n_j} - w_{n_j})\right\rangle \\ &= \left\langle v - w_{n_j}, Dv - Dw_{n_j} + Dw_{n_j} - Du_{n_j} + \frac{1}{s_n}(u_{n_j} - w_{n_j})\right\rangle \\ &\geq 0 + \left\langle v - w_{n_j}, Dw_{n_j} - Du_{n_j}\right\rangle + \left\langle v - w_{n_j}, \frac{1}{s_n}(u_{n_j} - w_{n_j})\right\rangle.\end{aligned}$$

It follows from the fact that $\lim_{j\to\infty} ||w_{n_j} - u_{n_j}|| = 0$, $\lim_{j\to\infty} ||Dw_{n_j} - Du_{n_j}|| = 0$ and $\lim_{j\to\infty} w_{n_j} = z$ (since $\lim_{j\to\infty} ||w_{n_j} - x_{n_j}|| = 0$, and $\lim_{j\to\infty} x_{n_j} = z$) that $\lim_{j\to\infty}\langle v - w_{n_j}, g\rangle = \langle v - z, g\rangle \geq 0$. Using the maximal monotonicity of $M + D$, we obtain $\theta \in (M + D)(z)$ and this implies that $z \in I(D, M)$.

Further, we show that $z \in GMEP(F_1, A, \varphi_1)$. Since $z_n := T_{r_n}^{(F_1,\varphi_1)}(x_n - r_n A x_n)$, $\ n \geq 1$, we have for any $y \in K$ that

$$F_1(z_n, y) + \varphi_1(y) - \varphi_1(z_n) + \langle Ax_n, y - z_n\rangle + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq 0.$$

Moreover, replacing n by n_j in the last inequality and using $(A2)$, we obtain

$$\varphi_1(y) - \varphi_1(z_{n_j}) + \langle Ax_{n_j}, y - z_{n_j}\rangle + \frac{1}{r_{n_j}}\langle y - z_{n_j}, z_{n_j} - x_{n_j}\rangle \geq F_1(y, z_{n_j}).$$

Let $z_t := ty + (1-t)z$ for all $t \in (0, 1]$ and $y \in K$. This implies that $z_t \in K$. Then, we have

$$\begin{aligned}\langle z_t - z_{n_j}, Az_t\rangle &\geq \varphi_1(z_{n_j}) - \varphi_1(z_t) + \langle z_t - z_{n_j}, Az_t\rangle - \langle z_t - z_{n_j}, Ax_{n_j}\rangle \\ &\quad - \left\langle z_t - z_{n_j}, \frac{z_{n_j} - x_{n_j}}{r_{n_j}}\right\rangle + F_1(z_t, z_{n_j}) \\ &= \varphi_1(z_{n_j}) - \varphi_1(z_t) + \langle z_t - z_{n_j}, Az_t - Az_{n_j}\rangle \\ &\quad + \langle z_t - z_{n_j}, Az_{n_j} - Ax_{n_j}\rangle - \left\langle z_t - z_{n_j}, \frac{z_{n_j} - x_{n_j}}{r_{n_j}}\right\rangle + F_1(z_t, z_{n_j}).\end{aligned}$$

Since $||x_{n_j} - z_{n_j}|| \to 0,\ j \to \infty$, we obtain $||Ax_{n_j} - Az_{n_j}|| \to 0,\ j \to \infty$. Furthermore, by the monotonicity of A, we obtain $\langle z_t - z_{n_j}, Az_t - Az_{n_j}\rangle \geq 0$. Then, by $(A4)$ we obtain (noting that $z_{n_j} \to z$)

$$\langle z_t - z, Az_t\rangle \geq \varphi_1(z) - \varphi_1(z_t) + F_1(z_t, z),\ j \to \infty. \tag{3.10}$$

Using $(A1)$, $(A4)$ and (3.10) we also obtain

$$\begin{aligned}0 &= F_1(z_t, z_t) + \varphi_1(z_t) - \varphi_1(z_t)\\ &\leq tF_1(z_t, y) + (1-t)F_1(z_t, z) + t\varphi_1(y) + (1-t)\varphi_1(z) - \varphi_1(z_t)\\ &\leq t[F_1(z_t, y) + \varphi_1(y) - \varphi_1(z_t)] + (1-t)\langle z_t - z, Az_t\rangle\\ &= t[F_1(z_t, y) + \varphi_1(y) - \varphi_1(z_t)] + (1-t)t\langle y - z, Az_t\rangle\end{aligned}$$

and hence

$$0 \leq F_1(z_t, y) + \varphi_1(y) - \varphi_1(z_t) + (1-t)\langle y - z, Az_t\rangle.$$

Letting $t \to 0$, we have, for each $y \in K$,

$$0 \leq F_1(z, y) + \varphi_1(y) - \varphi_1(z) + \langle y - z, Az\rangle. \tag{3.11}$$

This implies that $z \in GMEP(F_1, A, \varphi_1)$. By using similar arguments, we can show that $z \in GMEP(F_2, B, \varphi_2)$. Therefore, $z \in \cap_{i=1}^{\infty} F(T_i) \cap GMEP(F_1, A, \varphi_1) \cap GMEP(F_2, B, \varphi_2) \cap I(D, M)$.

Noting that $x_n = P_{C_n}x_0$, we have by inequality (2.3) that

$$\langle x_0 - x_n, y - x_n\rangle \leq 0,$$

for all $y \in C_n$. Since $F \subset C_n$ and by the continuity of inner product, we obtain from the above inequality that

$$\langle x_0 - z, y - z\rangle \leq 0,$$

for all $y \in F$. By inequality (2.3) again, we conclude that $z = P_F x_0$. This completes the proof. □

Corollary 3.6. *Let K be a nonempty closed and convex subset of a real Hilbert space H. For each $m = 1, 2$, let F_m be a bi-function from $K \times K \to \mathbb{R}$ satisfying $(A1) - (A4)$, $\varphi_m : K \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function with assumption $(B1)$ or $(B2)$, A be an α-inverse-strongly monotone mapping of K into H, B be a β-inverse-strongly monotone mapping of K into H and let $T : K \to K$ be an asymptotically nonexpansive mapping. Let D be a γ-inverse-strongly monotone mapping of K into H. Suppose $F := F(T) \cap GMEP(F_1, A, \varphi_1) \cap GMEP(F_2, B, \varphi_2) \cap VI(K, D) \neq \emptyset$ and bounded. Let $\{z_n\}_{n=1}^{\infty}$, $\{u_n\}_{n=1}^{\infty}$, $\{w_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=0}^{\infty}$ be generated by $x_0 \in K$, $C_1 = K$, $x_1 = P_{C_1}x_0$*

$$\begin{cases} z_n = T_{r_n}^{(F_1,\varphi_1)}(x_n - r_n Ax_n)\\ u_n = T_{\lambda_n}^{(F_2,\varphi_2)}(z_n - \lambda_n Bz_n)\\ w_n = P_K(u_n - s_n Du_n)\\ y_n = \alpha_n x_0 + (1-\alpha_n)T^n w_n\\ C_{n+1} = \{z \in C_n : ||y_n - z||^2 \leq ||x_n - z||^2 - \alpha_n(1-\alpha_n)||x_n - T^n y_n||^2 + \theta_n\}\\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \geq 1, \end{cases} \tag{3.12}$$

where $\theta_n = (1-\alpha_n)(k_{n,i}^2 - 1)(\sup_{x^*\in F}\{||x_n - x^*||^2\})$. *Assume that* $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$, $\{r_n\}_{n=1}^{\infty} \subset [0, 2\alpha]$ *and* $\{\lambda_n\}_{n=1}^{\infty} \subset [0, 2\beta]$ *satisfy* (i) $0 < a \leq r_n \leq b < 2\alpha$, (ii) $0 < c \leq \lambda_n \leq f < 2\beta$, (iii) $\lim_{n\to\infty} \alpha_n = 0$, (iv) $0 < h \leq s_n \leq j < 2\gamma$.
Then, $\{x_n\}_{n=0}^{\infty}$ *converges strongly to* $P_F x_0$.

4 Applications

We study here, the problem of finding a minimizer of a continuously Fréchet differentiable convex functional in a Hilbert space.

Theorem 4.1. *For each* $m = 1, 2$, *let* F_m *be a bi-function from* $H \times H \to \mathbb{R}$ *satisfying* $(A1) - (A4)$, $\varphi_m : H \to \mathbb{R} \cup \{+\infty\}$ *be a proper lower semicontinuous and convex function with assumption* $(B1)$ *or* $(B2)$, A *be an* α*-inverse-strongly monotone mapping of* H *into itself,* B *be a* β*-inverse-strongly monotone mapping of* H *into itself and for each* $i = 1, 2, \ldots,$ *let* $T_i : H \to H$ *be asymptotically nonexpansive mappings such that* $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$. *Suppose* f *is a functional on* H *which satisfies the following conditions:*

1. f *is a continuously Fréchet differentiable convex functional on* H *and* ∇f *is* $\frac{1}{\gamma}$*-Lipschitz continuous,*
2. $(\nabla f)^{-1}0 = \{z \in H : f(z) = \min_{y\in H} f(y)\} \neq \emptyset$.

Suppose $F := \cap_{i=1}^{\infty} F(T_i) \cap GMEP(F_1, A, \varphi_1) \cap GMEP(F_2, B, \varphi_2) \cap (\nabla f)^{-1}(0) \neq \emptyset$ *and bounded. Let* $\{z_n\}_{n=1}^{\infty}$, $\{u_n\}_{n=1}^{\infty}$, $\{w_n\}_{n=1}^{\infty}$, $\{y_{n,i}\}_{n=1}^{\infty}$ $(i = 1, 2, \ldots)$ *and* $\{x_n\}_{n=0}^{\infty}$ *be generated by* $x_0 \in K$, $C_{1,i} = K$, $C_1 = \cap_{i=1}^{\infty} C_{1,i}$, $x_1 = P_{C_1} x_0$

$$\begin{cases} z_n = T_{r_n}^{(F_1,\varphi_1)}(x_n - r_n A x_n) \\ u_n = T_{\lambda_n}^{(F_2,\varphi_2)}(z_n - \lambda_n B z_n) \\ w_n = (u_n - s_n \nabla f u_n) \\ y_{n,i} = \alpha_n x_n + (1-\alpha_n) T_i^n w_n \\ C_{n+1,i} = \{z \in C_{n,i} : ||y_{n,i} - z||^2 \leq ||x_n - z||^2 \\ \qquad\qquad -\alpha_n(1-\alpha_n)||x_n - T_i^n y_{n,i}||^2 + \theta_{n,i}\} \\ C_{n+1} = \cap_{i=1}^{\infty} C_{n+1,i} \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \geq 1, \end{cases} \tag{4.1}$$

where $\theta_{n,i} = (1-\alpha_n)(k_{n,i}^2 - 1)(\sup_{x^*\in F}\{||x_n - x^*||^2\}), i = 1, 2, \ldots$. *Assume that* $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$ $(i = 1, 2, \ldots)$, $\{r_n\}_{n=1}^{\infty} \subset [0, 2\alpha]$ *and* $\{\lambda_n\}_{n=1}^{\infty} \subset [0, 2\beta]$ *satisfy* (i) $0 < a \leq r_n \leq b < 2\alpha$, (ii) $0 < c \leq \lambda_n \leq f < 2\beta$, (iii) $\lim_{n\to\infty} \alpha_n = 0$, (iv) $0 < h \leq s_n \leq j < 2\gamma$.
Then, $\{x_n\}_{n=0}^{\infty}$ *converges strongly to* $P_F x_0$.

Proof. We know from condition (1) and Lemma 2.2 that ∇f is an γ-inverse-strongly monotone operator from H into H. Using Theorem 3.5 we have the desired conclusion. □

We now study a kind of multi-objective optimization problem with nonempty set of solutions:

$$\begin{cases} \min h_1(x) \\ \min h_2(x) \\ x \in K \end{cases} \tag{4.2}$$

where K is a nonempty closed convex subset of a real Hilbert space H and $h_i : K \to \mathbb{R},\ \ i = 1, 2$ is a convex and a lower semicontinuous functional. Let us denote the set of solutions to (4.2) by Ω and assume that $\Omega \neq \emptyset$.

We shall denote the set of solutions of the following two optimization problems by Ω_1 and Ω_2 respectively.

$$\begin{cases} \min\limits_{x \in K} h_1(x) \end{cases}$$

and

$$\begin{cases} \min\limits_{x \in K} h_2(x). \end{cases}$$

Clearly, if we find a solution $x \in \Omega_1 \cap \Omega_2$, then one must have $x \in \Omega$.

Now, for each $i = 1, 2$, let $F_i : K \times K \to \mathbb{R}$ be defined by $F_i(x, y) := h_i(y) - h_i(x)$. We consider now the following equilibrium problem: find $x \in K$ such that

$$F_i(x, y) \geq 0, \quad i = 1, 2, \tag{4.3}$$

for all $y \in K$. It is obvious that F_i satisfies conditions $(A1) - (A4)$ and $EP(F_i) = \Omega_i,\ \ i = 1, 2$, where $EP(F_i)$ is the set of solutions to (4.3). By Theorem 3.5, we have the following.

Theorem 4.2. *Let K be a nonempty closed and convex subset of a real Hilbert space H. For each $i = 1, 2$, let h_i be a lower semicontinuous and convex function such that $\Omega_1 \cap \Omega_2 \neq \emptyset$. Let $\{z_n\}_{n=1}^\infty,\ \{u_n\}_{n=1}^\infty,\ \{y_n\}_{n=1}^\infty$ and $\{x_n\}_{n=0}^\infty$ be generated by $x_0 \in K,\ C_1 = K,\ x_1 = P_{C_1} x_0$*

$$\begin{cases} h_1(y) - h_1(z_n) + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \quad \forall y \in K, \\ h(y) - h(u_n) + \frac{1}{\lambda_n}\langle y - u_n, u_n - z_n\rangle \geq 0, \quad \forall y \in K, \\ y_n = \alpha_n x_0 + (1 - \alpha_n) u_n \\ C_{n+1,i} = \{z \in C_{n,i} : ||y_{n,i} - z||^2 \leq ||x_n - z||^2 \\ \qquad\qquad -\alpha_n(1 - \alpha_n)||x_n - T_i^n y_{n,i}||^2 + \theta_{n,i}\} \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \geq 1, \end{cases}$$

where $\theta_{n,i} = (1 - \alpha_n)(k_{n,i}^2 - 1(\sup_{x^* \in F}\{||x_n - x^*||^2\}), i = 1, 2, \ldots.$ Assume that $\{\alpha_n\}_{n=1}^\infty \subset (0, 1),\ \{r_n\}_{n=1}^\infty \subset (0, \infty)$ and $\{\lambda_n\}_{n=1}^\infty \subset (0, \infty)$ satisfy
(i) $\liminf_{n\to\infty} r_n > 0$, (ii) $\liminf_{n\to\infty} \lambda_n > 0$, (iii) $\lim_{n\to\infty} \alpha_n = 0$.
Then, $\{x_n\}_{n=0}^\infty$ converges strongly to $P_{\Omega_1\cap\Omega_2} x_0$.

Remark 4.3. *Our results in this paper extend many important recent results, in particular, the results of [18, 28] were extended from the class of non expansive mappings to a more general class of asymptotically nonexpansive mappings. Also, there is no boundedness assumption on the domain of the operator.*

Prototypes. The prototypes of our iteration parameters are:
$\alpha_n := \frac{1}{n},\ n \geq 1;\ r_n := \alpha(\frac{n}{n+2}); \lambda_n := \beta(\frac{n}{n+1}),\ n \geq 1;$ and $s_n := \gamma(\frac{n}{n+1})\ n \geq 1,$
$a = \frac{\alpha}{4};\ c = \frac{\beta}{4};\ h = \frac{\gamma}{4},\ b = \alpha,\ f = \beta,\ j = \gamma.$

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