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Common Fixed Point Results in Uniformly Convex Metric Spaces

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Abstract : We obtain some common fixed point results in uniformly convex metric spaces. Our results improve and extend some results of Laowang and Panyanak [W. Laowang, B. Panyanak, A note on common fixed point results in uniformly convex hyperbolic spaces, J. Math., Vol. 2013 (2013), Article ID 503731, 5 pages] and Itoh and Takahashi [S. Itoh, W. Takahashi, The common fixed point theory of singlevalued mappings and multivalued mappings, Pacific J. Math. 79 (1978) 493–508].

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1 Introduction

Let E be a nonempty subset of a metric space X. We shall denote by 2^E the family of nonempty subsets of E, by $CB(E)$ the family of nonempty closed and bounded subsets of E, by $K(E)$ the family of nonempty compact subsets of E, and by $KC(E)$ the family of nonempty compact convex subsets of E. Let H be the Hausdorff distance on $CB(E)$, that is,

$$H(A,B) := \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\}, \qquad A,B \in CB(E).$$

where $d(a, B) := \inf\{\|a - b\| : b \in B\}$ is the distance from a point a to a subset B. Let t be a mapping of E into E and T a multivalued mapping of E into 2^E. The set of fixed points of t and T will be denoted by $Fix(t) := \{x \in E : tx = x\}$ and $Fix(T) := \{x \in E : x \in Tx\}$ respectively. A nonempty subset C of E is said to be t-invariant if $t(C) \subset C$. C is said to be T-invariant if $Tx \cap C \neq \emptyset$ for all $x \in C$. t and T is said to commute if for each $x \in E$, $t(Tx) \subset T(tx)$. A sequence $\{x_n\}$ in E is called an approximate fixed point sequence (afps for short) for T if $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Definition 1.1. A multivalued mapping $T : E \to 2^E$ is said to satisfy

(i) condition (E) if there exists $\mu \geq 1$ such that for each $x, y \in E$,

$$\lambda d(x, Ty) \leq \mu d(x, Tx) + d(x, y);$$

(ii) condition (C_λ) if there exists $\lambda \in (0, 1)$ such that for each $x, y \in E$,

$$\lambda d(x, Tx) \leq d(x, y) \text{ implies } H(Tx, Ty) \leq d(x, y).$$

Recall that $I - T$ is strongly demiclosed if for every sequence $\{x_n\}$ in E which converges to $x \in E$ and such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, we have $x \in Tx$. Notice that if T satifies condition (E) then $I - T$ is strongly demiclosed (see [1, Proposition 2.10]).

Recently, Akkasriworn et al. [2] defined a condition on single-valued mapping which is weaker than quasi-nonexpansiveness and asymptotically nonexpansiveness and proved the existence of common fixed points in uniformly convex Banach spaces.

Definition 1.2 ([2, Definition 1.1])**.** A mapping t on a set E is said to satisfy condition (K) if

(i) $Fix(t)$ is nonempty closed and convex;

(ii) for each $x \in Fix(t)$ and any closed convex t-invariant subset C of E, the nearest point of $x \in C$ must be contained in $Fix(t)$.

Theorem 1.3 ([2, Theorem 3.4])**.** *Let E be a nonempty bounded closed convex subset of a uniformly convex Banach space X. Let $t : E \to E$ be a mapping satisfying condition (K), and let $T : E \to KC(E)$ be a multivalued mapping satisfying conditions (E) and (C_λ) for some $\lambda \in (0, 1)$. If t and T are commute, then $Fix(t) \cap Fix(T) \neq \emptyset$.*

Laowang and Panyanak [3] generalized Theorem 1.3 to uniformly convex hyperbolic spaces and they also proved that the condition (K) is weaker than asymptotically quasi-nonexpansiveness and is weaker than asymptotically pointwise nonexpansiveness.

Theorem 1.4 ([3, Theorem 12])**.** *Let E be a bounded closed convex subset of a complete uniformly convex hyperbolic space X and $t : E \to E$ a mapping satisfying condition (K). Suppose that $T : E \to KC(E)$ satisfies condition (C_λ) and $I - T$ is strongly demiclosed. If t and T commute, then $Fix(t) \cap Fix(T) \neq \emptyset$.*

García-Falset et al. [4] defined a class of multivalued mappings.

Definition 1.5 ([4, Definition 1])**.** A mapping $T : E \to CB(E)$ satisfies condition (L) on a set E if

(i) every T-invariant closed convex subset possesses an afps;

(ii) for an afps $\{x_n\}$ for T in E and $x \in E$,

$$\limsup_{n\to\infty} d(x_n, Tx) \leq \limsup_{n\to\infty} d(x_n, x).$$

In this paper, we show that every total asymptotically nonexpansive mapping satisfies condition (K) and a multivalued mapping T satisfies condition (L) when T satisfies condition (C_λ) for some $\lambda \in (0, 1)$ and $I - T$ is strongly demiclosed. Moreover, we extend Theorem 1.4 to uniformly convex metric spaces while a class of single-valued mappings is no longer finite and a multivalued mapping satisfies condition (L). Moreover, we obtain a common fixed point theorem for a family of weakly commuting single-valued mappings and a multivalued mapping satisfying condition (L).

2 Preliminaries

Let (X, d) be a metric space. A geodesic path from x to y is a mapping $c : [0, l] \subset \mathbb{R} \to X$ with $c(0) = x, c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for every $t, t' \in [0, l]$. The image $c([0, l])$ of c forms a geodesic segment which joins x and y and is not necessarily unique. We will use $[x, y]$ to denote a geodesic segment joining x and y. (X, d) is a (uniquely) geodesic space if every two points $x, y \in X$ can be joined by a (unique) geodesic path. A point $z \in X$ belongs to the geodesic segment $[x, y]$ if and only if there exists $t \in [0, 1]$ such that $d(z, x) = td(x, y)$ and $d(z, y) = (1-t)d(x, y)$ and we will write $z = (1-t)x+ty$. A subset C of a geodesic space is said to be convex if $[x, y] \subset C$ for any $x, y \in C$.

Definition 2.1. A geodesic metric space (X, d) is called uniformly convex if for any $r > 0$ and any $\varepsilon \in (0, 2]$ there exists $\delta \in (0, 1]$ such that for all $a, x, y \in X$ with $d(x, a) \leq r, d(y, a) \leq r$ and $d(x, y) \geq \varepsilon r$ it is the case that

$$d(m, a) \leq (1 - \delta)r,$$

where m stands for any midpoint of any geodesic segment $[x, y]$. A mapping $\delta : (0, \infty) \times (0, 2] \to (0, 1]$ providing such a $\delta = \delta(r, \varepsilon)$ for a given $r > 0$ and $\varepsilon \in (0, 2]$ is called a modulus of uniform convexity.

From the definition, it is clear that uniformly convex metric spaces are uniquely geodesic. The mapping δ is monotone (resp. lower semi-continuous from the right) if for every fixed ε it decreases (resp. is lower semi-continuous from the right) with respect to r (see [5]). Examples of uniformly convex metric spaces are uniformly convex Banach spaces, CAT(0) spaces and CAT(1) spaces with small diameters (see [6]).

Throughout this paper, we assume that all uniformly convex metric spaces have monotone or lower semi-continuous from the right modulus of uniform convexity.

Let X be a metric space and $\mathcal{F}$ a family of subset of X. Then, $\mathcal{F}$ defines a convexity structure on X if it contains the closed balls and is stable by intersection.

Remark 2.2. *It is noted in [5] that if $\mathcal{F}$ stands for the collection of nonempty closed and convex subsets of a complete uniformly convex metric space, then $\mathcal{F}$ is a nested compact convexity structure, that is, if any decreasing chain $(A_\alpha)_{\alpha\in\Gamma}$ of nonempty bounded elements of $\mathcal{F}$ has nonempty intersection.*

For a bounded sequence $\{x_n\}$ in X and $x \in X$, define

$$r(x, x_n) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, x_n) : x \in X\},$$

and the asymptotic center of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, x_n) = r(\{x_n\}).$$

In [5], the authors proved that every bounded sequence in a complete uniformly convex metric space has a unique asymptotic center.

A bounded sequence $\{x_n\}$ is regular if $r(\{x_n\}) = r(\{x_{n_k}\})$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Every bounded sequence $\{x_n\}$ in a complete uniformly convex metric space has a regular subsequence $\{x_{n_k}\}$ and thus every subsequence of $\{x_{n_k}\}$ has the same asymptotic center as $\{x_{n_k}\}$ (see [7]). The following lemma can be found in [1].

Lemma 2.3. *Let E be a nonempty closed convex subset of a complete uniformly convex metric space X. Then for each $x \in X$, there exists a unique point x_0 in E such that*

$$d(x, x_0) = d(x, E) := \inf\{d(x, y) : y \in E\}.$$

Definition 2.4. A single-valued mapping $t : E \to E$ is said to be

(i) quasi-nonexpansive if for $Fix(t) \neq \emptyset$ and for $x \in E$ and $y \in Fix(t)$,

$$d(tx, y) \leq d(x, y);$$

(ii) asymptotically nonexpansive if there exists sequence $\{k_n\}$ in $[1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$d(t^n x, t^n y) \le k_n d(x,y) \text{ for all } x, y \in E \text{ and } n \in \mathbb{N};$$

(iii) asymptotically pointwise nonexpansive if there exists sequence $\alpha_n : E \to [0,\infty)$ with $\lim_{n\to\infty} \alpha_n(x) = 1$, such that

$$d(t^n x, t^n y) \le \alpha_n(x) d(x,y) \text{ for all } x, y \in E \text{ and } n \in \mathbb{N};$$

(iv) asymptotically quasi-nonexpansive if there exists sequence $\{k_n\}$ in $[1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$d(t^n x, p) \le k_n d(x,p) \text{ for all } x \in E, p \in Fix(t) \text{ and } n \in \mathbb{N};$$

(v) generalized asymptotically nonexpansive ([8]) if there exist sequences $\{k_n\}$ in $[1,\infty)$ and $\{s_n\}$ in $[0,\infty)$ with $\lim_{n\to\infty} k_n = 1$, $\lim_{n\to\infty} s_n = 0$ such that

$$d(t^n x, t^n y) \le k_n d(x,y) + s_n \text{ for all } x, y \in E \text{ and } n \in \mathbb{N};$$

(vi) total asymptotically nonexpansive ([9]) if there exist nonnegative real sequence $\{k_n^{(1)}\}$ and $\{k_n^{(2)}\}$ with $\lim_{n\to\infty} k_n^{(1)} = \lim_{n\to\infty} k_n^{(2)} = 0$, and strictly increasing and continuous functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that

$$d(t^n x, t^n y) \le d(x,y) + k_n^{(1)} \phi(d(x,y)) + k_n^{(2)} \text{ for all } x, y \in E \text{ and } n \in \mathbb{N}.$$

Before passing to main results, we briefly recall the notations. Let E be a nonempty subset of metric space X, C a nonempty subset of E and S a family of self-mappings of E. The set of common fixed points of S in E will be denoted by $F(S)$. We denote by $\partial_E C$ the relative boundary of C, that is, $\partial_E C = \overline{C} \cap \overline{E \setminus C}$. Let t be a mapping of E into E and T a multivalued mapping of E into 2^E. t and T is said to commute weakly if for each $x \in E$, $t(\partial_E Tx) \subset T(tx)$. S and T is said to commute (weakly) if each $t \in S$ and T commute (weakly).

3 Main Results

First we will show that every total asymptotically nonexpansive mapping satisfies condition (K).

Proposition 3.1. *Let E be a bounded closed convex subset of a complete uniform metric space X. If $t : E \to E$ is a continuous total asymptotically nonexpansive mapping, then t satisfies condition (K).*

Proof. The proof of $Fix(t) \neq \emptyset$ and closed convex follows similar patterns as in [5, Theorem 3.11], so we omit it. Next, we need to show that for each $x \in Fix(t)$ and any closed convex t-invariant subset C of E, the nearest point of $x \in C$ must be contained in $Fix(t)$. Now, let $x \in Fix(t)$, C be a closed convex t-invariant subset of E and $u \in C$ be such that $d(x,u) = d(x,C)$. We will show that $u \in Fix(t)$. If $x \in C$, then $u = x \in Fix(t)$. Suppose that $x \notin C$. Since t is total asymptotically nonexpansiveness, we have

$$\limsup_{n\to\infty} d(x, t^n u) \le d(x,u). \tag{3.1}$$

We will prove that $\{t^n u\}$ is a Cauchy sequence. Suppose not, there exists a separated subsequence $\{t^{m_i}u\}$ of $t^m u$, that is, there exists $\varepsilon > 0$ such that $d(t^{m_k}u, t^{m_h}u) \ge \varepsilon$ for every $k \ne h$ in $\mathbb{N}$.

For the monotone case, let m_{kh} be the midpoint of the segment $[t^{m_k}u, t^{m_h}u]$, $c = diam(E)$ and $\varepsilon_0 = \varepsilon/c$. The uniform convexity of the space implies that for every k and h in $\mathbb{N}$

$$\begin{aligned} d(m_{kh}, x) &\le (1 - \delta(\max\{d(t^{m_h}u, x), d(t^{m_k}u, x)\}, \varepsilon_0)) \max\{d(t^{m_h}u, x), d(t^{m_k}u, x)\} \\ &\le (1 - \delta(c, \varepsilon_0)) \max\{d(t^{m_h}u, x), d(t^{m_k}u, x)\}. \end{aligned}$$

Since u is a nearest point of x in C and (3.1),

$$\begin{aligned} d(u,x) &\le d(m_{kh}, x) \\ &\le (1 - \delta(c, \varepsilon_0)) d(u,x), \end{aligned}$$

and so $d(u,x) = 0$ which contradicts to $x \notin C$.

For the lower semicontinuous case, let m_{kh} be the midpoint of the segment $[t^{m_k}u, t^{m_h}u]$, $\varepsilon_1 = \varepsilon/d(u,x)$ and $p \in \mathbb{N}$. Then there exists $N \in \mathbb{N}$ such that $\max\{d(t^{m_h}u, x), d(t^{m_k}u, x)\} \le d(u,x) + p^{-1}$ for each $h, k \ge N$. By uniform convexity,

$$d(u,x) \le d(m_{kh}, x) \le (1 - \delta(d(u,x) + p^{-1}, \varepsilon)(d(u,x) + p^{-1}). \tag{3.2}$$

Note that if p is large enough,

$$\begin{aligned} \delta(d(u,x) + p^{-1}, \varepsilon) &\ge \frac{1}{2}\delta(d(u,x) + p^{-1}, \varepsilon) \\ 1 - \delta(d(u,x) + p^{-1}, \varepsilon) &\le 1 - \frac{1}{2}\delta(d(u,x) + p^{-1}, \varepsilon) \end{aligned}$$

From (3.2) and taking p to infinity, $d(u,x) < d(u,x)$, which is a contradiction. Therefore, $\{t^m u\}$ is a Cauchy sequence and its limit, by (3.1), is u. Then, from the continuity of t, $tu = u$. □

Proposition 3.2. *Let E be a bounded closed convex subset of a uniformly convex metric space X. If $T : E \to CB(E)$ satisfies condition (C_λ) and $I - T$ is strongly demiclosed, then T satisfies condition (L).*

Proof. According to [1, Lemma 3.1], if $C \subset E$ is closed convex and T -invariant, we can assure the existence of an afps for T in C. Next, Let $\{x_n\}$ be an afps for

T in E and $x \in E$. If $x_n \to x$, then $x \in Tx$ because $I - T$ is strongly demiclosed. So, for all $n \in \mathbb{N}$, $d(x_n, Tx) = \inf_{y \in Tx} d(x_n, y) \leq d(x_n, x)$. Otherwise, there exist $N \in \mathbb{N}$ and $y_n \in Tx_n$ such that for all $n \geq N$

$$\lambda d(x_n, Tx_n) = \lambda d(x_n, y_n) \leq d(x_n, x).$$

By condition (C_λ), $H(Tx_n, Tx) \leq d(x_n, x)$ for all $n \geq N$. We know that

$$d(x_n, Tx) \leq d(x_n, Tx_n) + H(Tx_n, Tx), \text{ for } n \geq N.$$

Taking upper limits on n we obtain

$$\limsup_{n\to\infty} d(x_n, Tx) \leq \limsup_{n\to\infty} d(x_n, x).$$

Therefore, T satisfies condition (L). □

Theorem 3.3. *Let E be a bounded closed convex subset of a complete uniformly convex metric space X and $S = \{t_i\}_{i\in I}$ a family of commuting mappings satisfying condition (K) on E. Then $F(S)$ is nonempty closed and convex.*

Proof. Since t_1 satisfies condition (K), $Fix(t_1)$ is nonempty closed and convex. Suppose that $F := \cap_{i=1}^{k-1} Fix(t_i)$ is nonempty closed and convex for some $1 < k \leq n$. For $x \in F$ and $1 \leq i < k$, since $t_k \circ t_i = t_i \circ t_k$, we have

$$t_k x = t_k \circ t_i x = t_i \circ t_k x,$$

thus $t_k x$ is a fixed point of t_i for every $i = 1, 2, ..., k-1$, that is, $t_k x \in F$. Hence $t_k(F) \subset F$. Again by condition (K) of t_k, t_k has a fixed point in F, that is, $\cap_{i=1}^{k} Fix(t_i)$ is nonempty closed and convex. By induction, we obtain $\cap_{i=1}^{n} Fix(t_i)$ is nonempty closed and convex.

Let $\Gamma = 2^I = \{\beta : \beta \subset I\}$. It is obvious that Γ is downward directed (the order on Γ is the set inclusion). The above proof implies that for every $\beta \in \Gamma$, the set $F_\beta = \cap_{i\in\beta} Fix(t_i)$ is nonempty closed and convex. Clearly the family $(F_\beta)_{\beta\in\Gamma}$ is decreasing. By using Remark 2.2, $F(S) = \cap_{i\in I} Fix(t_i)$ is nonempty. It is clear that it is closed and convex. □

As direct consequences of Proposition 3.1 and Theorem 3.3, we obtain the following corollaries.

Corollary 3.4 ([10, Theorem 3.2])**.** *Let E be a bounded closed convex subset of a complete CAT(0) space X and $\{t_i\}_{i\in I}$ any family of commuting asymptotic pointwise nonexpansive mappings on E. Then $\cap_{i\in I} Fix(t_i)$ is nonempty closed and convex.*

Corollary 3.5 ([8])**.** *Let E be a bounded closed convex subset of a complete CAT(0) space X and $\{t_i\}_{i\in\mathbb{N}}$ a countable infinite family of commuting continuous generalized asymptotically nonexpansive mappings on E. Then $\cap_{i=1}^{\infty} Fix(t_i)$ is nonempty closed and convex.*

Theorem 3.6. *Let E be a bounded closed convex subset of a complete uniformly convex metric space X. If $T : E \to K(E)$ satisfies condition (L), then $Fix(T) \neq \emptyset$.*

Proof. By the condition (L), T has an afps in E, say $\{x_n\}$. By passing through a subsequence, we may assume that $\{x_n\}$ is regular. Let $A(\{x_n\}) = \{x\}$. We are going to show that x is a fixed point of T. By compactness of Tx implies that for each n we can take $z_n \in Tx$ for each n such that

$$d(x_n, z_n) = d(x_n, Tx).$$

By compactness of Tx, there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $\lim_{k\to\infty} z_{n_k} = z \in Tx$. Note that

$$\begin{aligned} d(x_{n_k}, z) &\leq d(x_{n_k}, z_{n_k}) + d(z_{n_k}, z) \\ &= d(x_{n_k}, Tx) + d(z_{n_k}, z) \\ &\leq d(x_{n_k}, x) + d(z_{n_k}, z). \end{aligned}$$

It follows that $\limsup_k d(x_{n_k}, z) \leq \limsup_k d(x_{n_k}, x)$. Since $A(\{x_n\}) = \{x\}$, we have $x = z \in Tx$. □

Remark 3.7. *Notice that in the above proof, condition (i) of Definition 1.5 can be replaced by the weaker assumption*

$$(A') \qquad T \text{ has an afps in } E.$$

Corollary 3.8 ([1, Theorem 3.2])**.** *Let E be a bounded closed convex subset of a complete uniformly convex metric space X. If $T : E \to K(E)$ satisfies condition (C_λ) and $I - T$ is strongly demiclosed, then $Fix(T) \neq \emptyset$.*

Next, we will prove our main results which generalize Theorems 1.3 and 1.4.

Theorem 3.9. *Let E be a bounded closed convex subset of a complete uniformly convex metric space X and S a family of commuting mappings satisfying condition (K) on E. Suppose that $T : E \to KC(E)$ satisfies condition (L). If S and T commute, then $F(S) \cap Fix(T) \neq \emptyset$.*

Proof. By Theorem 3.3, $F(S)$ is nonempty closed and convex. Since S and T commute, we can see that for $t \in S$ and $x \in F(S)$,

$$t(Tx) \subset T(tx) = Tx.$$

Let u be the nearest point of x in Tx. Since t satisfies condition (K) and Tx is closed convex t-invariant, $u \in Fix(t)$ for every $t \in S$. Hence $Tx \cap F(S) \neq \emptyset$ for all $x \in F(S)$. Define a multivalued mapping $U : F(S) \to KC(F(S))$ by $Ux = Tx \cap F(S)$ for every $x \in F(S)$. It is easily seen that $d(u, Uv) = d(u, Tv)$ for all $u, v \in F(S)$. Let us show that U satisfies condition (L). First, let C be a closed convex U-invariant subset of $F(S)$. Thus C is also T-invariant which assures that T has an afps in C, say $\{x_n\}$. Consider

$$\lim_{n\to\infty} d(x_n, Ux_n) = \lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

We obtain that U has an afps in C. Next, let $\{x_n\}$ be an afps for T in $F(S)$ and $x \in F(S)$. Since T satisfies condition (L),

$$\limsup_{n\to\infty} d(x_n, Ux) = \lim_{n\to\infty} d(x_n, Tx) \le \limsup_{n\to\infty} d(x_n, x).$$

By Theorem 3.6, we obtain a fixed point in $F(S)$ of U and thus of T and we are done. □

Theorem 3.10. *Let E be a bounded closed convex subset of a complete uniformly convex metric space X and S a family of commuting mappings on E. Suppose that $T : E \to KC(E)$ satisfies condition (L) such that for all $t \in S$*

$$\emptyset \ne Tx \cap F(S) \subset Z(t) = \{z \in E : d(z, tx) \le d(z, x)\} \text{ and}$$

$$t(\partial_E(Tx)) \subset Tx \text{ for all } x \in F(S).$$

If $F(S)$ is closed and convex, then $F(S) \cap Fix(T) \ne \emptyset$.

Proof. Define a multivalued mapping $U : F(S) \to KC(F(S))$ by $Ux = Tx \cap F(S)$ for every $x \in F(S)$. We claim that $dist(u, Uv) = dist(u, Tv)$ for all $u, v \in F(S)$. Let $a \in Uu$ and $b \in Tv$ such that $d(a, b) = d(a, Tv)$. For $t \in S$, since $a \in Uu \subset Z(t)$, we have $d(a, tb) \le d(a, b)$. By the uniqueness of b as the closest point to a, $b = tb$ for $t \in S$. Therefore $b \in Tv \cap F(S) = Uv$. This shows that $dist(u, Uv) = dist(u, Tv)$. The proof now follows as Theroem 3.9. □

Corollary 3.11 ([11, Theorem 8])**.** *Let E be a bounded closed convex subset of a uniformly convex Banach space X and S a family of commuting quasi-nonexpansive mappings on E for which $F(S) \ne \emptyset$. Suppose that $T : E \to KC(E)$ is nonexpansive mapping. If S and T commute weakly, then $F(S) \cap Fix(T) \ne \emptyset$.*

Corollary 3.12 ([1, Theorem 3.4])**.** *Let E be a bounded closed convex subset of a complete uniformly convex metric space X and $t : E \to E$ a quasi-nonexpansive mapping whose $Fix(t) \ne \emptyset$. Suppose that $T : E \to KC(E)$ satisfies condition (C_λ) and $I - T$ is strongly demiclosed. If t and T commute weakly, then $Fix(t) \cap Fix(T) \ne \emptyset$.*

Remark 3.13. *Theorems 3.9 and 3.10 hold if T is a nonself multivalued mapping and $Tx \cap E \ne \emptyset$ for each $x \in X$ where commuting of S and T is denoted by $t(Tx \cap E) \subset Ttx \cap E$ for each $x \in X$ and $t \in S$.*

Corollary 3.14 ([12, Theorem 3.3]). *Let E be a bounded closed convex subset of a complete CAT(0) space X, $t : E \to E$ a mapping and $T : E \to KC(X)$ a nonexpansive mapping such that $Tx \cap E \neq \emptyset$ for each $x \in X$. Suppose that t and T satisfy the condition*

$$\emptyset \neq Tx \cap Fix(t) \subset Z(t), t(\partial_E(Tx) \cap E) \subset Tx \cap E \text{ for all } x \in Fix(t).$$

If $Fix(t)$ is closed and convex, then $Fix(t) \cap Fix(T) \neq \emptyset$.

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