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# Some Contributions to Modals Analysis

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**Abstract :** In the past decades, modal analysis has become a major technology in the quest for determining, improving and optimizing dynamic characteristics of engineering structures. Not only has it been recognized in mechanical and aeronautical engineering, but modal analysis has also been discovered in profound applications for civil and building structures, biomedical problems, space structures, acoustical instruments, transportation and nuclear problems, (for more information, see [1]). Shortly, modals analysis relies on mathematics to establish theoretical models for a dynamic system and to analyze data in various forms. Since modals are used in different branches of engineering in order to contribute to modals analysis, we have constructed some sequence spaces of modals and introduced the null, convergent and bounded sequence spaces of interval numbers which are denoted $c_0(gI)$, $c(gI)$ and $\ell_\infty(gI)$, respectively, consisting of all sequences $\tilde{u} = (\tilde{u}_k)$ such that $(\tilde{u}_k)$ is a sequence of modals. Also, we have given some new definitions and theorems about sequence spaces of the modals. Thus, we have contributed to the modal analysis.

**Keywords :** sequence space; Banach space; modal; solid; Schauder interval basis.
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## 1 Introduction and Preliminaries

Interval arithmetic was first suggested by Dwyer [2] in 1951. Development of interval arithmetic as a formal system and evidence of its value as a computational

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device was provided by Moore [3, 4] in 1959 and 1962. Furthermore, Moore and others [2, 5–7] have developed applications to differential equations.

Recently Chiao [8] introduced sequence of interval numbers and Şengönül and Eryılmaz [9] defined not only usual convergence of sequences of interval numbers but also showed that these spaces are complete metric spaces, [9]. We take courage from them and we defined bounded, convergent and null sequences spaces of modals.

Let's denote the set of all real valued closed intervals by $\mathtt{I}$, the set of all positive integers by $\mathbf{N}$ and the set of all real numbers by $\mathbf{R}$, through all the text. Any elements of $\mathtt{I}$ is called interval number and it is denoted by $\hat{x}$, [8]. Let $\underline{x}$ and $\overline{x}$ be first and last points of $\hat{x}$ interval number, respectively. Since $\hat{x}-\hat{x}\neq[0,0]=\hat{0}$, $-\hat{x}$ is not an additive inverse for $\hat{x}$ in the $\mathtt{I}$. Thus the algebraic structure of $(\mathtt{I},+)$ is a semigroup with respect to addition. This is very big deficiency because of so many reasons. In interval analysis, as you will recall, an interval number is defined by $\hat{x}=\{x\in\mathbf{R}:\underline{x}\leq x\leq\overline{x},\}$. Therefore, when $\underline{x}>\overline{x}$, the $\hat{x}$ is not interval number. But in modal analysis, a modal is no longer restricted to the ordered bound condition of $\underline{x}\leq\overline{x}$. That is $[\overline{x},\underline{x}]$ is also a valid interval. A modal $\tilde{x}=\{[\underline{x},\overline{x}]:\underline{x},\overline{x}\in\mathbf{R}\}$ is defined by a pair of real numbers $\underline{x},\overline{x}$. Let's denote the set of all modals by $gI$. Let us suppose that $\tilde{x},\tilde{y}\in gI$. Then algebraic operations between $\tilde{x}$ and $\tilde{y}$ are defined in the Kaucher arithmetic, [10]. For a modal $\tilde{x}=[\underline{x},\overline{x}]$, *dual* operator is defined as $dual\tilde{x}=[\overline{x},\underline{x}]$. Thus, if $\tilde{x}\in gI$ then $\tilde{x}-dual\tilde{x}=[0,0]=\tilde{0}$ and $dual\tilde{x}\in gI$. If we consider above algebraic properties then we see that the couple $(gI,+)$ is a group and the triple $(gI,+,\cdot)$ is a ring, [11]. Let us suppose that $\tilde{x}\in gI$ then $\tilde{x}$ is called symmetric modal if $\underline{x}=-\overline{x}$ or vice versa.

The set of all modals $gI$ is a metric space defined by the metric

$$d(\tilde{x}_1,\tilde{x}_2)=\max\{|\underline{x}_1-\underline{x}_2|,|\overline{x}_1-\overline{x}_2|\}. \tag{1.1}$$

If $\tilde{x},\tilde{y}\in gI$ and $\underline{x}\leq\overline{x}$, $\underline{y}\leq\overline{y}$ then the set $gI$ is reduced ordinary set of interval numbers which is complete metric space with the metric $d$ defined in (1.1), [3]. Moreover it is known that the set $gI$ is a complete metric space with the metric $d$. If we take $\tilde{x}_1=[a,a]$ and $\tilde{x}_2=[b,b]$, we obtain the usual metric of $\mathbf{R}$ with $d(\tilde{x}_1,\tilde{x}_2)=|a-b|$, where $a,b\in\mathbf{R}$.

Let's define transformation $f$ from $\mathbf{N}$ to $gI$ by $k\to f(k)=\tilde{x}$, $\tilde{x}=(\tilde{x}_k)$. Then, $(\tilde{x}_k)$ is called sequence of modals. The $\tilde{x}_k$ is called $k^{th}$ term of the sequence of modals. Let us denote the set of all sequences of modals by $w(gI)$.

For two sequences of modals $(\tilde{x}_k)$ and $(\tilde{y}_k)$, the addition, scalar product and multiplication are defined as follows $\tilde{x}_k+\tilde{y}_k=[\underline{x}_k+\underline{y}_k,\overline{x}_k+\overline{y}_k];\alpha\tilde{x}_k=[\alpha\overline{x}_k,\alpha\underline{x}_k],\alpha\in\mathbf{R}$; $(\tilde{x_k})(\tilde{y_k})=[\underline{x}_k\underline{y}_k,\overline{x}_k\overline{y}_k]$, respectively.

The set $w(gI)$ is a vector space since the vector space rules are clearly provided. The zero element of $w(gI)$ is the sequence $\tilde{\theta}=(\tilde{\theta}_k)=([0,0])$ all terms of which are zero interval. If $(\tilde{x}_k)\in w(gI)$ then inverse of $(\tilde{x}_k)$, according to addition, is $dual(\tilde{x}_k)$.

**Definition 1.1.** Let $\lambda(gI) \subset w(gI)$. A modal norm is a function $||.|| : \lambda(gI) \to \mathbb{R}$ such that (1) $||\tilde{x}||_{\lambda(gI)} = 0 \Leftrightarrow \tilde{x} = \tilde{\theta}$, (2) $||\alpha\tilde{x}||_{\lambda(gI)} = |\alpha|||\tilde{x}||_{\lambda(gI)}$, (3) $||\tilde{x} + \tilde{y}||_{\lambda(gI)} \leq ||\tilde{x}||_{\lambda(gI)} + ||\tilde{y}||_{\lambda(gI)}$. So the pair $(\lambda(gI), ||.||_{\lambda(gI)})$ is called normed space.

If a normed space $\lambda(gI)$ contains a sequence $(\tilde{e}_n)$ of modals with the property that for every $\tilde{u} \in \lambda(gI)$ there is a unique sequence of scalars $(t_n)$ such that $\lim_n ||\tilde{u} - (\tilde{t}_1\tilde{e}_1 + \cdots + \tilde{t}_n\tilde{e}_n)||_{\lambda(gI)} \to \tilde{0}$ then $(\tilde{e}_n)$ is called a Schauder modal basis for $\lambda(gI)$. The series $\sum_{k=0}^{\infty} \tilde{t}_k\tilde{e}_k$ which has the sum $\tilde{u}$ is then called the expansion of $\tilde{u}$ with respect to $(\tilde{e}_n)$, and we write $\tilde{u} = \sum_{k=1}^{\infty} \tilde{t}_k\tilde{e}_k$.

Let $\lambda(gI)$ and $\mu(gI)$ be linear spaces of modals. Then a function $\tilde{A} : \lambda(gI) \to \mu(gI)$ is called a linear transformation if and only if, for all $\tilde{u}_1, \tilde{u}_2 \in \lambda(gI)$ and all $\tilde{t}_1, \tilde{t}_2 \in gI$, $\tilde{A}(\tilde{t}_1\tilde{u}_1 + \tilde{t}_2\tilde{u}_2) = \tilde{t}_1\tilde{A}\tilde{u}_1 + \tilde{t}_2\tilde{A}\tilde{u}_2$.

A linear transformation $\tilde{A} : \lambda(gI) \to \mu(gI)$ is called bounded if and only if there exists a constant $M$ such that, for all $\tilde{u} \in \lambda(gI)$, $\left\|\tilde{A}(\tilde{u})\right\|_{\mu(gI)} \leq M \left\|\tilde{u}\right\|_{\lambda(gI)}$.

**Proposition 1.2.** *If $(\tilde{u}_k), (\tilde{v}_k), (\tilde{r}_k)$ are sequences of symmetric modal then the following equality holds:*

$$(\tilde{u}_k)\{(\tilde{v}_k) - (\tilde{r}_k)\} = (\tilde{u}_k)(\tilde{v}_k) - (\tilde{u}_k)(\tilde{r}_k). \tag{1.2}$$

**Definition 1.3.** A sequence $\tilde{u} = (\tilde{u}_k)$ of modals is said to be convergent to the modal $\tilde{u}_0$ if for each $\varepsilon > 0$ there exists a positive integer $n_0$ such that $d(\tilde{u}_k, \tilde{u}_0) < \varepsilon$ for all $k \geq n_0$, and we denote it by writing $\lim_k \tilde{u}_k = \tilde{u}_0$.

Thus, $\lim_{k\to\infty} \tilde{u}_k = \tilde{u}_0 \Leftrightarrow \lim_{k\to\infty} \underline{u}_k = \underline{u}_0$ and $\lim_{k\to\infty} \overline{u}_k = \overline{u}_0$.

**Definition 1.4.** A sequence of modals, $\tilde{u} = (\tilde{u}_k) \in w(gI)$, is said to be modal fundamental sequence if for every $\varepsilon > 0$ there exists $k_0 \in \mathbf{N}$ such that $d(\tilde{u}_n, \tilde{u}_k) < \varepsilon$ whenever $n, k > k_0$.

**Definition 1.5.** Let $\lambda(gI)$ be a linear space over $gI$ and $\mu(gI) \subset \lambda(gI)$. Then $\mu(gI)$ is called

1. Convex $\Leftrightarrow \forall\tilde{\zeta} = [\zeta, \zeta] \subset [0, 1] : \tilde{\zeta}\lambda(gI) + (\tilde{1} - \tilde{\zeta})\lambda(gI) \subset \lambda(gI)$, $\tilde{1} = [1, 1]$,
2. Balanced $\Leftrightarrow \forall\tilde{\zeta} \in gI : |\tilde{\zeta}| = \max\{|\underline{\zeta}|, |\overline{\zeta}|\} \leq \tilde{1} \Rightarrow \tilde{\zeta}\mu(gI) \subset \mu(gI)$
3. Absolutely convex $\Leftrightarrow \forall\tilde{\zeta}, \tilde{\psi} \in gI : |\tilde{\zeta}| + |\tilde{\psi}| = \max\{|\underline{\zeta}|, |\overline{\zeta}|\} + \max\{|\underline{\psi}|, |\overline{\psi}|\} \leq \tilde{1} \Rightarrow \tilde{\zeta}\mu(gI) + \tilde{\psi}\mu(gI) \subset \mu(gI)$,
4. Absorbing $\Leftrightarrow \forall\tilde{u} \in \mu(gI) \;\; \exists\rho > 0 \;\; \forall\tilde{\zeta} \in gI : |\tilde{\zeta}| = \max\{|\underline{\zeta}|, |\overline{\zeta}|\} \geq \rho \Rightarrow \tilde{u} \in \tilde{\zeta}\mu(gI)$.

In Section 2, we compute Schauder interval basis of these spaces and we show that these sets are Banach spaces. Also null, convergent and bounded sequence spaces of modals which are denoted by $c_0(gI)$, $c(gI)$ and $\ell_\infty(gI)$ respectively are defined in Section 2. Furthermore in this section some interesting theorems and definitions are given about the sequence spaces of $c_0(gI)$, $c(gI)$ and $\ell_\infty(gI)$.

In Section 3 we define the $\alpha-$, $\beta-$ and $\gamma-$ duals of the spaces $c_0(gI)$, $c(gI)$ and $\ell_\infty(gI)$. In this section we give and prove some important theorems about $\alpha-$, $\beta-$ and $\gamma-$ duals of the spaces $c_0(gI)$, $c(gI)$ and $\ell_\infty(gI)$. Finally, in the last section, we characterize some matrix transformations on sequence spaces of modals $c_0(gI)$, $c(gI)$ and $\ell_\infty(gI)$.

## 2 Some Sequence Spaces of the Modals

In this section we define null, convergent, bounded, convergent series, bounded series and $p-$absolute convergent series of sequences spaces of the symmetric modals which are denoted $c_0(gI)$, $c(gI)$, $\ell_\infty(gI)$, $cs(gI)$, $bs(gI)$ and $\ell_p(gI)$ respectively, that is

$$c_0(gI) = \{\tilde{u} = (\tilde{u}_k) \in w(gI) : \lim_k d(\tilde{u}_k, \tilde{0}) = 0,\ where\ \tilde{0} = [0,0]\}, \quad (2.1)$$

$$c(gI) = \{\tilde{u} = (\tilde{u}_k) \in w(gI) : \lim_k d(\tilde{u}_k, \tilde{u}_0) = 0, \quad \tilde{u}_0 \in gI\}, \quad (2.2)$$

$$\ell_\infty(gI) = \{\tilde{u} = (\tilde{u}_k) \in w(gI) : \sup_k d(\tilde{u}_k, \tilde{0}) < \infty\}, \quad (2.3)$$

$$cs(gI) = \{\tilde{u} = (\tilde{u}_k) \in w(gI) : \lim_n \{d(\sum_{k=0}^{n} \tilde{u}_k, \tilde{u}_0)\} = \tilde{0}\}, \quad (2.4)$$

$$bs(gI) = \{\tilde{u} = (\tilde{u}_k) \in w(gI) : \sup_n \{d(\sum_{k=0}^{n} \tilde{u}_k, \tilde{0})\} < \infty\}, \quad (2.5)$$

$$\ell_p(gI) = \{\tilde{u} = (\tilde{u}_k) \in w(gI) : \left(\sum_k (d(\tilde{u}_k, \tilde{0}))^p\right)^{\frac{1}{p}} < \infty,\ p \geq 1\}. \quad (2.6)$$

Clearly we see that the spaces $c_0(gI)$, $c(gI)$, $\ell_\infty(gI)$, $cs(gI)$, $bs(gI)$ and $\ell_p(gI)$ are subvector spaces in accordance with scalar product and addition on $w(gI)$. Besides, for all $(\tilde{x}_k), (\tilde{y}_k) \in c_0(gI)$ $(or\ c(gI),\ \ell_\infty(gI))$ the function $\tilde{d}$ defined by

$$\tilde{d}(\tilde{u}_k, \tilde{v}_k) = \sup_k \max\{|\underline{u}_k - \underline{v}_k|, |\overline{u}_k - \overline{v}_k|\} \quad (2.7)$$

which satisfies the metric axioms.

Thus, $(c_0(gI), \tilde{d})$ $\quad (or \quad (c(gI), \tilde{d}) \quad and \quad (\ell_\infty(gI), \tilde{d}))$ is a metric space.

Let us suppose that $\tilde{v} \in w(gI)$, $\tilde{v} = ([\underline{v}_k, \overline{v}_k])$ and $\underline{v}_k = \overline{v}_k$. Then, the sequence $\tilde{v} = (\tilde{v}_k)$ is reduced sequence of real numbers. In this case, the sets $c_0(gI), c(gI)$ and $\ell_\infty(gI)$ are reduced the classical sequence spaces (i.e., null, convergent and bounded sequences of the real or complex numbers). We shall denote $\ell_\infty$, $c$ and $c_0$ for the classical spaces of all bounded, convergent and null sequences of real numbers, respectively.

Let suppose that $\lambda(gI) = \{c_0(gI), c(gI), \ell_\infty(gI)\}$. Also the metric $\tilde{d}$ satisfies following properties: For all $\tilde{u} = (\tilde{u}_k), \tilde{v} = (\tilde{v}_k), \tilde{r} = (\tilde{r}_k) \in \lambda(gI)$ and for all $\alpha \in \mathbf{R}$, (1) $\tilde{d}(\tilde{u} + \tilde{r}, \tilde{v} + \tilde{r}) = \tilde{d}(\tilde{u}, \tilde{v})$, (2) $\tilde{d}(\alpha\tilde{u}, \alpha\tilde{v}) = |\alpha|\tilde{d}(\tilde{u}, \tilde{v})$.

Thus, we have

$$\tilde{d}(\tilde{u},\tilde{0}) = \sup_k \max\{|\underline{u}_k|, |\overline{u}_k|\} = ||u||_{\lambda(gI)}. \tag{2.8}$$

Based on the above explanations and definitions, we give following theorem :

**Theorem 2.1.** *The sets $c_0(gI)$, $c(gI)$ and $\ell_\infty(gI)$ are Banach spaces with the norm defined by in* (2.8).

*Proof.* We only give the proof for $\ell_\infty(gI)$. It is easy to see that norm conditions are provided easily.

Let $(\tilde{x}^n) = (\tilde{x}^n_k) = (\tilde{x}^n_0, \tilde{x}^n_1, \tilde{x}^n_2, \ldots) \in \ell_\infty(gI)$ for each $n$ and $(\tilde{x}^n)$ be a fundamental sequence. Then, for every $\varepsilon > 0$ there exist a $k_0 \in \mathbf{N}$ such that $||\tilde{x}^n - \tilde{x}^m|| < \varepsilon$ whenever $n, m \geq k_0$. Hence, we have $|\underline{x}^n_k - \underline{x}^m_k| < \varepsilon$ and $|\overline{x}^n_k - \overline{x}^m_k| < \varepsilon$. This means that $(\underline{x}^n)$ and $(\overline{x}^n)$ are Cauchy sequences in $\mathbf{R}$. Since $\mathbf{R}$ is a Banach space, $(\underline{x}^n)$ and $(\overline{x}^n)$ are convergent i.e, $(\tilde{x}^n_k)$ is convergent.

Now, let $\lim_{n\to\infty} \tilde{x}^n_k = \tilde{x}_k$ for each $k \in \mathbf{N}$. Since $\tilde{d}(\tilde{x}^n, \tilde{x}^m) < \varepsilon$ for all $n, m \geq k_0$, $\lim_{m\to\infty} \tilde{d}(\tilde{x}^n_k, \tilde{x}^m_k) = \tilde{d}(\tilde{x}^n_k, \lim_{m\to\infty} \tilde{x}^m_k) = \tilde{d}(\tilde{x}^n_k, \tilde{x}_k) < \varepsilon$. This implies that $\tilde{x}^n \to \tilde{x}$, $(n \to \infty)$ for all $n \geq k_0$ in $\ell_\infty(gI)$. On the other hand, since $\tilde{d}(\tilde{x}_k, \tilde{x}^n_k - \tilde{x}^n_k) = \sup_k \max\{|\underline{x}_k - (\underline{x}^n_k - \underline{x}^n_k)|, |\overline{x}_k - (\overline{x}^n_k - \overline{x}^n_k)|\} \leq \sup_k \max\{|\underline{x}_k - \underline{x}^n_k| + |\underline{x}^n_k|, |\overline{x}_k - \overline{x}^n_k| + |\overline{x}^n_k|\} \leq \sup_k \max\{|\underline{x}_k - \underline{x}^n_k|, |\overline{x}_k - \overline{x}^n_k|\} + \sup_k \max\{|\underline{x}_k|, |\overline{x}_k|\}$ this shows that $\tilde{x} \in \ell_\infty(gI)$. □

**Definition 2.2.** Let $\lambda(gI)$ is a sequence space of the modals. Then $\lambda(gI)$ is called solid if $\tilde{v} \in \lambda(gI)$ whenever $||\tilde{v}_k||_{\lambda(gI)} \leq ||\tilde{u}_k||_{\lambda(gI)}$, $(k \in \mathbf{N})$ for some $\tilde{u} \in \lambda(gI)$.

**Theorem 2.3.** *The sets $c_0(gI)$ and $c(gI)$which are sequence spaces of modals are solid.*

*Proof.* We consider only $c_0(gI)$. Now, let $||\tilde{v}_k||_{\lambda(gI)} \leq ||\tilde{u}_k||_{\lambda(gI)}$, for all $(k \in \mathbf{N})$ and for some $\tilde{u} \in c_0(gI)$. Then we have, $\tilde{d}(\tilde{v}_k, \tilde{0}) \leq \tilde{d}(\tilde{u}_k, \tilde{0})$, that is $\{|\underline{v}_k - \underline{0}|, |\overline{v}_k - \overline{0}|\} \leq \{|\underline{u}_k - \underline{0}|, |\overline{u}_k - \overline{0}|\}$. Thus we obtain $\underline{v}_k \leq \underline{u}_k$ and $\overline{v}_k \leq \overline{u}_k$ i.e., $\tilde{v} \leq \tilde{u}$. It is clear that $\tilde{v} \in c_0(gI)$. Therefore $c_0(gI)$ is solid. □

**Theorem 2.4.** *The inclusion $w \subset w(gI)$ holds, where $w$ denotes the space of all real or complex valued sequences.*

*Proof.* The proof is clear since every element of $w$ is a degenerate modal sequence. □

Also, the inclusions $\ell_\infty \subset \ell_\infty(gI)$, $c \subset c(gI)$ and $c_0 \subset c_0(gI)$ holds.

**Theorem 2.5.** *The inclusion $c_0(gI) \subset c(gI)$ holds.*

*Proof.* If we take any $\tilde{x} \in c_0(gI)$ then we see that $\tilde{x} \in c(gI)$ since $\tilde{d}(\tilde{x}_k, \tilde{0}) = \sup_k \max\{|\underline{x}_k - \underline{0}|, |\overline{x}_k - \overline{0}|\} < \varepsilon$. Furthermore, the convergent sequence of the modals $\tilde{y} = ([1, 1 + \frac{1}{n}]) \in c(gI)$ but $\tilde{y} \notin c_0(gI)$ since $\lim_n \underline{y}_n = 1$ and $\lim_n \overline{y}_n = 1$. □

**Theorem 2.6.** *The space $c_0(gI)$ which is the set of symmetric modals is convex and balanced.*

*Proof.* Let suppose that $\tilde{\zeta} = [\underline{\zeta}, \overline{\zeta}] \subset [0,1]$ and $\tilde{u} = (\tilde{u}_k)$ an arbitrary element of $c_0(gI)$. Then $\lim_k \tilde{d}(\tilde{\zeta}\tilde{u}_k + (\tilde{1} - \tilde{\zeta})\tilde{u}_k, \tilde{0}) = \lim_k \max\{|\underline{\zeta}\underline{u}_k + (1 - \underline{\zeta})\underline{u}_k|, |\overline{\zeta}\overline{u}_k + (1 - \overline{\zeta})\overline{u}_k|\} = \lim_k \max\{|\underline{u}_k|, |\overline{u}_k|\}$, which says to us $\tilde{\zeta}\tilde{u}_k + (\tilde{1} - \tilde{\zeta})\tilde{u}_k \in c_0(gI)$.

For the second part of the proof, let us suppose that $|\tilde{\zeta}| \leq \tilde{1}$ and $\tilde{u} \in c_0(gI)$.

(i) If $-\tilde{1} \leq \tilde{\zeta} < \tilde{0}$ then $\lim_k \tilde{d}(\tilde{\zeta}\tilde{u}_k, \tilde{0}) = \lim_k \tilde{d}(\tilde{\zeta}[\underline{u}_k, \overline{u}_k], \tilde{0}) \leq M \lim_k \tilde{d}([\overline{u}_k, \underline{u}_k], \tilde{0})$ which it says to us $\tilde{\zeta}\tilde{u} \in c_0(gI)$, where $M = \max\{|-\underline{\zeta}|, |-\overline{\zeta}|\}$.

(ii) If $\tilde{\zeta} = \tilde{0}$ then there is no need to prove.

(iii) If $\tilde{0} < \tilde{\zeta} \leq \tilde{1}$ the proof is similar to the (i). Thus $c_0(gI)$ is balanced. □

**Theorem 2.7.** *The sequences $(\tilde{e}_0, \tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_k, \ldots)$ and $(\tilde{e}, \tilde{e}_0, \tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_k, \ldots)$ are Schauder modal bases for $c_0(gI)$ and $c(gI)$, respectively, where*

$$\tilde{e}_k = (\tilde{0}, \tilde{0}, \ldots, [\underbrace{\ 1, 1\ }_{k^{th}\quad case}], \tilde{0}, \ldots).$$

*Proof.* Let $\tilde{u} = (\tilde{u}_k) \in c_0(gI)$. Therefore for every $\epsilon > 0$ there exists $n \in \mathbf{N}$ such that for $k \geq n$, $||\tilde{u}||_{c_0(gI)} = \sup_k \tilde{d}(\tilde{u}_k, \tilde{0}) < \epsilon$. Now we should show the following statement.

$$\lim_n \left\| \tilde{u}_k - \sum_{k=0}^{n} \tilde{e}_k \tilde{u}_k \right\|_{c_0(gI)} = \tilde{0}.$$

From here we can write next steps;

$$\begin{aligned}
\left\| \tilde{u_k} - \sum_{k=0}^{n} \tilde{e_k} \tilde{u_k} \right\|_{c_0(gI)} &= ||([\underline{x}_0, \overline{x}_0], [\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n], [\underline{x}_{n+1}, \overline{x}_{n+1}], \ldots) \\
&\qquad - \{([\underline{x}_0, \overline{x}_0], [\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n], \tilde{0}, \tilde{0}, \ldots)\}||_{c_0(gI)} \\
&= ||(\tilde{0}, \tilde{0}, \ldots, [\underline{x}_{n+1}, \overline{x}_{n+1}], [\underline{x}_{n+2}, \overline{x}_{n+2}], \ldots)||_{c_0(gI)} \\
&= \sup_{k \geq n+1} \max\{|\underline{u}_k|, |\overline{u}_k|\} \to \tilde{0}, \ (n \to \infty),
\end{aligned}$$

so we have

$$\tilde{u}_k = \sum_k \tilde{u}_k \tilde{e}_k. \tag{2.9}$$

Let us show uniqueness of the representation given by (2.9) for $\tilde{u}_k \in c_0(\mathtt{gI})$. Suppose that there exists a representation $\tilde{u}_k = \sum_k \tilde{v}_k \tilde{e}_k$. Then, for $n \to \infty$, we have

$$\begin{aligned}
&\left\| \sum_{k=0}^{n} (\tilde{v}_k - \tilde{u}_k) \tilde{e}_k \right\|_{c_0(gI)} \\
&\qquad = \tilde{d}((\tilde{v}_k - \tilde{u}_k), \tilde{0}) = \sup_{k \geq n+1} \max\{|(\underline{v}_k - \underline{u}_k) - \underline{0}|, |(\overline{v}_k - \overline{u}_k) - \overline{0}|\} \to \tilde{0}.
\end{aligned}$$

This shows that for $k \geq n+1$, $|\underline{v}_k - \underline{u}_k| \to \underline{0}$ and $|\overline{v}_k - \overline{u}_k| \to \overline{0}$. Therefore we have, $\underline{v}_k = \underline{u}_k$ and $\overline{v}_k = \overline{u}_k$, i.e., $\tilde{v} = \tilde{u}$. □

# 3 $\alpha-$, $\beta-$ and $\gamma-$ Duals of Sequence Spaces of the Modals

In this section, we have stated and proved the theorems determining the $\alpha-$, $\beta-$ and $\gamma-$ duals of the spaces $c(gI)$, $c_0(gI)$, $\ell_\infty(gI)$. For the sequence spaces $\lambda(gI)$ and $\mu(gI)$, we define the set $S(\lambda(gI), \mu(gI))$ by

$$S(\lambda(gI), \mu(gI)) = \{(\tilde{v}_k) \in w(gI) : (\tilde{u}_k \tilde{v}_k) \in \mu(gI) \; for \; all \; (\tilde{u}_k) \in \lambda(gI)\}. \quad (3.1)$$

With the notation of (3.1), the $\alpha-$, $\beta-$ and $\gamma-$duals of a sequence space $\lambda(gI)$, which are respectively denoted by $\lambda^\alpha(gI)$, $\lambda^\beta(gI)$ and $\lambda^\gamma(gI)$ are defined by with $\lambda^\alpha(gI) = S(\lambda(gI), \ell_1(gI)), \lambda^\beta(gI) = S(\lambda(gI), cs(gI))$ and $\lambda^\gamma(gI) = S(\lambda(gI), bs(gI))$.

**Theorem 3.1.** *The $\beta-$ dual of sequence spaces $c_0(gI)$, $c(gI)$ and $\ell_\infty(gI)$ are the set $\ell_1(gI)$.*

*Proof.* Since the proofs are similar, we will give the proof only for the space $c_0(gI)$. Let us suppose that $\tilde{u} = (\tilde{u}_k) \in c_0(gI)$ and $\tilde{a} = (\tilde{a}_k) \in w(gI)$. Then we can write

$$\begin{aligned}
\lim_n d(\sum_{k=0}^n [\underline{a}_k, \overline{a}_k][\underline{u}_k, \overline{u}_k], \tilde{0}) &= \lim_n d(\sum_{k=0}^n [\underline{a}_k \underline{u}_k, \overline{a}_k \overline{u}_k], \tilde{0}) \\
&= \lim_n \max\{|\sum_{k=0}^n \underline{a}_k \underline{u}_k|, |\sum_{k=0}^n \overline{a}_k \overline{u}_k|\} \\
&\leq M \lim_n \max\{\sum_{k=0}^n |\underline{a}_k|, \sum_{k=0}^n |\overline{a}_k|\} = M \lim_n d(\sum_{k=0}^n \tilde{a}_k, \tilde{0}) \\
&= M d(\sum_k \tilde{a}_k, \tilde{0}),
\end{aligned}$$

where $M = \max\{M_1, M_2\}$, $M_1 = \sup_k |\underline{u}_k|$ and $M_2 = \sup_k |\overline{u}_k|$. From here, we see that to get $(\tilde{a}_k \tilde{u}_k) \in cs(gI)$, $(\tilde{a}_k)$ should be in $\in \ell_1(gI)$. □

Using similar techniques, we can calculate the $\alpha-$ and $\gamma-$ duals of the sequence spaces $c_0(gI)$, $c(gI)$ and $\ell_\infty(gI)$ which are identical to the set $\ell_1(gI)$.

# 4 Matrix Transformations on Sequence Spaces of Modals

Let $\lambda(gI)$ and $\mu(gI)$ be two sequence spaces of generalized intervals and $\tilde{A} = (\tilde{a}_{nk})$ be an infinite matrix of generalized intervals and $\tilde{u} = (\tilde{u}_k) \in \lambda(gI)$, where

$n, k \in \mathbf{N} = \{0, 1, 2, \ldots\}$. Then, we can say that $\tilde{A}$ defines a matrix mapping from $\lambda(gI)$ to $\mu(gI)$, and we denote it by writing $\tilde{A} : \lambda(gI) \to \mu(gI)$, if for every sequence $\tilde{u} = (\tilde{u}_k) \in \lambda(gI)$ the sequence $\tilde{A}\tilde{u} = \left\{(\tilde{A}\tilde{u})_n\right\}$, the $\tilde{A}$-transform of $\tilde{u}$, is in $\mu(gI)$, where

$$\tilde{A}_n(\tilde{u}) = \sum_k \tilde{a}_{nk}\tilde{u}_k = \sum_k [\underline{a}_{nk}, \overline{a}_{nk}][\underline{u}_k, \overline{u}_k] = \sum_k [\underline{a}_{nk}\underline{u}_k, \overline{a}_{nk}\overline{u}_k], \tag{4.1}$$

and $\underline{a}_{nk}, \underline{u}_k, \overline{a}_{nk}, \overline{u}_k \in gI$. For simplicity in notation, here and in what follows, the summation without limit runs from 0 to $\infty$. By $(\lambda(gI) : \mu(gI))$, we denote the class of matrices $\tilde{A}$ such that $\tilde{A} : \lambda(gI) \to \mu(gI)$. Thus, $\tilde{A} \in (\lambda(gI) : \mu(gI))$ if and only if the series on the right side of (4.1) converges for each $n \in \mathbf{N}$ and every $\tilde{u} \in \lambda(gI)$, we have $\tilde{A}\tilde{u} = \left\{(\tilde{A}\tilde{u})_n\right\}_{n\in\mathbf{N}} \in \mu(gI)$ for all $\tilde{u} \in \lambda(gI)$.

In this section, we will seek answers, when does $\tilde{A} \in (\ell_\infty(gI) : \ell_\infty(gI))$ and $\tilde{A} \in (c_0(gI) : c_0(gI))$? Firstly, for second the question, the necessary and sufficient condition is given by the following theorem:

**Theorem 4.1.** *Let, for all fix $k \in \mathbf{N}$, $\lim_n \tilde{a}_{nk} = \tilde{0}$ and suppose that $M = \sup_n \sum_k \tilde{d}(\tilde{a}_{nk}, \tilde{0}) < \infty$. Then $\tilde{A}$ defines a bounded linear operator from $c_0(gI)$ to $c_0(gI)$.*

*Proof.* Firstly, we will show that $\tilde{A}$ defines a bounded linear operator from $c_0(gI)$ to $c_0(gI)$. Let us suppose that $\tilde{u} \in c_0(gI)$. If $\tilde{u} = \tilde{\theta} = (\tilde{0}, \tilde{0}, \ldots, \tilde{0}, \ldots)$ then $\tilde{A}\tilde{u} = \sum_k \tilde{d}(\tilde{a}_{nk}\tilde{u}_k, \tilde{0}) = \tilde{\theta}$ which there is no need to prove. Now, let us suppose that $\tilde{u} \neq \tilde{\theta}$. Under conditions of the hypothesis, that is, since $\tilde{u} \in c_0(gI)$ and $\sum_k \tilde{d}(\tilde{a}_{nk}, \tilde{0}) < \infty$, for all $n \in \mathbf{N}$ the series $\tilde{A}_n(\tilde{u}) = \sum_k [\underline{a}_{nk}\underline{u}_k, \overline{a}_{nk}\overline{u}_k]$ in $\ell_1(gI)$. On the other hand, since $\|\tilde{u}\|_{c_0(gI)} = \sup_k \max\{|\underline{u}_k|, |\overline{u}_k|\}$ we can write $\|\tilde{u}\|_{c_0(gI)} \geq \{|\underline{u}_k|, |\overline{u}_k|\}$. Accordingly,

$$\begin{aligned}
\left\|\tilde{A}_n(\tilde{u})\right\|_{c_0(gI)} = \left\|\sum_k \tilde{a}_{nk}\tilde{u}_k\right\|_{c_0(gI)} &= \left\|\sum_{k=1}^{N} \tilde{a}_{nk}\tilde{u}_k + \sum_{k\geq N+1} \tilde{a}_{nk}\tilde{u}_k\right\|_{c_0(gI)} \\
&\leq \sum_{k=1}^{N} \|\tilde{a}_{nk}\tilde{u}_k\|_{c_0(gI)} + \sum_{k\geq N+1} \|\tilde{a}_{nk}\tilde{u}_k\|_{c_0(gI)} \\
&\leq \sum_{k=1}^{N} \|\tilde{a}_{nk}\|_{c_0(gI)} \|\tilde{u}_k\|_{c_0(gI)} + M\|\tilde{u}_{k>N}\|_{c_0(gI)}.
\end{aligned}$$

Since $\tilde{u} \in c_0(gI)$, we take $k > N$ so large that $\|\tilde{u}_{k>N}\|_{c_0(gI)} < \frac{\epsilon}{M}$ and from $\lim_n \tilde{a}_{nk} = \tilde{0}$ ( $k$ fixed) we take $n$ so large that $\sum_{k=1}^{N} \|\tilde{a}_{nk}\|_{c_0(gI)} \leq \frac{\epsilon}{2\|\tilde{u}_k\|_{c_0(gI)}}$. Hence, we have shown that $\tilde{A}\tilde{u} \in c_0(gI)$. Finally, we will show that $\tilde{A}$ is bounded:

$$||\tilde{A}\tilde{u}||_{c_0(gI)} \quad = \sup_n \tilde{d}(\textstyle\sum_k \tilde{a}_{nk}\tilde{u}_k, \tilde{0}) \leq M||\tilde{u}||_{c_0(gI)},$$

that is $\tilde{A}$ is bounded. Now let us suppose that $\tilde{u}, \tilde{v} \in c_0(gI)$ and $\rho$ be an element of gI. $\tilde{A}(\rho\tilde{u} + \tilde{v}) = \sum_k [\underline{a}_{nk}(\underline{\rho u}_k + \underline{u}_k), \overline{a}_{nk}(\overline{\rho u}_k + \overline{v}_k)] = [\underline{\rho}, \overline{\rho}] \sum_k [\underline{a}_{nk}\underline{u}_k, \overline{a}_{nk}\overline{u}_k] + \sum_k [\underline{a}_{nk}\underline{v}_k, \overline{a}_{nk}\overline{v}_k] = \rho\tilde{A}\tilde{u} + \tilde{A}\tilde{v}$. □

The above-mentioned theorem shows that a certain type of matrix of $gI$'s defines a linear operator on $c_0(gI)$ into itself.

**Example 4.1.** *Now let us show that there exists a matrix $\tilde{A} = ([\underline{a}_{nk}, \overline{a}_{nk}])$ which satisfies the condition of Theorem 4.3. Define a matrix $\tilde{A} = ([\underline{a}_{nk}, \overline{a}_{nk}])$ by*

$$\tilde{a}_{nk} = \begin{cases} [-\frac{1}{n+1}, \frac{1}{n+1}], & 0 \le k \le n \\ \tilde{0}, & otherwise. \end{cases}$$

*The matrix $\tilde{A} = ([\underline{a}_{nk}, \overline{a}_{nk}])$ satisfies to the condition $\sup_n \sum_k \tilde{d}(\tilde{a}_{nk}, \tilde{0}) = [1,1] < \infty$.*

**Theorem 4.2.** *Let $\tilde{A}$ be any bounded linear transformation from $c_0(gI)$ to $c_0(gI)$. Then $\tilde{A}$ determines a matrix $(\tilde{a}_{nk})$ of modals such that*

$$(\tilde{A}\tilde{u})_n = \sum_k \tilde{a}_{nk}\tilde{u}_k \tag{4.2}$$

*for every $\tilde{u} \in c_0(gI)$.*

*Proof.* Since $(\tilde{e}_k)$ is a basis in $c_0(gI)$, every $\tilde{u} \in c_0(gI)$ may be written as $\tilde{u} = \sum_k \tilde{u}_k\tilde{e}_k$, where $\tilde{e}_k = (\tilde{0}, \tilde{0}, \ldots, [\underbrace{1,1}_{k^{th}\ \ case}], \tilde{0}, \ldots)$. Linearity and continuity of $\tilde{A}$ yield

$$\tilde{A}\tilde{u} = \sum_k \tilde{u}_k\tilde{A}\tilde{e}_k. \tag{4.3}$$

Since $\tilde{A}\tilde{e}_k = (\tilde{a}_k^0, \tilde{a}_k^1, \tilde{a}_k^2, \ldots) \in c_0(gI), (k \in \mathbf{N})$ from (4.3) we can write $\tilde{A}\tilde{u} = \sum_k \tilde{u}_k\tilde{a}_{nk}$. That is (4.2) holds. □

**Theorem 4.3.** *$\tilde{A} = ([\underline{a}_{nk}, \overline{a}_{nk}]) \in (\ell_\infty(gI) : \ell_\infty(gI))$ if and only if*

$$||\tilde{A}||_{\ell_\infty(gI)} = \tilde{d}(\tilde{A}\tilde{u}, \tilde{0}) = \sup_n \sum_k \tilde{d}(\tilde{a}_{nk}, \tilde{0}) < \infty. \tag{4.4}$$

*Proof.* Let us suppose that (4.4) holds and $\tilde{u} \in \ell_\infty(gI)$. Then,

$$\left\|\tilde{A}\tilde{u}\right\|_{\ell_\infty(gI)} = \sup_n \tilde{d}(\sum_k \tilde{a}_{nk}\tilde{u}_k, \tilde{0}) \le \sup_n \sum_k \tilde{d}(\tilde{a}_{nk}, \tilde{0})\tilde{d}(\tilde{u}_k, \tilde{0}) \le M \left\|\tilde{u}\right\|_{\ell_\infty(gI)} < \infty,$$

that is $\tilde{A}\tilde{u} \in \ell_\infty(gI)$.

Conversely, let us suppose that $\tilde{A} = ([\underline{a}_{nk}, \overline{a}_{nk}]) \in (\ell_\infty(gI) : \ell_\infty(gI))$ and $\tilde{u} \in \ell_\infty(gI)$. Then, since $\tilde{A}\tilde{u} \in \ell_\infty(gI)$ exists, the series $\sum_k [\underline{a}_{nk}, \overline{a}_{nk}][\underline{u}_k, \overline{u}_k]$ converges for each fixed $n \in \mathbf{N}$. And hence $\tilde{A} \in \ell_\infty^\beta(gI)$. This holds for the sequence $(\tilde{u}_k) = ([-1, 1]) \in \ell_\infty(gI)$. Then, we can write

$$\left\|\tilde{A}\tilde{u}\right\|_{\ell_\infty(gI)} = \sup_n \tilde{d}(\textstyle\sum_k \tilde{a}_{nk}\tilde{u}_k, \tilde{0}) \leq \sup_n \sum_k \tilde{d}(\tilde{a}_{nk}, \tilde{0})\tilde{d}(\tilde{u}_k, \tilde{0})\} < \infty$$

which means that (4.4) holds. □

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