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Semi-Complementary Graphs

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Abstract : In communication networks, a secret message is being sent by means of adjacency matrix associated with a simple graph G. As it is easily traceable instead of adjacency, non adjacency matrix(that is associated with the complementary graph G^c) is being preferred. Now, we introduce another type of graph called as semi-complementary graph G^{sc} of G. This is a spanning subgraph of G^c and hence more secrecy can be achieved by using this in defence problems.

Already semi complete graphs have been introduced ([1, 2]) and it is observed that for such graphs $G^{sc} = G^c$. Semi complete graphs are playing a vital role in sharing a secret code in parts, by two individuals, instead of one. Thus these are useful in bank transactions.

Keywords : semi complete graph; dominating set; vertex cover; restrained dominating set.
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1 Introduction

In transportation problems the concept of complementary graphs is very much useful in providing a substitute network (hidden) between the sources and destinations in connecting each source/destination to all the sources/destinations that are not adjacent to the former so that the system remains connected at times of

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need. We have introduced the concept semi-complementary graphs which serves the above mentioned purpose in a more efficient way (minimizes the cost).

2 Preliminaries

A set D of vertices in a graph $G = (V, E)$ is said to be a dominating set of G if and only if every vertex in $V - D$ is adjacent to some vertex in D [3]. A set D of vertices in a graph $G = (V, E)$ is said to be a restrained dominating set if and only if it is a dominating set of G and further every vertex in $V - D$ is adjacent to some other vertex in $V - D$ [3]. A set S of vertices in a graph $G = (V, E)$ is said to be an independent set of G if and only if no two vertices in S are adjacent G [4]. The number of vertices in a maximum independent set of G is called the independence number of G and is denoted by $\alpha(G)$ [4]. A set S of vertices in a graph $G = (V, E)$ is said to be a vertex cover of G if and only if for each edge uv in G, either $u \in S$ or $v \in S$ [4]. A set S of vertices in a graph $G = (V, E)$ is said to be a neighbourhood cover of G if and only if $G = \bigcup_{v \in S} < N[v] >$, where $N[v] = \{u \in V(G)/uv \in E(G)\} \bigcup \{v\}$ [5]. The girth of a graph G is defined as the length of the shortest cycle in G. A graph G is said to be semi complete if and only if it is simple and for any two vertices u, v of G there is a vertex w of G such that w is adjacent to both u and v(in G) (i.e, $\{u, w, v\}$ is a path in G) [1, 2].

All graphs considered in this paper are simple, finite, undirected and connected. For standard terminology and notation we refer Bondy and Murthy [4].

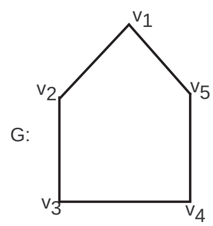
3 Main Results

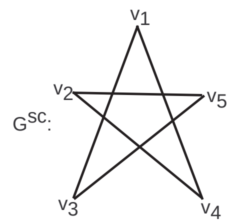
Now, we introduce a new type of graph.

Definition 3.1. Let G be a graph with vertex set $V(= V(G))$. Then the graph whose vertex set is V and the edge set being $\{uv : u, v \in V, uv \notin G \textit{ and there is a } w \textit{ in } V \textit{ such that } < uwv > \textit{ is a path in } G\}$ is called the semi-complementary graph of G and is denoted by G^{sc}.

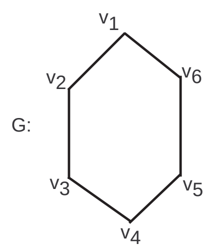
Note. By definition, there is no interest with empty graph, complete graph K_n, multi graph and disconnected graph with regard to this concept. Hence, throughout this work, by a graph we mean a simple, connected graph with atleast three vertices and is not complete.

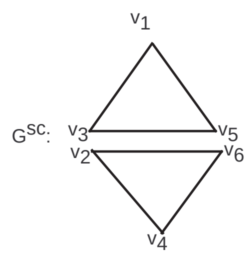
Given below are the examples of some graphs and their corresponding semi-complementary graphs.





(connected)





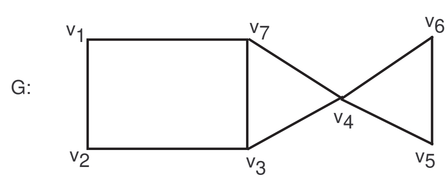
(disconnected)

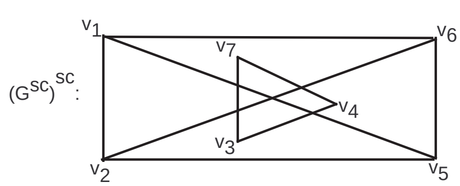
Observations:

(i) It is taken, for convenience, G to be connected; but G^{sc} need not be connected(in view of the above examples).

(ii) G^{sc} is clearly a spanning subgraph of $G^c \Rightarrow$ If G is a finite graph then $|E(G^c)| \geq |E(G^{sc})|$.
We know that $(G^c)^c = G$ but,

(iii) $(G^{sc})^{sc}$ need not be G even if G^{sc} is connected, in view of the following:





(iv) If G is finite, then $|E(G^{sc})| = |\{\{u,v\} : u,v \in V \ and \ d_G(u,v) = 2\}|(\Rightarrow$ $u \ and \ v$ are nonadjacent in G).

(v) If $G = K_n$, then G^{sc} is a null graph.

(vi) If G is a star graph with $n(\geq 3)$ vertices then $G^{sc} = K_1 \bigcup K_{n-1}$.

(vii) If $G = P_n(n \geq 3)$, then

$$G^{sc} = P_{\frac{n}{2}} \bigcup P_{\frac{n}{2}} \text{ if } n \text{ is even,}$$
$$= P_{\frac{n+1}{2}} \bigcup P_{\frac{n-1}{2}} \text{ if } n \text{ is odd.}$$

(If $\{v_1, v_2, \ldots, v_n\}$ are the vertices of G then the first $P_{\frac{n}{2}}$ is formed by $\{v_1, v_3, \ldots, v_{n-1}\}$ and the second $P_{\frac{n}{2}}$ is formed by the vertices $\{v_2, v_4, \ldots, v_n\}$ (when n is even); $P_{\frac{n+1}{2}}$ is formed by $\{v_1, v_3, \ldots, v_n\}$ and $P_{\frac{n-1}{2}}$ is formed by $\{v_2, v_4, \ldots, v_{n-1}\}$ (when n is odd)).

Thus ${P_n}^{sc}$ is disconnected.

(viii) If $G = C_n(n \geq 4)$, then

$$G^{sc} = C_{\frac{n}{2}} \bigcup C_{\frac{n}{2}} \text{ if } n \text{ is even,}$$
$$= C_n \text{ if } n \text{ is odd.}$$

(The first $C_{\frac{n}{2}}$ is formed by $\{v_1, v_3, \ldots, v_{n-1}, v_1\}$ and the second $C_{\frac{n}{2}}$ is formed by $\{v_2, v_4, \ldots, v_n, v_2\}$, when n is even; C_n is formed by $\{v_1, v_3, \ldots, v_n, v_2, \ldots, v_{n-1}, v_1\}$, when n is odd).

Thus ${C_{2n+1}}^{sc}$ is isomorphic to C_{2n+1} when $n \geq 2$ and is connected, Eulerian and Hamiltonian.

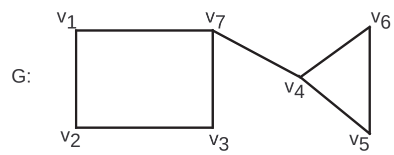
(ix) G^{sc} is Eulerian if and only if $\forall u \in V, |\{v \in V/d_G(u,v) = 2\}|$ is even.

(x) G is a connected graph with vertex set V and $S \subseteq V$. Then S is an independent set of G and G^{sc} if and only if $d_G(u,v) \geq 3$ for every $u, v \in S$(u, v are adjacent in $G \Leftrightarrow d_G(u,v) = 1$ and u, v are adjacent in $G^{sc} \Leftrightarrow d_G(u,v) = 2$).

Theorem 3.2. *If G is a connected bipartite graph, then G^{sc} is disconnected and is a union of two components.*

Proof. Let G be a connected, bipartite graph with bipartition X, Y. Since G is connected any vertex of X is connected to any vertex of Y and vice-versa. Also the distance between them is always an odd integer. So by definition, in G^{sc} no vertex of X is connected to a vertex of Y and vice-versa. Further between any two vertices of X or that of Y the distance is always an even integer. Hence there is a path between the vertices of X and similarly the vertices of Y in G^{sc}. Thus G^{sc} is disconnected and has components formed by the corresponding graph in X and that of in Y. □

Remark 3.3. *The converse of the above Theorem is false in view of the following example:*



Here G^{sc} has exactly two components, G is connected but not bipartite (G contains an odd cycle).

Corollary 3.4. *If* $G = K_{m,n}(m+n \geq 3)$ *then* $G^{sc} = K_m \bigcup K_n$*(If* $\{X, Y\}$ *is the bipartition of* $K_{m,n}$ *then any two vertices in* X *or that of* Y *are adjacent in* G^{sc}*).*

Corollary 3.5. *If* G *is a tree (with atleast three vertices), then* G^{sc} *is disconnected and is a union of two components (since* G *is bipartite).*

Theorem 3.6. G *be a connected graph with vertex set* V*. Then* $G^c = G^{sc}$ *if and only if the distance between any pair of non adjacent vertices is 2.*

Proof. Let G be a connected graph. Since G, G^{sc} have the same vertex set, $G^c = G^{sc}$ is equivalent to, $uv \in G^c \Leftrightarrow uv \in G^{sc}$. This is same as, u and v are not adjacent in $G \Leftrightarrow d_G(u, v) = 2$. □

Corollary 3.7. G *is a semi complete graph. Then* $G^{sc} = G^c$*.*

Proof. Since G is a semi complete graph the distance between any pair of nonadjacent vertices is two, then by the above theorem the proof follows. (observe that, if two vertices are adjacent in G, then also there must be a path of length 2 in G as it is a semi complete graph). □

Observation: $G^{sc} = G^c$ for $G = C_4$, but $G = C_4$ is not semi complete. Hence the converse fails.

Theorem 3.8. G *be a connected graph . Then* $G \subseteq (G^{sc})^{sc}$ *if and only if for each* uv *in* G *there is* w *in* V *such that* $d_G(u, w) = d_G(w, v) = 2$.

Proof. Assume that $G \subseteq (G^{sc})^{sc}$. Let $uv \in E(G)$
$\Rightarrow uv \in E((G^{sc})^{sc})$
$\Rightarrow d_{G^{sc}}(u, v) = 2$
$\Rightarrow$ there is w in $V(G)$ such that uw, wv are in $E(G^{sc})$
$\Rightarrow \ d_G(u, w) = d_G(w, v) = 2$.
Conversely assume that $uv \in E(G)$. Then by our assumption
there is w in V such that $d_G(u, w) = d_G(w, v) = 2$
$\Rightarrow uw, wv \in E(G^{sc})$ *and further* $uv \notin E(G^{sc})$
$\Rightarrow d_{G^{sc}}(u, v) = 2$
$\Rightarrow uv \in E((G^{sc})^{sc})$.
Thus $E(G) \subseteq E((G^{sc})^{sc})$.
Hence the result.

Now, we prove a necessary and sufficient condition for a connected graph G to have a connected G^{sc} □

Theorem 3.9. *G is a connected graph with vertex set V. Then G^{sc} is connected if and only if for any pair of distinct vertices u, v of G with $d_G(u,v) \neq 2$ there is a sequence $\{w_s : s = 1, 2, \ldots, n_0\}$($n_0$ being a positive integer)of distinct vertices in V and paths $P_s(w_{s-1} \to w_s)(s = 1, 2, \ldots, m+1)$ with the convention $w_0 = u$ and $w_{m+1} = v$ in G of even length each such that no pair of vertices at consecutive odd places in any of the paths P_s are adjacent in G.*

Proof. Let G be a connected graph with vertex set V. Suppose G^{sc} is connected. Let $u, v \in V$ with $u \neq v$ and $d_G(u,v) \neq 2 \Rightarrow uv \notin E(G^{sc})$. Since G^{sc} is connected there is a $u-v$ path, say $\{u = u_0, u_1, \ldots, u_n = v\}(n \geq 2)$ in G^{sc}. By the definition of G^{sc}, for each $j \in \{1, 2, \ldots, n\}$, there is an $x_j \in V$ such that $\{u_{j-1}, x_j, u_j\}$ is a path in G. Now follows that $W = \{u_0, x_1, u_1, \ldots, u_{n-1}, x_n, u_n = v\}$ is a $u-v$ walk in G. If this is a path then taking $w_1 = u_j$ for any $j \in \{1, 2, \ldots, n-1\}$, we get two paths $P_1(u = w_0 \to w_1)$ and $P_2(w_1 \to w_2 = v)$ in G of the required property, since the length of any $u_{j-1} - u_j$ path in G is even for $j \in \{1, 2, \ldots, n\}$ and u_{j-1} and u_j are not adjacent in G since they are adjacent in G^{sc}. Otherwise there is a subsequence $\{x_{j_s}; s = 1, 2, \ldots, m\}$ of $\{x_j; j = 1, 2, \ldots, n\}$ such that each x_{j_s} is a vertex in the sub walk $u_0 - u_{j_{s-1}}(s = 1, 2, \ldots, m)$. Taking $w_s = u_{j_{s-1}}$, we get the required paths with the specified property.

This proves the necessary part.

Conversely, assume that the specified condition of the theorem holds. Let $u, v \in V$ be such that $d_G(u,v) = 2$. Now follows that $uv \in E(G^{sc})$. Thus u and v are connected in G^{sc}. Otherwise, by hypothesis from the condition follows that $W = \bigcup_{s=1}^{m+1} P_s$ is a $u-v$ walk in G such that any pair of vertices in consecutive odd places in W are nonadjacent in G and hence adjacent in G^{sc}. So they give rise to a $u-v$ walk in G^{sc} and hence a $u-v$ path in G^{sc}. Thus G^{sc} is connected. This completes the proof of the theorem. □

Corollary 3.10. *In the characterization Theorem G^{sc} is "connected" is replaced by "a tree" and "sequence" is replaced by "a unique sequence", then the corresponding result holds.*

Proof. Necessary part is obvious since a tree is connected. In the sufficiency part the condition implies that between any pair of vertices there is a unique path and hence follows that it is a tree. □

Corollary 3.11. *In the characterization Theorem G is "connected" is replaced by "semi complete" and "$d_G(u,v) \neq 2$" is replaced by $uv \in E(G)(\Rightarrow d_G(u,v) = 1)$ then the corresponding result holds.*

Corollary 3.12. *G is a connected graph with vertex set V and girth of G is greater than 3. Then G^{sc} is connected if and only if for any pair of distinct vertices u, v of G with $d_G(u,v) \neq 2$ there is a sequence $\{w_s : s = 1, 2, \ldots, n_0\}$($n_0$ being a positive integer)of distinct vertices in V and paths $P_s(w_{s-1} \to w_s)(s = 1, 2, \ldots, m+1)$ with the convention $w_0 = u$ and $w_{m+1} = v$ in G of even length in G.*

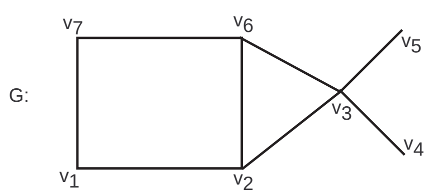
Proof. Since girth of G is greater than 3, there cannot be any cycles of length 3 in G. Hence the proof follows from the Characterization Theorem. □

Now we give an elegant sufficient condition for the graph G^{sc} of the connected graph G to be connected.

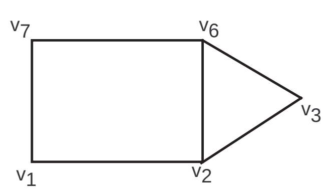
Theorem 3.13. *G is a connected graph with vertex set V such that for each pair of distinct vertices u, v in G there is a $u - v$ path P of even length, in G, further the subgraph induced by the vertices of P is acyclic. Then G^{sc} is connected.*

Proof. Under the given hypothesis let P be the $u - v$ path of even length say $P = \{u = u_0, u_1, \ldots, u_{2n} = v\}$($n$ being a positive integer)with the specified property; Now it follows that no u_j is adjacent with $u_i(0 \leq i < j)$ for $i, j \in \{0, 1, 2, \ldots, 2n\}$. Thus in this path any two vertices at consecutive odd places are adjacent in G^{sc}. This gives rise to the $u - v$ path $\{u_0, u_2, \ldots, u_{2n} = v\}$ in G^{sc}. Thus G^{sc} is connected. □

Observation: The converse of the above Theorem is false in view of the following:
Counter Example: Consider the graph given under:



Between the vertices v_1 and v_2 there is only one path of even length (4) namely $\{v_1, v_7, v_6, v_3, v_2\}$. The subgraph induced by the vertices of this path is



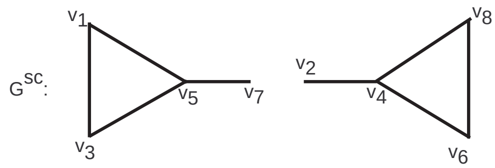
It contains three cycles namely $\{v_1, v_2, v_3, v_6, v_7, v_1\}$, $\{v_1, v_2, v_6, v_7, v_1\}$ and $\{v_2, v_3, v_6, v_2\}$. So it is not acyclic. But the graph G^{sc} is connected.

Theorem 3.14. *G is a connected graph such that G^{sc} is connected; then G is cyclic.*

Proof. Under the given hypothesis, let $e = uv \in E(G)$. Now u, v are the vertices of G and hence G^{sc}. Since G^{sc} is connected there is a shortest $u - v$ path in G^{sc}. This induces a path P in G. Now $P \bigcup \{e\}$ is a cycle in G. Thus G is cyclic. □

Remark 3.15. *The converse of the above Theorem is false in view of the following:*

Counter Example: Consider the connected cyclic graph given under:



G^{sc} is a union of two isomorphic components and so disconnected (observe that G is bipartite).

Theorem 3.16. *G and G^{sc} are connected with vertex set V. Then $\alpha(G) \geq 2$.*

Proof. Under the given hypothesis, if $\alpha(G) = 1$ there is a $v_0 \in V$ such that v_0 is adjacent with all other vertices of V. So $d_G(v, v_0) = 1$ for all $v \in V - \{v_0\}$. Hence v_0 is not adjacent with all other vertices of V in G^{sc}. So v_0 is an isolated vertex of G. Hence G^{sc} is disconnected. This is a contradiction. Hence $\alpha(G) \geq 2$. □

Observation: The converse of the above theorem is false in view of P_8. $\alpha(P_8) = 4 \geq 2$. Clearly $\{v_1, v_3, v_5, v_7\}$ and $\{v_2, v_4, v_6, v_8\}$ are maximum independent sets in P_8 but ${P_8}^{sc}$ is not connected.

Proposition 3.17. *G be a connected graph with vertex set V and $D \subset V$. Then D is a dominating set for G^{sc} if and only if for each $v \in V - D$ there is $u \in D \ni d_G(u, v) = 2$.*

Proof. D is a dominating set for $G^{sc} \Leftrightarrow$ for $v \in V - D$ there is a $u \in D$ such that $uv \in E(G^{sc}) \Leftrightarrow d_G(u, v) = 2$. □

Proposition 3.18. *If D is a dominating set for G, G^{sc}, then $|D| \geq 2$.*

Proof. If $|D| = 1$,then D is not a dominating set for G^{sc},which is a contradiction to the hypothesis. Hence $|D| \geq 2$. □

Proposition 3.19. *G be a connected graph with vertex set V and $D \subset V$. If D is an independent dominating set for G, G^{sc}, then D is a restrained dominating set for G.*

Proof. Suppose that the hypothesis holds. Let $u \in V - D$. Since D is a dominating set for G^{sc} there is $v \in D$ such that $d_G(u, v) = 2 \Rightarrow \exists w \in V(G) \ni \{u, w, v\}$ is a path in $G \Rightarrow uw \in E(G)$ and $w \in V - D$. Since D is a dominating set for G, there is an $x \in D \ni xu \in E(G)$. Hence D is a restrained dominating set for G. □

Proposition 3.20. *G be a connected graph with vertex set V . Then $S \subset V$ is a vertex cover for G, G^{sc} if and only if for any $u, v \in V(G) \ni d_G(u, v) \leq 2 \Rightarrow$ either $u \in S$ (or) $v \in S$.*

Proof. Suppose that the hypothesis holds. Assume that S is a vertex cover for G, G^{sc}. Let $u, v \in V \ni d_G(u, v) \leq 2$.

If $d_G(u, v) = 1 \Rightarrow uv \in E(G) \Rightarrow u \in S\ (or)\ v \in S$ (by the nature of S).

If $d_G(u, v) = 2 \Rightarrow uv \in E(G^{sc}) \Rightarrow u \in S\ (or)\ v \in S$ (by the nature of S).

Assume that the stated condition holds. Now, $uv \in E(G) \Rightarrow d_G(u, v) = 1 \leq 2 \Rightarrow u \in S\ (or)\ v \in S \Rightarrow S$ is a vertex cover for G.

Let $uv \in E(G^{sc}) \Rightarrow d_G(u, v) = 2 \Rightarrow u \in S\ (or)\ v \in S \Rightarrow S$ is a vertex cover for G^{sc}. Hence the proof. □

Proposition 3.21. *G be a connected graph. Then $S \subset V$ is a neighbourhood cover for G^{sc} if and only if for u, v in $V - S$ with $d_G(u, v) = 2$ there is w in S such that $d_G(u, w) = d_G(w, v) = 2$.*

Proof. Suppose that the hypothesis holds. Assume that S is a neighbourhood cover for G^{sc}. Let u, v be two vertices in $V - S$ such that $d_G(u, v) = 2$. This implies $uv \in E(G^{sc})$. Since S is a neighbourhood cover for G^{sc} there is w in S such that uv is in $< N_{G^{sc}}[w] >$. Hence $d_G(u, w) = d_G(w, v) = 2$.

Assume that that the stated condition holds. Let $e = uv$ be an edge in $E(G^{sc})$ $(d_G(u, v) = 2)$. Then by our assumption there is $w \in S$ such $d_G(u, w) = d_G(w, v) = 2$. Hence uw, wv are in $E(G^{sc})$. This implies uv is in $< N_{G^{sc}}[w] >$. Therefore S is a neighbourhood cover for G^{sc}. □

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