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On Graded Weakly Semiprime Submodules

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Abstract : Let G be a group with identity e. Let R be a G-graded commutative ring and M be a graded R-module. In this paper we study the concepts of graded weakly prime and graded weakly semiprime submodules of M.

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1 Introduction

Graded prime and graded primary ideals of a commutative graded ring R with a non-zero identity have been introduced and studied by Refaei and Alzobi in [1]. Graded prime submodules of a graded R-module have been studied by Ebrahimi Atani and Farzalipour in [2, 3]. Also, graded weakly prime submodules of graded R-modules has been studied in [4]. Here we study a number of results of graded weakly prime submodules (see Sec. 2). Also, we define the graded weakly semiprime submodules of a graded R-module and we give some results concerning this class of submodules (see Sec. 3). Before we state some results let us introduce some notation and terminology. Let G be a group. A ring (R, G) is called a G-graded ring if there exists a family $\{R_g : g \in G\}$ of additive subgroups of R

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such that $R = \bigoplus_{g\in G} R_g$ such that $1 \in R_e$ and $R_g R_h \subseteq R_{gh}$ for each g and h in G. For simplicity, we will denote the graded ring (R, G) by R. If R is G-graded, then an R-module M is said to be G-graded if it has a direct sum decomposition $M = \bigoplus_{g\in G} M_g$ such that for all $g, h \in G$; $R_g M_h \subseteq M_{gh}$. Any element of R_g or M_g for any $g \in G$, is said to be a homogeneous element of degree g. A submodule $N \subseteq M$, where M is G-graded, is called G-graded if $N = \bigoplus_{g\in G}(N \cap M_g)$ or if, equivalently, N is generated by homogeneous elements. Moreover, M/N becomes a G-graded module with g-component $(M/N)_g = (M_g + N)/N$ for $g \in G$. We write $h(R) = \cup_{g\in G} R_g$ and $h(M) = \cup_{g\in G} M_g$. A graded ring R is called graded integral domain, if whenever $ab = 0$ for $a, b \in h(R)$, then $a = 0$ or $b = 0$. A graded ring R is called a graded field, if every homogeneous element of R is unit. A graded ideal I of R is said to be a graded prime ideal if $I \neq R$; and whenever $ab \in I$, we have $a \in I$ or $b \in I$, where $a, b \in h(R)$ [1]. The set of all graded prime ideas of R is denoted by $Spec^g(R)$ ([5]). A graded ideal I of R is said to be graded maximal if $I \neq R$ and there is no graded ideal J of R such that $I \subsetneq J \subsetneq R$. A graded ring R with a unique graded maximal ideal P is called graded local and denoted by (R, P). Let N be a graded R-submodule of M, then $(N :_R M) = \{r \in R : rM \subseteq N\}$ is a graded ideal of R (see [2]). A proper graded submodule N of M is called graded prime, if whenever $rm \in N$ where $r \in h(R)$ and $m \in h(M)$, then $m \in N$ or $r \in (N : M)$. A graded R-module M is said to be graded prime, if the zero graded submodule of M is a graded prime submodule. A graded module M over a G-graded ring R is called graded finitely generated if $M = \sum_{i=1}^{n} Rx_{g_i}$ where $x_{g_i} \in h(M)$. A graded R-module M is called graded cyclic if $M = Rx_g$ where $x_g \in h(M)$. A graded module M over a G-graded ring R is called graded free if it has a basis $\{x_g\}_{g\in G}$ of homogeneous elements of $h(M)$.

2 On Graded Weakly Prime Submodules

A proper graded submodule of a graded R-module M is said to be graded weakly prime submodule if whenever $0 \neq rm \in N$ where $r \in h(R)$ and $m \in h(M)$, then $m \in N$ or $r \in (N : M)$ (see [4]). Clearly, every graded prime submodule of a graded R-module M is a graded weakly prime, but the converse is not true in general. In fact the zero graded submodule of a graded R-module M is always graded weakly prime (by definition), but it is not necessarily graded prime. For example, let R be a graded ring that is not graded integral domain and M a faithful graded R-module. If the zero graded submodule of M is graded prime, then $(0 : M) = 0$ is a graded prime ideal of R by [6, Proposition 2.5], which is not the case.

The following Lemma is known, but we write it here for the sake of references.

Lemma 2.1. *Let M be a graded module over a graded ring R. Then the following hold:*

(i) If I and J are graded ideals of R, then $I + J$ and $I \bigcap J$ are graded ideals.

(ii) If N is a graded submodule, $r \in h(R)$ and $x \in h(M)$, then Rx, IN and rN are graded submodules of M.

(iii) If N and K are graded submodules of M, then $N + K$ and $N \bigcap K$ are also graded submodules of M and $(N : M)$ is a graded ideal of R.

(iv) Let $\{N_\lambda\}$ be a collection of graded submodules of M. Then $\sum_\lambda N_\lambda$ and $\bigcap_\lambda N_\lambda$ are graded submodues of M.

We know that if N is a graded prime submodule of a graded R-module M, then $(N : M)$ is a graded prime ideal of R by [6, Proposition 2.5]. This is not true for the case of graded weakly prime submodules. For example, let $R = \mathbb{Z} \bigoplus \mathbb{Z}$ and $M = \mathbb{Z}_3 \bigoplus \mathbb{Z}_4$ and $G = \mathbb{Z}_2$. Then R is a G-graded ring with $R_0 = \mathbb{Z}$ and $R_1 = \mathbb{Z}$ and M is a G-graded R-module with $M_0 = \mathbb{Z}_3$ and $M_1 = \mathbb{Z}_4$. Let $N =< (0, 0) >$. Certainly, N is a graded weakly prime submodule of M, but $(N : M) = (0 : M) =< (3, 4) >$ is not a graded weakly prime ideal of R, because $0 \neq (3, 8) = (1, 4)(3, 2) \in (N : M)$, but $(1, 4) \notin (N : M)$ and $(3, 2) \notin (N : M)$.

In the following results we give some conditions under which $(N :_R M)$ is a graded weakly prime ideal of R for a graded weakly prime submodule N of the graded R-module M.

Proposition 2.2. *Let M be a graded free R-module with a basis of homogeneous elements $\{x_\lambda\}_{\lambda\in\Lambda}$. If N is a graded weakly prime submodule of M, then $(N : M)$ is a graded weakly prime ideal of R.*

Proof. It is clear that for every $\lambda \in \Lambda$, $Ann(x_\lambda) = 0$. Let $a, b \in h(R)$ and $0 \neq ab \in (N : M)$ with $a \notin (N : M)$. So $aM \nsubseteq N$, hence $ax_\mu \notin N$ for some $\mu \in \Lambda$. Now $b(ax_\mu) = (ab)x_\mu \in N$ and clearly $(ab)x_\mu \neq 0$. Thus $b \in (N : M)$, that is, $(N : M)$ is a graded weakly prime ideal of R. □

Theorem 2.3. *If F is a graded free R-module and P a graded weakly prime ideal of R, then PF is a graded weakly prime submodule of F and $(PF : F) = P$.*

Proof. Let $\{x_\lambda\}_{\lambda\in\Lambda}$ be a basis of homogeneous elements for F. Then it is clear that the graded submodule PF is of the form $PF = \{\sum_{i=1}^n s_i x_i : s_i \in P \bigcap h(R), x_i \in \{x_\lambda\}_{\lambda\in\Lambda}\}$. Let $0 \neq rx \in PF$ where $r \in h(R)$, $x \in h(F)$ and $r \notin (PF : F)$, so $r \notin P$. Since $x \in F$, so $x = \sum_{i=1}^n r_{\lambda_i} x_{\lambda_i}$ where $r_{\lambda_i} \in h(R)$ and $x_{\lambda_i} \in \{x_\lambda\}_{\lambda\in\Lambda}$. But $rx = \sum_{i=1}^n (rr_{\lambda_i})x_{\lambda_i} \in PF$ implies that $rx = \sum_{i=1}^n (s_{\lambda_i})x_{\lambda_i}$ where $s_{\lambda_i} \in h(R) \bigcap P$. Therefore, $\sum_{i=1}^n (rr_{\lambda_i})x_{\lambda_i} = \sum_{i=1}^n (s_{\lambda_i})x_{\lambda_i}$ and since $\{x_\lambda\}_{\lambda\in\Lambda}$ is a basis for F we must have $rr_{\lambda_i} = s_{\lambda_i}$ for all i. Clearly for some λ_i, $s_{\lambda_i} \neq 0$. Thus $0 \neq rr_{\lambda_i} = s_{\lambda_i} \in P$ and $r \notin P$, then $r_{\lambda_i} \in P$. Hence $x = \sum_{i=1}^n r_{\lambda_i} x_{\lambda_i} \in PF$. Therefore PF is a graded weakly prime submodule of F. It is clear that $P \subseteq (PF : F)$. Let $r \in (PF : F)$ and $x_\mu \in \{x_\lambda\}_{\lambda\in\Lambda}$. Then $rx_\mu \in PF$ and by the way PF is defined, we have $r \in P$. Hence $P = (PF : F)$. □

Corollary 2.4. *Let M be a graded free graded multiplication R-module. Then a proper graded submodule N of M is graded weakly prime if and only if the graded ideal $(N : M)$ is a graded weakly prime.*

Proof. This is clear by using Proposition 2.2, Theorem 2.3 and the fact that $(N : M)M = N$ □

Let M be a graded multiplication module over a G-graded ring R. Let N and K be graded submodules of M with $N = IM$ and $K = JM$ for some graded ideals of R. The product of N and K is defined by $NK = IJM$. Moreover, for $a, b \in h(M)$, by ab, we mean the product of Ra and Rb (see [7]). Clearly, NK is a graded submodule of M by Lemma 2.1 and $NK \subseteq N \cap K$. Now we show that the product of graded submodules N and K is independent of the presentation of N and K. Let $N = I_1M = I_2M$ and $K = J_1M = J_2M$ where I_i, J_i are graded ideals of R for $i = 1, 2$. We have $NK = I_1J_1M = I_1(J_1M) = I_1(J_2M) = J_2(I_1M) = J_2(I_2M) = J_2(I_2M) = I_2J_2M$.

3 Graded Weakly Semiprime Submodules

Definition 3.1.

(i) A proper graded submodule N of a graded R-module M is said to be *graded semiprime* if whenever $r^k m \in N$ where $r \in h(R)$, $m \in h(M)$ and $k \in \mathbb{Z}^+$, then $rm \in N$.

(ii) A proper graded submodule N of a graded R-module M is said to be *graded weakly semiprime* if whenever $0 \neq r^k m \in N$ where $r \in h(R)$, $m \in h(M)$ and $k \in \mathbb{Z}^+$, then $rm \in N$.

Since the graded ring R is a graded R-module over itself, according to our definition, a proper graded ideal I of R is a graded weakly semiprime ideal, if whenever $0 \neq a^k b \in I$ for some $a, b \in h(R)$ and $k \in \mathbb{Z}^+$, then $ab \in I$. It is clear that every graded semiprime submodule is graded weakly semiprime, but the converse is not true in general. In fact the zero graded submodule is always graded weakly semiprime (by definition), but is not necessarily graded semiprime. Also, if N is a graded weakly prime submodule of a graded R-module M, then N is a graded weakly semiprime. Because if $0 \neq r^k m \in N$ where $r \in h(R)$, $m \in h(M)$ and $k \in \mathbb{Z}^+$, so we have $r(r^{k-1}m) \in N$ and hence $r^{k-1}m \in N$ or $r \in (N : M)$. If $r \in (N : M)$ then $rm \in N$. If $r \notin (N : M)$ then $r^{k-1}m \in N$. We write $r(r^{k-2}m) = r^{k-1}m \in N$ and since $r^{k-1}m \neq 0$ we must have $r^{k-2}m \in N$. In this way we obtain $rm \in N$.

Proposition 3.2. *A graded R-module M is a graded prime if and only if, for every $0 \neq m \in h(M)$, $Ann(M) = Ann(m)$.*

Proof. Let M be a graded prime R-module and $0 \neq m \in h(M)$. Clearly $Ann(M) \subseteq Ann(m)$. Let $r \in Ann(m)$, so $rm = 0$. So we can write $rm = (r_{g_1} + \cdots + r_{g_n})m = 0$ where $0 \neq r_{g_i} \in h(R)$, so $r_{g_i}m = 0$ for any i. Since M is graded prime and $m \neq 0$, so $r_{g_i} \in Ann(M)$ for any i. Therefore $r \in Ann(M)$, and hence $Ann(m) = Ann(M)$. Conversely, let for every $0 \neq m \in h(M)$, $Ann(M) = Ann(m)$. Let

$rm = 0$ and $m \neq 0$ where $r \in h(R)$ and $m \in h(M)$. Then $r \in Ann(m) = Ann(M)$. So the proof is complete. □

In the following results we give some conditions under which $(N :_R M)$ is a graded weakly semiprime ideal of R for a graded weakly semiprime submodule N of the graded R-module M.

Proposition 3.3. *Let M be a faithful graded cyclic R-module and N a graded weakly semiprime submodule of M. Then $(N : M)$ is a graded weakly semiprime ideal of R.*

Proof. Assume that $M = Rx$ for some $x \in h(M)$ and let $0 \neq a^k b \in (N : M)$ for some $a, b \in h(R)$ and $k \in \mathbb{Z}^+$. So $a^k bM \subseteq N$ and since M is faithful, $0 \neq a^k bM$. Hence $0 \neq a^k bx \in N$, N graded weakly semiprime gives $a(bx) \in N$. Therefore, for every $r \in R$ we have $r(abx) = (ab)(rx) \in N$, that is, $(ab)M \subseteq N$. So $(N : M)$ is a graded weakly semiprime ideal of R. □

Remark 3.4. *If M is a graded free R-module with a basis $\{e_\lambda\}_{\lambda \in \Lambda}$ ($e_\lambda \in h(M)$), then M is faithful. Because if $r \in Ann(M)$, then $rM = 0$ and so for every $\lambda \in \Lambda$, $re_\lambda = 0$, hence $r = 0$ and M is faithful. It is clear that for every graded submodule N of M, $(N : M) = \bigcap_{\lambda \in \Lambda}(N : e_\lambda)$. Also for every $\lambda \in \Lambda$ it can be shown that $Ann(e_\lambda) = Ann(M)$.*

Proposition 3.5. *Let M be a graded free R-module and N a graded weakly semiprime submodule of M. Then $(N : M)$ is a graded weakly semiprime ideal of R.*

Proof. Let $0 \neq a^k b \in (N : M)$ where $a, b \in h(R)$ and $k \in \mathbb{Z}^+$. Let $\{e_\lambda\}_{\in \Lambda}$ ($e_\lambda \in h(M)$) be a basis for M. By Remark 3.4, it is enough to prove that $ab \in \bigcap_{\lambda \in \Lambda}(N : e_\lambda)$. We have $a^k bM \subseteq N$ and since M is faithful, $0 \neq a^k bM$. Hence for every $\lambda \in \Lambda$, $0 \neq a^k be_\lambda \in N$, N graded weakly semiprime gives $abe_\lambda \in N$, that is $abe_\lambda \in N$. □

Proposition 3.6. *Let M be a P-graded prime R-module and N a graded weakly semiprime submodule of M. Then $(N : M)$ is a graded weakly semiprime ideal of R.*

Proof. Let $0 \neq a^k b \in (N : M)$ where $a, b \in h(R)$ and $k \in \mathbb{Z}^+$. We show that $ab \in (N : M)$. Let x be an arbitrary element of M. If $x \in N$, then $abx \in N$. Let $x \in M - N$. Then $x = \sum_{g \in G} x_g$, $x_g \in h(M)$ and $x_g = 0$ for almost all $g \in G$. There exists $g \in G$ such that $x_g \notin N$, then from $a^k bM \subseteq N$ we have $a^k bx_g \in N$. Let $a^k bx_g = 0$. Then $a^k b \in Ann(x_g) = Ann(M)$ by Proposition 3.2. So $a^k b \in (0 : M) = P$, hence $ab \in P \subseteq (N : M)$ since P is graded prime. If $a^k bx_g \neq 0$. Then from $a^k bx_g \in N$ we conclude $abx_g \in N$ for any $g \in G$. In any case $abx_g \in N$ for any $g \in G$. Therefore $abx = ab(\sum_{g \in G} x_g) \in N$. Hence $ab \in (N : M)$. □

Theorem 3.7. *Let M be a graded R-module and N a proper graded submodule of M. If for every graded ideal I of R, graded submodule K of M and $t \in \mathbb{Z}^+$, $0 \neq I^tK \subseteq N$ implies that $IK \subseteq N$, then N is a graded weakly semiprime submodule of M.*

Proof. Let $0 \neq r^l m \in N$ where $r \in h(R)$, $m \in h(M)$ and $l \in \mathbb{Z}^+$. We take $I = Rr$ and $K = Rm$. Now $0 \neq I^l K \subseteq N$, then by hypothesis, $IK \subseteq N$ which implies that $rm \in N$. Therefore N is a graded weakly semiprime submodule of M. □

Lemma 3.8. *Let $\{N_\lambda\}_{\lambda\in\Lambda}$ be a non-empty family of graded weakly semiprime submodules of a graded R-module M. Then $N = \bigcap_{\lambda\in\Lambda} N_\lambda$ is a graded weakly semiprime submodule of M. If $\{N_\lambda\}_{\lambda\in\Lambda}$ is a totally ordered by inclusion, then $T = \bigcup_{\lambda\in\Lambda} N_\lambda$ is a graded weakly semiprime submodule of M.*

Proof. The proof is straightforward. □

If R is a graded ring and M a graded R-module, the subset $T^g(M)$ of M is defined by $T^g(M) = \{m \in M : rm = 0 \text{ for some } 0 \neq r \in h(R)\}$. If R is a graded integral domain, then $T^g(M)$ is a graded submodule of M and is called graded torsion submodule. Also if $T(M) = 0$, then we say that M is a graded torsion free and if $T^g(M) = M$, we say that M is a graded torsion module (see [2]).

Proposition 3.9. *Let R be a graded integral domain and M a graded torsion free R-module. Then every graded weakly semiprime submodule of M is graded semiprime.*

Proof. Let N be a graded semiprime submodule of M. Suppose that $r^k m \in N$ where $r \in h(R)$, $m \in h(M)$ and $k \in \mathbb{Z}^+$. If $r^k m \neq 0$, then $rm \in N$. Let $r^k m = 0$ with $m \neq 0$. Then $r^k \in T^g(M) = 0$, and since R is graded integral domain, so $r = 0$. In any case we find $rm \in N$ and hence N is a graded weakly semiprime submodule of M. □

Proposition 3.10. *Let M be a graded R-module. Assume that N and K are graded submodules of M such that $K \subseteq N$ with $N \neq M$. Then the following hold:*

- *(i) If N is a graded weakly semiprime submodule of M, then N/K is a graded weakly semiprime submodule of the graded R-module M/K.*
- *(ii) If K and N/K are graded weakly semiprime submodules of M and M/K, respectively, then N is a graded weakly semiprime submodule of M.*

Proof. (i) Let $0 \neq r^k(m + K) = r^k m + K \in N/K$ where $r \in h(R)$, $m \in h(M)$ and $k \in \mathbb{Z}^+$. If $r^k m = 0$, then $r^k(m + K) = 0$, which is a contradiction. If $r^k m \neq 0$, N graded weakly semiprime gives $rm \in N$; hence $r(m + K) \in N/K$, as needed.

(ii) Let $0 \neq r^k m \in N$ where $r \in h(R)$, $m \in h(M)$ and $k \in \mathbb{Z}^+$, so $r^k(m + K) = r^k m + K \in N/K$. If $r^k m \in K$, then K graded weakly semiprime gives $rm \in K \subseteq N$. So we may assume that $r^k m \notin K$. Then $0 \neq r^k(m + K) \in N/K$. Since N/K is graded weakly semiprime, we get $rm \in N$, as required. □

Theorem 3.11. *Let R be a graded ring and M a graded R-module. Assume that N and K are graded weakly semiprime submodules of M such that $N+K \neq M$. Then $N+K$ is a graded weakly semiprime submodule of M.*

Proof. By Lemma 3.8, $N \bigcap K$ is a graded weakly semiprime submodule of M and $N \bigcap K \subseteq K$. Since $(N+K)/K \cong N/(N \bigcap K)$, so $(N+K)/K$ is a graded weakly semiprime submodule by Proposition 3.10 (i). Now the assertion follows from Proposition 3.10 (ii). □

Let R be a G-graded ring and $S \subseteq h(R)$ be a multiplicatively closed subset of R. Then the ring of fractions $S^{-1}R$ is a graded ring which is called the graded ring of fractions. Indeed, $S^{-1}R = \bigoplus_{g\in G}(S^{-1}R)_g$ where $(S^{-1}R)_g = \{r/s : r \in h(R), s \in S \text{ and } g = (degs)^{-1}(degr)\}$.

Let M be a graded module over a graded ring R and $S \subseteq h(R)$ be a multiplicatively closed subset of R. The module of fraction $S^{-1}M$ over a graded ring $S^{-1}R$ is a graded module which is called the module of fractions, if $S^{-1}M = \bigoplus_{g\in G}(S^{-1}M)_g$ where $(S^{-1}M)_g = \{m/s : m \in h(M), s \in S$ and $g = (degs)^{-1}(degm)\}$. We write $h(S^{-1}R) = \bigcup_{g\in G}(S^{-1}R)_g$ and $h(S^{-1}M) = \bigcup_{g\in G}(S^{-1}M)_g$. One can prove that the graded submodules of $S^{-1}M$ are of the form $S^{-1}N = \{\beta \in S^{-1}M : \beta = m/s \text{ for } m \in N \text{ and } s \in S\}$ and that $S^{-1}N \neq S^{-1}M$ if and only if $S \cap (N : M) = \phi$. Let P be any graded prime ideal of a graded ring R and consider the multiplicatively closed subset of $S = h(R) - P$. We denote the graded ring of fraction $S^{-1}R$ of R by R_P^g and we call it the graded localization of R. This graded ring is graded local with the unique graded maximal ideal $S^{-1}P$ which will be denoted by PR_P^g. Moreover, R_P^g-module $S^{-1}M$ is denoted by M_P^g (see [8]).

Proposition 3.12. *Let M be a graded R-module and $S \subseteq h(R)$ be a multiplicatively closed subset of R. Let N be a graded weakly semiprime submodule of M such that $(N : M) \bigcap S = \emptyset$. Then $S^{-1}N$ is a graded weakly semiprime submodule of $S^{-1}M$.*

Proof. Let $0/1 \neq (r/s)^k.m/t \in S^{-1}N$ where $r/s \in h(S^{-1}R)$, $m/t \in h(S^{-1}M)$ and $k \in \mathbb{Z}^+$. So $0/1 \neq r^k m/s^k t = n/t'$ for some $n \in N \bigcap h(M)$ and $t' \in S$, hence there exists $s' \in S$ such that $0 \neq s't'r^k m = s's^k tn \in N$ (because if $s't'r^k m = 0$, $r^k m/s^k t = s't'r^k m/s't's^k t = 0/1$, a contradiction). Since $s't'r^k m = r^k(s't'm) \in N$ and N is graded weakly semiprime, we have $rs't'm \in N$. Therefore $rm/st \in S^{-1}N$, as needed. □

Proposition 3.13. *Let M be a graded module over a graded local ring (R, P) with $PM = 0$. Then every proper graded submodule N of M is graded weakly semiprime.*

Proof. Let N be a proper graded submodule of M and $0 \neq r^k m \in N$ where $r \in h(R)$, $m \in h(M)$ and $k \in \mathbb{Z}^+$. If r is a unit element of R, then $m \in N$ and so $rm \in N$. If r is not a unit element of R, then $rm \in PM = 0$ and so $r^k m = 0$, a contradiction. Hence N is graded weakly semiprime. □

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