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Vector-Valued FK-Spaces Defined by a Modulus Function and an Infinite Matrix[1]

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Abstract : The present paper is devoted to studying on the sequence space $\lambda(A, X_k, r, f, s)$ defined by a modulus function f and an infinite matrix A and constructed its FK-structure under some conditions. Finally, we exposed some inclusion relations among the variations of the space. The vector-valued sequence space $\lambda(A, X_k, r, f, s)$ as a paranormed space which is a most general form of the space investigated in [1].

Keywords : vector-valued FK-spaces; paranormed spaces; sequence spaces; modulus function.
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1 Introduction

A sequence $(b_n)_{n=0}^{\infty}$ in a linear metric space X is called Schauder basis if, for every $x \in X$, there exists a unique sequence $(\lambda_n)_{n=0}^{\infty}$ of scalars such that $x = \sum_{n=0}^{\infty} \lambda_n b_n$.

By w we denote the space of all real or complex-valued sequences $x = (x_k)_{k=0}^{\infty}$. Any vector subspace of w is called a sequence space. As usual, we write c_0, c and l_∞ denote the sets of sequences that are convergent to zero, convergent and bounded, respectively. Also by l_1 and l_p; we denote the spaces of absolutely and $p-$absolutely convergent series, respectively; where $1 < p < \infty$. We write e and

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$e^{(n)}$ $(n = 0, 1, \ldots)$ for the sequences with $e_k = 1$ $(k = 0, 1, \ldots)$ and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ $(k \neq n)$. If $x \in w$ then $x^{[m]} = \sum_{k=0}^{m} x_k e^{(k)}$ denotes the $m-$section of x.

A sequence space λ with a linear topology is called a $K-$space provided each of the maps $P_i : \lambda \to \mathbb{C}$ defined by $P_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$; where $\mathbb{C}$ and $\mathbb{N}$ denote the complex field and the set of all natural numbers, respectively. Let λ be a $K-$space. Then, λ is called $FK-$space provided λ is a complete linear metric space. An $FK-$space whose topology is normable is called a $BK-$space (see Choudhary and Nanda [2], pp. 272-273).

Let (X_k, q_k) be an infinite sequence of seminormed spaces. Then we may construct the most general sequence spaces $s(X_k)$ such that $x = (x_k) \in s(X_k)$ iff $x_k \in X_k$ for each $k \in \mathbb{N}$. Taking $X_k = \mathbb{C}$ for each $k \in \mathbb{N}$, we get w, the space of all complex-valued sequences. This case is called scalar-valued case. Easily check that $s(X_k)$ is a linear space (over $\mathbb{C}$) under the natural coordinatewise operations.

Now, let us establish a semimetrizable topology on $s(X_k)$ using by seminorm topologies of the sequence (X_k, q_k). Define functions $g_k : s(X_k) \to \mathbb{R}$, $g_k(x) = q_k(x_k)$, then each g_k is a seminorm on $s(X_k)$. But there exists a topology on $s(X_k)$ such that it is larger than that of g_k for each $k \in \mathbb{N}$ [3]. This is a paranorm topology, say g , and is obtained from Frechet combination of the sequence (g_k) by

$$g(x) = \sum_{n=0}^{\infty} \frac{g_n(x)}{2^n[1 + g_n(x)]} = \sum_{n=0}^{\infty} \frac{q_n(x_k)}{2^n[1 + q_n(x_k)]}.$$

Also, $d(x, y) = g(x - y)$ is the invariant semimetric giving this topology, and for a sequence $(x^n) \subset s(X_k)$, $g(x^n) \to 0$ iff $g_k(x^n) = q_k(x_k^n) \to 0$ in X_k for each k. So, $s(X_k)$ is a product space, i.e., $s(X_k) = \prod X_k$, and g is the weakest topology such that the projections

$$P_k : s(X_k) \to X_k;\ P_k(x) = x_k, \qquad k = 1, 2, \ldots$$

are continuous. Totality of g and completeness of $s(X_k)$ with this paranorm depends on the sequence (X_k, q_k). Therefore proving the following assertion is not hard: $(s(X_k), g)$ be a Frechet space if and only if each (X_k, q_k) is a Banach space.

From above discussion, it is natural to define FK structure on $s(X_k)$ as in scalar-valued case. Remember that $FK-$spaces corresponding $H = w$ [3]. More generally, we say here an $FK-$spaces, we must assume that each X_k is a Banach space. $l_\infty(X_k)$, $c_0(X)$ is a $BK-$space with the norm $\|x\|_p = \left(\sum \|x_k\|^p\right)^{\frac{1}{p}}$, $p \geq 1$ where the norm of X_k denoted by only a symbol $\|\cdot\|$ for each k. Moreover, if $\cap X_k \neq \phi$ then we can define $c(X_k)$ by $x \in c(X_k)$ iff there exists an $l \in \cap X_k$ such that $\|x_k - l\| \to 0$. An $FK-$space E is called to have $AK-$property, or is called an $AK-$space if for each $x^{[n]} \to x$ in E where $x^{[n]} = (x_1, x_2, \ldots, x_n, 0, \ldots)$, the n^{th} section of x. In addition if E is a $BK-$space then is called an $AK-BK$ space.

The scalar-valued sequence space λ is called normal or solid if $y \in \lambda$ whenever $|y_i| \leq |x_i|$, for some $x \in \lambda$. Also λ is called a sequence algebra if it is closed under the multiplication defined by $xy = (x_i y_i)$, $i \geq 1$. Should λ is both normal and sequence algebra then it is called a nornal sequence algebra. For example, c

is a sequence algebra but not normal. w, l_∞, c_0 and l_p $(0 < p < \infty)$ are normal sequence algebras [4].

A linear topological space X over the real field $\mathbb{R}$ is said to be a paranormed space if there is a subadditive function $g : X \to \mathbb{R}$ such that $g(\theta) = 0$, $g(x) = g(-x)$, and scalar multiplication is continuous, i.e., $|\alpha_n - \alpha| \to 0$ and $g(x_n - x) \to 0$ imply $g(\alpha_n x_n - \alpha x) \to 0$ for all α's in $\mathbb{R}$ and all x's in X, where θ is the zero vector in the linear space X. Assume here and after that (r_k) be a bounded sequence of strictly positive real numbers. Then, the linear space $l(r)$ was defined by Maddox [5] (see also Simons [6] and Nakano [7]) as follows:

$$l(r) = \left\{ x = (x_k) \in w : \sum_k |x_k|^{r_k} < \infty \right\}$$

which is a complete space paranormed by

$$g(x) = \left(\sum_k |x_k|^{r_k} \right)^{\frac{1}{M}}$$

where $M = \max(1, \sup r_k)$.

A paranorm p on a normal sequence space λ is said to be absolutely monotone whenever $p(x) \le p(y)$ for $x, y \in \lambda$ with $|x_i| \le |y_i|$ for each i [4]. The norm $\|x\| = \sup |x_k|$ which makes the l_∞, c, c_0 into a BK-space is absolutely monotone, also so is the norm $\|x\| = (\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}}$ on l_p, $p \ge 1$.

Now, we shall contruct a vector-valued sequence space (subspace of $s(X_k)$) using a modulus function, an infinite matrix and normal sequence algebra λ. Notation of modulus function introduced by Nakano [8] in 1953 and used to solve some structural problems of the scalar FK-spaces theory. For example, the question; "is there an FK-space in which the sequence of coordinate vectors is bounded", exposed by A. Wilansky, was solved by W. H. Ruckle with negative answer [9]. The Problem was solved by consructing a class of scalar FK-spaces $L(f)$ where f is a modulus function. $L(f)$, in fact, is a generalization of the spaces l_p $(0 < p \le 1)$. Another extension of l_p, $(p > 0)$ spaces with respect to a positive real sequence $r = (r_k)$ was given by Simons [6]. We shall introduce and generalize vector-valued FK-spaces with this respects. For the definition of modulus function and some related results we refer the reader to [9]. In recent years, many authors have made many studies, using modulus function [1, 10–14].

Some definitions and conventions are made in this section will be given in the next sections.

2 The Sequence Spaces $\lambda(A, X_k, r, f, s)$

Let $A = (a_{mk})$ be a nonnegative matrix, λ be a scalar, normal $AK - BK$ sequence algebra with absolutely monotone norm $\|\cdot\|_\lambda$ and f be a modulus function. Also, suppose that $r = (r_k)$ be a bounded sequence of positive real numbers and $s \geq 0$. Then, let us define

$$\lambda\left(A, X_k, r, f, s\right) = \left\{x = (x_k) \in s\left(X_k\right) : \left(a_{mk} k^{-s}\left[\left(f \circ q_k\right)\left(x_k\right)\right]^{r_k}\right) \in \lambda\right\}$$

where each X_k is a seminormed space.

It is a verification to show that $\lambda\left(A, X_k, r, f, s\right)$ is a linear space over $\mathbb{C}$ under the coordinatewise operations.

Remark 2.1. *The argument s, that is, the factor k^{-s} ,was used by Bulut and Çakar [15], to generalize the Maddox sequence spaces $l\left(r\right)$ where $r = (r_k)$ be defined above. It performs an extension mission. For example, the space*

$$l\left(p, s\right) = \left\{x \in w : \sum_{k=1}^{\infty} k^{-s}\left|x_k\right|^{r_k} < \infty\right\}$$

contains $l\left(r\right)$ as a subspace for $s > 0$, and it is coincide with $l\left(r\right)$ only for $s = 0$. In a problem, if we need an FK-space containing $\lambda\left(X_k, r, f\right)$ as a subspace, then the space $\lambda\left(A, X_k, r, f, s\right)$ for $s > 0$ provides a quick example meeting the requirement (we show that below $\lambda\left(A, X_k, r, f, s\right)$ is an FK-space whenever each X_k is a Banach space).

Now let us give two lemmas to put a paranorm topology on $\lambda\left(A, X_k, r, f, s\right)$.

Lemma 2.2. *Let (X_k, q_k) be an infinite sequence of seminormed spaces, $A = (a_{mk})$ be a nonnegative matrix and λ be a normal $AK - BK$ space with absolutely monotone norm $\|\cdot\|_\lambda$. Suppose $r = (r_k)$ is a bounded sequence of positive real numbers. Then the mappings*

$$\tilde{x}_n : [0, \infty) \to [0, \infty) \;\; ; \;\; \tilde{x}_n\left(u\right) = \left\|\sum_{k=1}^{n} a_{mk} k^{-s}\left[f\left(u q_k\left(x_k\right)\right)\right]^{r_k} e_k\right\|_\lambda$$

defined by means of an $x = (x_k) \in \lambda\left(A, X_k, r, f, s\right)$, a positive integer n and for each m, are continuous, where (e_k) denotes the unit basis of λ.

Proof. Since the norm function is continuous it is sufficient to show that the mappings defined by

$$h_k : [0, \infty) \to \lambda, \quad h_k\left(u\right) = \left[a_{mk} k^{-s}\left[f\left(u q_k\left(x_k\right)\right)\right]^{r_k} e_k\right]$$

,for each m, are continuous. Let $u_i \to 0 \;\; (i \to \infty)$, then

$$h_k\left(u_i\right) \to (0, 0, \ldots) \;\; (i \to \infty)$$

for each k. Hence, each g_k is sequential continuous (it is equivalent to continuity here). □

Lemma 2.3. *Let λ be a normal sequence algebra and $\|\cdot\|_\lambda$ be an absolutely monotone seminorm on λ. Then for every $u = (u_n)$, $v = (v_n) \in \lambda$ and $p \geq 1$,*

$$\|(u+v)^p\|_\lambda^{\frac{1}{p}} \leq \|u^p\|_\lambda^{\frac{1}{p}} + \|v^p\|_\lambda^{\frac{1}{p}},$$

where $(u+v)^p = ((u_n+v_n)^p)$[16].

Theorem 2.1. *Define*

$$g(x) = \left\|a_{mk}k^{-s}\left[f(q_k(x_k))\right]^{r_k}\right\|_\lambda^{\frac{1}{M}}$$

where $M = \max(1, H)$, $H = \sup r_n$. Then g is a paranorm on $\lambda(A, X_k, r, f, s)$.

Proof. It is obvious that $g(\theta) = 0$ and $g(-x) = g(x)$. From the absolute monotonicity of $\|\cdot\|_\lambda$, properties of f and Lemma 2.3 that

$$\begin{aligned} g(x+y) &= \left\|\left(a_{mk}k^{-s}\left[f(q_k(x_k+y_k))\right]^{r_k}\right)\right\|_\lambda^{\frac{1}{M}} \\ &\leq \left\|\left(\left(a_{mk}k^{\frac{-s}{M}}\left[f(q_k(x_k))\right]^{\frac{r_k}{M}} + a_{mk}k^{\frac{-s}{M}}\left[f(q_k(y_k))\right]^{\frac{r_k}{M}}\right)^M\right)\right\|_\lambda^{\frac{1}{M}} \\ &\leq \left\|\left(a_{mk}k^{\frac{-s}{M}}\left[f(q_k(x_k))\right]^{r_k}\right)\right\|_\lambda^{\frac{1}{M}} + \left\|\left(a_{mk}k^{\frac{-s}{M}}\left[f(q_k(y_k))\right]^{r_k}\right)\right\|_\lambda^{\frac{1}{M}} \\ &= g(x) + g(y) \end{aligned}$$

for $x, y \in \lambda(A, X_k, r, f, s)$. For the continuity of scalar multiplication suppose that (μ^n) is a sequence of scalars such that $|\mu^n - \mu| \to 0$ and $g(x^n - x) \to 0$ for an arbitrary sequence $(x^n) \subset \lambda(A, X_k, r, f, s)$. We shall show that

$$g(\mu^n x^n - \mu x) \to 0 \quad (n \to \infty).$$

Say $\tau_n = |\mu^n - \mu|$ then

$$\begin{aligned} g(\mu^n x^n - \mu x) &= \left\|\left(a_{mk}k^{-s}\left[f(q_k(\mu^n x_k^n - \mu x_k))\right]^{r_k}\right)\right\|_\lambda^{\frac{1}{M}} \\ &\leq \left\|\left(\left\{a_{mk}^{\frac{1}{M}}k^{-\frac{s}{M}}\left[A(k,n)\right]^{\frac{r_k}{M}} + a_{mk}^{\frac{1}{M}}k^{-\frac{s}{M}}\left[B(k,n)\right]^{\frac{r_k}{M}}\right\}^M\right)\right\|_\lambda^{\frac{1}{M}}, \end{aligned}$$

where $A(k,n) = Rf(q(x_k^n - x_k))$, $B(k,n) = f(\tau_n q(x_k))$ and $R = 1 + \max\{1, \sup|\mu^n|\}$. Again by Lemma 2.3,

$$\begin{aligned} g(\mu^n x^n - \mu x) &\leq R^{\frac{H}{M}}\left\|\left(a_{mk}k^{-s}\left[\frac{A(k,n)}{R}\right]^{r_k}\right)\right\|_\lambda^{\frac{1}{M}} + \left\|\left(a_{mk}k^{-s}\left[B(k,n)\right]^{r_k}\right)\right\|_\lambda^{\frac{1}{M}} \\ &= R^{\frac{H}{M}}g(x^n - x) + \left\|\left(a_{mk}k^{-s}\left[B(k,n)\right]^{r_k}\right)\right\|_\lambda^{\frac{1}{M}}. \end{aligned}$$

Because of $g(x^n - x) \to 0$ we must only show that

$$\left\| \left(a_{mk} k^{-s} \left[B\left(k,n\right) \right]^{r_k} \right) \right\|_{\lambda}^{\frac{1}{M}} \to 0 \quad \left(n \to \infty\right).$$

There exist a positive integer n_0 such that $0 \leq \tau_n \leq 1$ for $n \geq n_0$. Write

$$\begin{aligned} &\left\| \left(a_{mk} k^{-s} \left[f\left(q_k\left(x_k\right)\right) \right]^{r_k} \right) - \sum_{k=1}^{t} a_{mk} k^{-s} \left[f\left(q_k\left(x_k\right)\right) \right]^{r_k} e_k \right\|_{\lambda} \\ &\qquad = \left\| \sum_{k=t+1}^{\infty} a_{mk} k^{-s} \left[f\left(q_k\left(x_k\right)\right) \right]^{r_k} e_k \right\|_{\lambda} \\ &\qquad \to 0 \ \left(t \to \infty\right) \end{aligned}$$

since λ is an $AK-$space, where (e_k) is the unit vector basis of λ. Hence, for every $\epsilon > 0$, there exist a positive integer t_0 such that

$$\left\| \sum_{k=t_0+1}^{\infty} a_{nk} k^{-s} \left[f\left(q_k\left(x_k\right)\right) \right]^{r_k} e_k \right\|_{\lambda}^{\frac{1}{M}} < \frac{\varepsilon}{2}.$$

For $n \geq n_0$, since $\tau_n q\left(x_k\right) \leq q\left(x_k\right)$, we get

$$a_{mk} k^{-s} \left[f\left(\tau_n q_k\left(x_k\right)\right) \right]^{r_k} \leq a_{mk} k^{-s} \left[f\left(q_k\left(x_k\right)\right) \right]^{r_k}$$

for each k. This implies

$$\left\| \sum_{k=t_0+1}^{\infty} a_{mk} k^{-s} \left[f\left(\tau_n q_k\left(x_k\right)\right) \right]^{r_k} e_k \right\|_{\lambda}^{\frac{1}{M}} \leq \left\| \sum_{k=t_0+1}^{\infty} a_{mk} k^{-s} \left[f\left(q_k\left(x_k\right)\right) \right]^{r_k} e_k \right\|_{\lambda}^{\frac{1}{M}} < \frac{\varepsilon}{2}.$$

Now, from Lemma 2.2, the function

$$\tilde{x}_{t_0}\left(u\right) = \left\| \sum_{k=1}^{t_0} a_{mk} k^{-s} \left[f\left(u q_k\left(x_k\right)\right) \right]^{r_k} e_k \right\|_{\lambda}$$

is continuous. Hence, there exists a δ $(0 < \delta < 1)$ such that

$$\tilde{x}_{t_0}\left(u\right) \leq \left(\frac{\varepsilon}{2}\right)^{M},$$

for $0 < u < \delta$. Also we can find a number Δ such that $\tau_n < \delta$ for $n > \Delta$. So, for $n > \Delta$, we have

$$\left(\tilde{x}_{t_0}\left(\tau_n\right)\right)^{\frac{1}{M}} = \left\| \sum_{k=1}^{t_0} a_{mk} k^{-s} \left[f\left(\tau_n q_k\left(x_k\right)\right) \right]^{r_k} e_k \right\|_{\lambda}^{\frac{1}{M}} < \frac{\varepsilon}{2},$$

so eventually,

$$\begin{aligned}\left\|\left(a_{mk}k^{-s}\left[f\left(\tau_n q_k\left(x_k\right)\right)\right]^{r_k} e_k\right)\right\|_{\lambda}^{\frac{1}{M}} &\leq \left\|\sum_{k=1}^{t_0} a_{mk}k^{-s}\left[f\left(\tau_n q_k\left(x_k\right)\right)\right]^{r_k} e_k\right\|_{\lambda}^{\frac{1}{M}} \\ &\quad + \left\|\sum_{k=t_0+1}^{\infty} a_{mk}k^{-s}\left[f\left(\tau_n q_k\left(x_k\right)\right)\right]^{r_k} e_k\right\|_{\lambda}^{\frac{1}{M}} \\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} \\ &= \varepsilon.\end{aligned}$$

This shows that $\|(a_{mk}k^{-s}\left[B\left(k,n\right)\right]^{r_k})\|_{\lambda}^{\frac{1}{M}} \to 0 \quad (n\to\infty)$. □

Lemma 2.4. *If* $a_k, b_k \in \mathbb{C}$ *and* $0 < p_k \leq \sup p_k = H$ *for each* k*, we have (see Maddox [17, p.346])*

$$|a_k+b_k|^{p_k} \leq D(|a_k|^{p_k}+|b_k|^{p_k})$$

where $D = \max\,(1, 2^{H-1})$.

Theorem 2.2. *If each* (X_k, q_k) *is complete then so is* $\lambda\left(A, X_k, r, f, s\right)$ *with the paranorm* g.

Proof. Let $\left(x^i\right)$ be a Cauchy sequence in $\lambda\left(A, X_k, r, f, s\right)$. Therefore

$$g\left(x^i-x^j\right) = \left\|\left(a_{mk}k^{-s}\left[f\left(q_k\left(x_k^i-x_k^j\right)\right)\right]^{r_k}\right)\right\|_{\lambda}^{\frac{1}{M}} \to 0 \quad (i,j\to\infty),$$

also, since λ is an $FK-$space, for each k

$$a_{mk}k^{-s}\left[f\left(q_k\left(x_k^i-x_k^j\right)\right)\right]^{r_k} \to 0 \quad (i,j\to\infty)$$

and so $q_k\left(x_k^i-x_k^j\right)\to 0 \quad (i,j\to\infty)$ from the continuity of f. Because of the completeness of each X_k, there exists an $x_k \in X_k$ such that $q_k\left(x_k^i-x\right)\to 0 \quad (i\to\infty)$ for each k. Construct the sequence $x=(x_k)$ with these points and define sequences $\left(a_{mk}k^{-s}\left[f\left(q_k\left(x_k^i-x_k\right)\right)\right]^{r_k}\right)_{i=1}^{\infty}$, $k=1,2,\ldots$. Then $a_{mk}k^{-s}\left[f\left(q_k\left(x_k^i-x_k\right)\right)\right]^{r_k}$ $\to 0 \ (i\to\infty)$ for each k. Now we can determine a sequence $\mu_k \in c_0 \left(0<\mu_i^k\leq 1\right)$ for each k, such that

$$a_{mk}k^{-s}\left[f\left(q_k\left(x_k^i-x_k\right)\right)\right]^{r_k} \leq \mu_i^k a_{mk}k^{-s}\left[f\left(q_k\left(x_k^i\right)\right)\right]^{r_k}.$$

On the other hand,

$$\left[f\left(q_k\left(x_k\right)\right)\right]^{r_k} \leq D\left\{\left[f\left(q_k\left(x_k^i-x_k\right)\right)\right]^{r_k}+\left[f\left(q_k\left(x_k^i\right)\right)\right]^{r_k}\right\},$$

where $D=\max\left(1,2^{H-1}\right)$; $H=\sup r_k$. From (1) we have

$$\begin{aligned} a_{mk}k^{-s}\left[f\left(q_k\left(x_k\right)\right)\right]^{r_k} &\leq D\left(1+\mu_i^k\right)k^{-s}\left[f\left(q_k\left(x_k^i\right)\right)\right]^{r_k} \\ &\leq 2Da_{mk}k^{-s}\left[f\left(q_k\left(x_k^i\right)\right)\right]^{r_k}. \end{aligned}$$

So we get $x\in\lambda\left(A,X_k,r,f,s\right)$ from the normality of λ. Now, for each $\varepsilon>0$ there exist $i_0\left(\varepsilon\right)$ such that

$$\left[g\left(x^i-x^j\right)\right]^M<\varepsilon^M \quad \text{for } i,j>i_0.$$

Also,

$$\begin{aligned} \left\|\sum_{k=1}^{t_0}a_{mk}k^{-s}\left[f\left(q_k\left(x_k^i-x_k^j\right)\right)\right]^{r_k}e_k\right\|_\lambda &\leq \left\|\sum_{k=1}^{\infty}a_{mk}k^{-s}\left[f\left(q_k\left(x_k^i-x_k^j\right)\right)\right]^{r_k}e_k\right\|_\lambda \\ &=\left[g\left(x^i-x^j\right)\right]^M. \end{aligned}$$

Letting $j\to\infty$ we have

$$\left\|\sum_{k=1}^{t_0}a_{mk}k^{-s}\left[f\left(q_k\left(x_k^i-x_k^j\right)\right)\right]^{r_k}e_k\right\|_\lambda \to \left\|\sum_{k=1}^{t_0}a_{mk}k^{-s}\left[f\left(q_k\left(x_k^i-x_k\right)\right)\right]^{r_k}e_k\right\|_\lambda<\varepsilon^M$$

for $i>i_0$. Since (e_k) is a Schauder basis for λ

$$\left\|\sum_{k=1}^{t_0}a_{mk}k^{-s}\left[f\left(q_k\left(x_k^i-x_k\right)\right)\right]^{r_k}e_k\right\|_\lambda \to \left\|\left(a_{mk}k^{-s}\left[f\left(q_k\left(x_k^i-x_k\right)\right)\right]^{r_k}\right)\right\|_\lambda<\varepsilon^M$$

as $j_0\to\infty$. Then we get $g\left(x^i-x\right)<\varepsilon$ for $i>i_0$ so $g\left(x^i-x\right)\to 0$ $(i\to\infty)$. □

Theorem 2.3. *$\lambda\left(A,X_k,r,f,s\right)$ is an $FK-$space iff each X_k is a Banach space. Moreover it has $AK-$property in this case since λ has.*

Proof. The condition is necessary and sufficient to the Frechet structure. Only we shall prove that, the projections

$$P_k:\lambda\left(A,X_k,r,f,s\right)\to X_k\ ;\ P_k\left(x\right)=x_k$$

are continuous under this condition. Let $(x^n)\subset\lambda\left(A,X_k,r,f,s\right)$ is a sequence such that $g\left(x^n\right)=\|(a_{mk}k^{-s}\left[f\left(q_k\left(x_k^n\right)\right)\right]^{r_k})\|\to 0$. Then

$$a_{mk}k^{-s}\left[f\left(q_k\left(x_k^n\right)\right)\right]^{r_k}\to 0\ \ (n\to\infty)$$

for each k, since λ is an $FK-$space. This implies

$$q_k\left(x_k^n\right)=q_k\left(P_k\left(x^n\right)\right)\to 0\ \ (n\to\infty)$$

for each k. This shows that each P_k is sequential continuous at 0, so is continuous. Also

$$\begin{aligned} g\left(x-x^{(n)}\right) &= g\left(0,0,\ldots,0,x_{n+1},x_{n+2},\ldots\right) \\ &= \left\| \sum_{k=n+1}^{\infty} a_{mk}k^{-s}\left[f\left(q_k\left(x_k\right)\right)\right]^{r_k} e_k \right\|_{\lambda} \\ &\to 0 \quad (n\to\infty) \end{aligned}$$

since λ is an $AK-$space. Hence $\lambda\left(A,X_k,r,f,s\right)$ is an $AK-$space. □

We obtain some sequence space in literature with some special choosing. For example, taking (X_k,q_k) as Banach spaces, $r_k=1$ for each k, $a_{mk}=1$ for all m,k and $s=0$, we get the space $F\left(E_k,f\right)$ investigated in [18]. Moreover taking $E_k=\mathbb{C}$ for each k and $a_{mk}=1$ for all m,k, in $F\left(E_k,f\right)$ we obtain the space $L\left(f\right)$ [9], and also can be reached the space $l\left(p_v\right)$ in ([6]) with the selections $a_{mk}=1$ for all m,k, $f\left(x\right)=x$, $\lambda=l_1$, $X_k=\mathbb{C}$ for each k and $s=0$.

Lemma 2.5. *Let f_1,f_2 are modulus function and $0<\delta<1$. If $f_1\left(t\right)>\delta$ for $t\in[0,\infty)$ then*

$$\left(f_2\circ f_1\right)\left(t\right)\leq\frac{2f_2\left(1\right)}{\delta}f_1\left(t\right)$$

[17].

Theorem 2.4. *Let f_1,f_2 are modulus function and $s,\ s_1,\ s_2>0$. Then*

- *i)* $\limsup\frac{f_1(t)}{f_2(t)}<\infty$ *implies* $\lambda\left(A,X_k,r,f_2,s\right)\subset\lambda\left(A,X_k,r,f_1,s\right)$,
- *ii)* $\lambda\left(A,X_k,r,f_1,s\right)\subset\lambda\left(A,X_k,r,f_2,s\right)\subseteq\lambda\left(A,X_k,r,f_1+f_2,s\right)$,
- *iii) If the matrix $A=(a_{mk})$ is a regular matrix and λ includes the sequence $\left(k^{-s}\right)$, then* $\lambda\left(A,X_k,r,f_1,s\right)\subseteq\lambda\left(A,X_k,r,f_1\circ f_2,s\right)$,
- *iv)* $s_1\leq s_2$ *implies* $\lambda\left(A,X_k,r,f_1,s_1\right)\subseteq\lambda\left(A,X_k,r,f_1,s_2\right)$.

Proof. i) Since there exist a $K>0$ such that $f_1\left(t\right)\leq f_2\left(t\right)$ by the hypothesis, we can write that

$$a_{mk}k^{-s}\left[f_1\left(q_k\left(x_k\right)\right)\right]^{r_k}\leq K^H a_{mk}k^{-s}\left[f_2\left(q_k\left(x_k\right)\right)\right]^{r_k}.$$

This proves the assertion from the normality of λ.

ii) The relation follows from the inequality

$$\begin{aligned} a_{mk}k^{-s}\left[\left(f_1+f_2\right)\left(q_k\left(x_k\right)\right)\right]^{r_k} &= a_{mk}k^{-s}\left[f_1\left(q_k\left(x_k\right)\right)+f_2\left(q_k\left(x_k\right)\right)\right]^{r_k} \\ &\leq Da_{mk}k^{-s}\left\{\left[f_1\left(q_k\left(x_k\right)\right)\right]^{r_k}+\left[f_2\left(q_k\left(x_k\right)\right)\right]^{r_k}\right\} \end{aligned}$$

where $C=\max\left(1,2^{H-1}\right)$.

iii) Let $0 < \delta < 1$, and define the sets $N_1 = \{k \in \mathbb{N} : f_1(q_k(x_k)) \leq \delta\}$ and $N_2 = \{k \in \mathbb{N} : f_1(q_k(x_k)) > \delta\}$. It follows from Lemma 2.5 that

$$(f_2 \circ f_1)(q_k(x_k)) \leq \frac{2f_2(1)}{\delta} f_1(q_k(x_k))$$

when $k \in N_2$. If $k \in N_1$ then

$$(f_2 \circ f_1)(q_k(x_k)) \leq f_2(\delta),$$

and so

$$k^{-s}[(f_2 \circ f_1)(q_k(x_k))]^{r_k} \leq \varepsilon_1 k^{-s}$$

for $x \in \lambda(A, X_k, r, f_1, s)$, where $\varepsilon_1 = \max\left\{[f_2(\delta)]^{\inf r_k}, [f_2(\delta)]^{\sup r_k}\right\}$. On the other hand

$$\begin{aligned} a_{mk}k^{-s}[(f_2 \circ f_1)(q_k(x_k))]^{r_k} &\leq a_{mk}k^{-s}\left[\frac{2f_2(1)}{\delta} f_1(q_k(x_k))\right]^{r_k} \\ &\leq \varepsilon_2 a_{mk}k^{-s}[f_1(q_k(x_k))]^{r_k} \end{aligned}$$

for $k \in N_2$. Where $\varepsilon_2 = \max\left\{\left[\frac{2f_2(1)}{\delta}\right]^{\inf r_k}, \left[\frac{2f_2(1)}{\delta}\right]^{\sup r_k}\right\}$. Now, say $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$ and we get

$$a_{mk}k^{-s}[(f_2 \circ f_1)(q_k(x_k))]^{r_k} \leq \varepsilon\left(a_{mk}k^{-s} + a_{mk}k^{-s}[f_1(q_k(x_k))]^{r_k}\right).$$

Then $(a_{mk}k^{-s} + a_{mk}k^{-s}[f_1(q_k(x_k))]^{r_k}) \in \lambda$ since λ includes the sequence (k^{-s}). Therefore

$$a_{mk}k^{-s}[(f_2 \circ f_1)(q_k(x_k))]^{r_k} \leq a_{mk}k^{-s} + a_{mk}k^{-s}[f_1(q_k(x_k))]^{r_k}$$

for $k \in N_1 \cup N_2 = \mathbb{N}$. This implies $x \in \lambda(A, X_k, r, f_1 \circ f_2, s)$ from normality.

iv) This follows from the inequality

$$a_{mk}k^{-s_2}[f_1(q_k(x_k))]^{r_k} \leq a_{mk}k^{-s_1}[f_1(q_k(x_k))]^{r_k}$$

for $s_1 \leq s_2$. □

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