Thai Journal of Mathematics
Volume 12 (2014) Number 1 : 145–154
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

On New Ostrowski Type Integral Inequalities

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Abstract : In this article, we give a new Montgomery type identity and using this identity establish a new Ostrowski type inequality and its perturbed inequality forms.

Keywords : Ostrowski's inequality; Čebyšev-Grüss inequality; Montgomery identity; Chebyshev functional.
2010 Mathematics Subject Classification : 26D07; 26D10; 26D15.

1 Introduction

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) and assume $|f'(x)| \leq M$ for all $x \in (a, b)$. Then the following inequality holds [1]:

$$|S(f; a, b)| \leq \frac{M}{b-a}\left[\left(\frac{b-a}{2}\right)^2 + \left(x - \frac{a+b}{2}\right)^2\right] \tag{1.1}$$

for all $x \in [a, b]$ where

$$S(f; a, b) = f(x) - \mathcal{M}(f; a, b)$$

and

$$\mathcal{M}(f;a,b)=\frac{1}{b-a}\int_a^b f(x)dx. \tag{1.2}$$

This inequality is well known in the literature as Ostrowski inequality.

In 1882, Čebyšev [2] gave the following inequality:

$$|T(f,g)|\leq \frac{1}{12}(b-a)^2\left\|f'\right\|_\infty \left\|g'\right\|_\infty, \tag{1.3}$$

where $f,g:[a,b]\to\mathbb{R}$ are absolutely continuous function, whose first derivatives f' and g' are bounded,

$$\begin{aligned} T(f,g) &= \frac{1}{b-a}\int_a^b f(x)g(x)dx-\left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right) \\ &= \mathcal{M}(fg;a,b)-\mathcal{M}(f;a,b)\,\mathcal{M}(g;a,b) \end{aligned} \tag{1.4}$$

and $\|.\|_\infty$ denotes the norm in $L_\infty[a,b]$ defined as $\|p\|_\infty = ess\sup_{t\in[a,b]}|p(t)|$.

In 1935, Grüss [3] proved the following inequality:

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx-\frac{1}{b-a}\int_a^b f(x)dx\frac{1}{b-a}\int_a^b g(x)dx\right|\leq\frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma), \tag{1.5}$$

provided that f and g are two integrable function on $[a,b]$ satisfying the condition

$$\varphi\leq f(x)\leq\Phi \quad\text{and}\quad \gamma\leq g(x)\leq\Gamma \text{ for all } x\in[a,b]. \tag{1.6}$$

The constant $\frac{1}{4}$ is best possible.

From [4], if $f:[a,b]\to\mathbb{R}$ is differentiable on $[a,b]$ with the first derivative f' integrable on $[a,b]$, then Montgomery identity holds:

$$f(x)=\frac{1}{b-a}\int_a^b f(t)dt+\int_a^b P(x,t)f'(t)dt, \tag{1.7}$$

where $P(x,t)$ is the Peano kernel defined by

$$P(x,t)=\begin{cases}\frac{t-a}{b-a}, & a\leq t\leq x\\ \frac{t-b}{b-a}, & x<t\leq b.\end{cases}$$

In [5], Pachpatte established new inequalities of the Cebysev type by using Pečarić's extension of the Montgomery identity [6].

Many researchers have given considerable attention to the inequalities (1.1), (1.3), (1.5) and various generalizations, extensions and variants of these inequalities have appeared in the literature, to mention a few, see ([7–11]) and the references cited therein [1–19]. The aim of this paper is to establish some new inequalities similar to the Ostrowski's inequality by using new extension of the Montgomery identity proved in the following Lemma 2.1.

2 Main Results

In order to prove our main results, we need the following identity which although of interest in itself, it will be used to obtain bounds.

Lemma 2.1. *Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous mapping. Denote by $P(x, \cdot) : [a, b] \to \mathbb{R}$ the kernel given by*

$$P(x,t) = \begin{cases} \dfrac{\alpha}{\alpha+\beta}\dfrac{(t-a)(x-t)}{x-a}, & t \in [a,x] \\ -\dfrac{\beta}{\alpha+\beta}\dfrac{(b-t)(x-t)}{b-x}, & t \in [x,b] \end{cases} \tag{2.1}$$

where $\alpha, \beta \in \mathbb{R}$ nonnegative and not both zero, then the identity

$$\begin{aligned} &\int_a^b P(x,t) f''(t)dt \\ &\quad = f(x) + \frac{\alpha f(a) + \beta f(b)}{\alpha+\beta} - \frac{2}{\alpha+\beta}\left[\frac{\alpha}{x-a}\int_a^x f(t)dt + \frac{\beta}{b-x}\int_x^b f(t)dt\right] \end{aligned} \tag{2.2}$$

holds.

Proof. From (2.1), we have

$$\begin{aligned} &\int_a^b P(x,t) f''(t)dt \\ &= \frac{\alpha}{\alpha+\beta}\int_a^x \frac{(t-a)(x-t)}{x-a} f''(t)dt - \frac{\beta}{\alpha+\beta}\int_x^b \frac{(b-t)(x-t)}{b-x} f''(t)dt \\ &= \frac{\alpha}{\alpha+\beta}\left[\frac{(t-a)(x-t)}{x-a} f'(t)\Big|_a^x - \int_a^x \frac{x+a-2t}{x-a} f'(t)dt\right] \\ &\quad - \frac{\beta}{\alpha+\beta}\left[\frac{(b-t)(x-t)}{b-x} f'(t)\Big|_x^b - \int_x^b \frac{2t-x-b}{b-x} f(t)dt\right] \\ &= -\frac{\alpha}{\alpha+\beta}\int_a^x \frac{x+a-2t}{x-a} f'(t)dt + \frac{\beta}{\alpha+\beta}\int_x^b \frac{2t-x-b}{b-x} f'(t)dt \\ &= -\frac{\alpha}{\alpha+\beta}\left[\frac{x+a-2t}{x-a} f(t)\Big|_a^x + \frac{2}{x-a}\int_a^x f(t)dt\right] \\ &\quad + \frac{\beta}{\alpha+\beta}\left[\frac{2t-x-b}{b-x} f(t)\Big|_x^b - \frac{2}{b-x}\int_x^b f(t)dt\right] \\ &= f(x) + \frac{\alpha f(a) + \beta f(b)}{\alpha+\beta} - \frac{2}{\alpha+\beta}\left[\frac{\alpha}{x-a}\int_a^x f(t)dt + \frac{\beta}{b-x}\int_x^b f(t)dt\right] \end{aligned}$$

where the integration by parts formula has been utilised on the separate intervals $[a, x]$ and $(x, b]$. This completes the proof. □

Theorem 2.2. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous mapping and define*

$$\mathcal{T}(x;\alpha,\beta):=f(x)+\frac{\alpha f(a)+\beta f(b)}{\alpha+\beta}-\frac{2}{\alpha+\beta}\left[\alpha\mathcal{M}(f;a,x)+\beta\mathcal{M}(f;x,b)\right] \quad (2.3)$$

where $\mathcal{M}(f;a,b)$ is the integral mean as defined by (1.2). Then

$$|\mathcal{T}(x;\alpha,\beta)| \leq \begin{cases} \left[\alpha\left(x-a\right)^{2}+\beta\left(b-x\right)^{2}\right]\dfrac{\left\|f''\right\|_{\infty}}{6\left(\alpha+\beta\right)}, & f''\in L_{\infty}\left[a,b\right]; \\ \left[\alpha^{q}\left(x-a\right)^{q+1}+\beta^{q}\left(b-x\right)^{q+1}\right]^{\frac{1}{q}}\dfrac{\left\|f''\right\|_{p}}{\left(\alpha+\beta\right)}\beta^{\frac{1}{q}}\left(q+1,q+1\right), & f''\in L_{p}\left[a,b\right], \\ & p>1, \\ & \frac{1}{p}+\frac{1}{q}=1; \\ \dfrac{\left\|f''\right\|_{1}}{4\left(\alpha+\beta\right)}\left[\dfrac{\alpha\left(x-a\right)+\beta\left(b-x\right)}{2}+\left|\dfrac{\alpha\left(x-a\right)-\beta\left(b-x\right)}{2}\right|\right], & f''\in L_{1}\left[a,b\right]; \end{cases} \quad (2.4)$$

where $\|h\|_{.}$ are the usual Lebesque norms for $h\in L_{.}[a,b]$ with

$$\|h\|_{\infty}:=ess\sup_{t\in[a,b]}|h(t)|<\infty$$

and

$$\|h\|_{p}:=\left(\int_{a}^{b}|h(t)|^{p}\,dt\right)^{\frac{1}{p}}, \qquad 1\leq p\leq\infty.$$

Proof. Taking the modulus of (2.2) we have from (2.3) and (1.2)

$$|\mathcal{T}(x;\alpha,\beta)|=\left|\int_{a}^{b}P\left(x,t\right)f''(t)dt\right|\leq\int_{a}^{b}\left|P\left(x,t\right)\right|\left|f''(t)\right|dt, \quad (2.5)$$

where we have used the well known properties of the integral and modulus.

Thus, for $f''\in L_{\infty}[a,b]$ from (2.5) gives

$$|\mathcal{T}(x;\alpha,\beta)|\leq\|f''\|_{\infty}\int_{a}^{b}\left|P\left(x,t\right)\right|dt$$

from which a simple calculation using (2.1) gives

$$\begin{aligned}\int_{a}^{b}\left|P\left(x,t\right)\right|dt &= \frac{\alpha}{\alpha+\beta}\int_{a}^{x}\frac{\left(t-a\right)\left(x-t\right)}{x-a}dt+\frac{\beta}{\alpha+\beta}\int_{x}^{b}\frac{\left(b-t\right)\left(t-x\right)}{b-x}dt \\ &= \frac{\alpha}{\alpha+\beta}\left(\frac{\left(x-t\right)^{2}}{2}-\frac{\left(x-t\right)^{3}}{3\left(x-a\right)}\right)\Bigg|_{a}^{x}+\frac{\beta}{\alpha+\beta}\left(\frac{\left(t-x\right)^{2}}{2}-\frac{\left(t-x\right)^{3}}{3\left(b-x\right)}\right)\Bigg|_{x}^{b} \\ &= \left[\alpha\left(x-a\right)^{2}+\beta\left(b-x\right)^{2}\right]\frac{1}{6\left(\alpha+\beta\right)}\end{aligned}$$

and hence the first inequality results.

Further, using Hölder's integral inequality, we have for $f'' \in L_p[a,b]$ from (2.5)

$$|\mathcal{T}(x;\alpha,\beta)| \leq \|f''\|_p \left(\int_a^b |P(x,t)|^q \, dt \right)^{\frac{1}{q}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$. Now

$$\begin{aligned}
(\alpha+\beta) & \left(\int_a^b |P(x,t)|^q \, dt \right)^{\frac{1}{q}} \\
&= \left[\alpha^q \int_a^x \left(\frac{(t-a)(x-t)}{x-a} \right)^q dt + \beta^q \int_x^b \left(\frac{(b-t)(t-x)}{b-x} \right)^q dt \right]^{\frac{1}{q}} \\
&= \left[\alpha^q (x-a)^{q+1} + \beta^q (b-x)^{q+1} \right]^{\frac{1}{q}} \left(\int_0^1 u^q (1-u)^q \, du \right)^{\frac{1}{q}}
\end{aligned}$$

and so the second inequality is obtained.

Finally, for $f'' \in L_1[a,b]$ we have from (2.5) and using (2.1)

$$|\mathcal{T}(x;\alpha,\beta)| \leq \sup_{t\in[a,b]} |P(x,t)| \, \|f''\|_1,$$

where

$$\begin{aligned}
(\alpha+\beta) \sup_{t\in[a,b]} |P(x,t)| &= \frac{1}{4} \sup\{\alpha(x-a), \beta(b-x)\} \\
&= \frac{1}{4} \left[\frac{\alpha(x-a)+\beta(b-x)}{2} + \left| \frac{\alpha(x-a)-\beta(b-x)}{2} \right| \right]
\end{aligned}$$

and so the theorem is now completely proven. □

Corollary 2.3. *Under the assumptions of Theorem 2.2 , we have*

$$\left| f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2} - \frac{2}{b-a} \int_a^b f(x)dx \right|$$
$$\leq \begin{cases} (b-a)^2 \dfrac{\|f''\|_\infty}{24}, & f'' \in L_\infty[a,b]; \\[2ex] (b-a)^{1+\frac{1}{q}} \dfrac{\|f''\|_p}{4} \beta^{\frac{1}{q}}(q+1,q+1), & f'' \in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex] (b-a) \dfrac{\|f''\|_1}{16}, & f'' \in L_1[a,b]. \end{cases}$$

Proof. If we take $x = \frac{a+b}{2}$ and $\alpha = \beta$ in (2.4), the above result could obtain. □

Corollary 2.4. *Under the assumptions of Theorem 2.2 , we have*

$$\left|f(x)+\frac{f(a)+f(b)}{2}-\left[\mathcal{M}(f;a,x)+\mathcal{M}(f;x,b)\right]\right|$$
$$\leq\begin{cases}\left[(x-a)^2+(b-x)^2\right]\dfrac{\|f''\|_\infty}{12}, & f''\in L_\infty[a,b];\\[2ex] \left[(x-a)^{q+1}+(b-x)^{q+1}\right]^{\frac{1}{q}}\dfrac{\|f''\|_p}{2}\beta^{\frac{1}{q}}(q+1,q+1), & f''\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \dfrac{\|f''\|_1}{4}\left[\dfrac{b-a}{4}+\dfrac{1}{2}\left|x-\dfrac{a+b}{2}\right|\right], & f''\in L_1[a,b].\end{cases}$$

Proof. The result is readily obtained on allowing $\alpha=\beta$ in (2.4) so that the left hand side is $\mathcal{T}(x;\alpha,\alpha)$ from (2.3). □

Corollary 2.5. *Under the assumptions of Theorem 2.2 , we have*

$$\left|f(\tfrac{a+b}{2})+\frac{\alpha f(a)+\beta f(b)}{\alpha+\beta}-\frac{2}{\alpha+\beta}\left[\alpha\mathcal{M}(f;a,\tfrac{a+b}{2})+\beta\mathcal{M}(f;\tfrac{a+b}{2},b)\right]\right|$$
$$\leq\begin{cases}(b-a)^2\dfrac{\|f''\|_\infty}{24}, & f''\in L_\infty[a,b];\\[2ex] \left(\dfrac{b-a}{2}\right)^{1+\frac{1}{q}}\left[\alpha^q+\beta^q\right]^{\frac{1}{q}}\dfrac{\|f''\|_p}{(\alpha+\beta)}\beta^{\frac{1}{q}}(q+1,q+1), & f''\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \dfrac{(b-a)\|f''\|_1}{16}\left[1+\left|\dfrac{\alpha-\beta}{\alpha+\beta}\right|\right], & f''\in L_1[a,b];\end{cases} \tag{2.6}$$

Proof. Placing $x=\frac{a+b}{2}$ (2.4) and (2.3) produces the results as stated in (2.6). □

Perturbed version of the results of Theorem 2.2 may be obtained by using Grüss type results involving the Chebychev functional

$$T(f,g)=\mathcal{M}(fg;a,b)-\mathcal{M}(f;a,b)\,\mathcal{M}(g;a,b)$$

with $\mathcal{M}(f;a,b)$ being the integral mean of f over $[a,b]$, namely

$$\mathcal{M}(f;a,b)=\frac{1}{b-a}\int_a^b f(t)dt.$$

Theorem 2.6. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous mapping and $\alpha\geq 0$, $\beta\geq 0$, $\alpha+\beta\neq 0$ then*

$$\begin{aligned}\left|\mathcal{T}(x;\alpha,\beta)-\left(\frac{\alpha(x-a)^2+\beta(b-x)^2}{(\alpha+\beta)}\right)\frac{\mathcal{K}}{6}\right| & \\ \leq (b-a)\mathcal{N}(x)\left[\frac{1}{b-a}\left\|f''\right\|_2^2-\mathcal{K}^2\right]^{\frac{1}{2}} & \\ \leq \frac{(b-a)\left(\Gamma-\gamma\right)}{8(\alpha+\beta)}\sup\left\{\alpha\left(x-a\right),\beta\left(b-x\right)\right\} &\end{aligned}\tag{2.7}$$

where, $\mathcal{T}(x;\alpha,\beta)$ is as given by (2.3),

$$\mathcal{K}=\frac{f'(b)-f'(a)}{b-a},\tag{2.8}$$

$$\mathcal{N}^2(x)=\left(\frac{\alpha}{\alpha+\beta}\right)^2\frac{(x-a)^2}{30}+\left(\frac{\beta}{\alpha+\beta}\right)^2\frac{(b-x)^2}{30}-\left(\frac{\alpha(x-a)^2+\beta(b-x)^2}{6(\alpha+\beta)(b-a)}\right)^2.\tag{2.9}$$

Proof. Associating $f(t)$ with $P(x,t)$ and $g(t)$ with $f''(t)$ then from (2.1) and (1.4) we obtain

$$T(P(x,\cdot),f^{''}(\cdot))=\mathcal{M}\left(P(x,\cdot)f''(\cdot);a,b\right)-\mathcal{M}\left(P(x,\cdot);a,b\right)\mathcal{M}\left(f''(\cdot);a,b\right)$$

and so, on using identity (2.2),

$$(b-a)T(P(x,\cdot),f^{''}(\cdot))=\mathcal{T}(x;\alpha,\beta)-(b-a)\mathcal{M}\left(P(x,\cdot);a,b\right)\mathcal{K}\tag{2.10}$$

where $\mathcal{K}$ is the secant slope of f' over $[a,b]$ as given in (2.8). Now, from (2.2),

$$\begin{aligned}(b-a)\mathcal{M}\left(P(x,\cdot);a,b\right)&=\int_a^b P(x,t)dt\\ &=\frac{\alpha}{\alpha+\beta}\int_a^x\frac{(t-a)\,(x-t)}{x-a}dt-\frac{\beta}{\alpha+\beta}\int_x^b\frac{(b-t)\,(x-t)}{b-x}dt\\ &=\frac{\alpha(x-a)^2+\beta(b-x)^2}{(\alpha+\beta)}\int_0^1 u(1-u)du\end{aligned}\tag{2.11}$$

and combining with (2.9) gives the left hand side of (2.7).

For $f,g:[a,b]\to\mathbb{R}$ and integrable on $[a,b]$, as is their product, then we know the following inequality

$$\begin{aligned}|T(f,g)|&\leq T^{\frac{1}{2}}(f,f)T^{\frac{1}{2}}(g,g)\quad (f,g\in L_2\left[a,b\right])\\ &\leq\frac{\Gamma-\gamma}{2}T^{\frac{1}{2}}(f,f)\qquad (\gamma\leq g(t)\leq\Gamma,\ t\in[a,b])\\ &\leq\frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma)\quad (\varphi\leq f\leq\Phi,\ t\in[a,b])\end{aligned}\tag{2.12}$$

exits [4].

Now, for the bounds on (2.10) from (2.12) we have to determine $T^{\frac{1}{2}}(P(x,\cdot),P(x,\cdot))$ and $\varphi \leq P(x,\cdot) \leq \Phi$.

Firstly, we note however that

$$\begin{aligned}0 &\leq T^{\frac{1}{2}}\left(f''(.),f''(.)\right) \\ &= \left[\mathcal{M}\left(f''(.)^2;a,b\right) - \mathcal{M}^2\left(f''(.);a,b\right)\right]^{\frac{1}{2}} \\ &= \left[\frac{1}{b-a}\int_a^b \left[f''(t)\right]^2 dt - \left(\frac{\int_a^b f''(t)dt}{b-a}\right)^2\right]^{\frac{1}{2}} \\ &= \left[\frac{1}{b-a}\left\|f''\right\|_2^2 - \mathcal{K}^2\right]^{\frac{1}{2}} \\ &\leq \frac{\Gamma-\gamma}{2}\end{aligned} \tag{2.13}$$

where $\gamma \leq f''(t) \leq \Gamma,\ t \in [a,b]$.

Now from (2.1), the definition of $P(x,t)$, we have

$$T(P(x,\cdot),P(x,\cdot)) = \mathcal{M}\left(P^2(x,\cdot);a,b\right) - \mathcal{M}^2\left(P(x,\cdot);a,b\right) \tag{2.14}$$

where from (2.11),

$$\mathcal{M}\left(P(x,\cdot);a,b\right) = \frac{\alpha(x-a)^2+\beta(b-x)^2}{6(\alpha+\beta)(b-a)}$$

and

$$\begin{aligned}&(b-a)\mathcal{M}\left(P^2(x,\cdot);a,b\right) \\ &\quad= \left(\frac{\alpha}{\alpha+\beta}\right)^2\int_a^x\left(\frac{(t-a)(x-t)}{x-a}\right)^2 dt + \left(\frac{\beta}{\alpha+\beta}\right)^2\int_x^b\left(\frac{(b-t)(t-x)}{b-x}\right)^2 dt \\ &\quad= \left(\frac{\alpha}{\alpha+\beta}\right)^2\frac{(x-a)^2}{30} + \left(\frac{\beta}{\alpha+\beta}\right)^2\frac{(b-x)^2}{30}.\end{aligned}$$

Thus, substituting the above results into (2.14) gives

$$0 \leq \mathcal{N}(x) = T^{\frac{1}{2}}(P(x,\cdot),P(x,\cdot))$$

which is given explicitly by (2.9).

Combining (2.10), (2.14) and (2.13) give from the first inequality in (2.12), the first inequality in (2.7). Also, utilising the inequality in (2.13) produces the second result in (2.7).

Further, it may be noticed from the definition of $P(x,t)$ in (2.1) that for $\alpha,\beta \geq 0$ and α and β not zero at the same time give

$$\Phi = \sup_{t\in[a,b]} P(x,t) \quad \text{and} \quad \varphi = \inf_{t\in[a,b]} P(x,t),$$

giving $\Phi = \dfrac{1}{4(\alpha+\beta)} \sup\{\alpha(x-a), \beta(b-x)\}$ and $\varphi = 0$.

Hence, from (2.10) and the last inequality in (2.12) gives the final result in (2.7) and the theorem is now completely proved. □

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(Received 7 March 2012)
(Accepted 30 October 2012)