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Embedding of Special Semigroup Amalgams[1]

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Abstract : After showing that any special semigroup amalgam in the class of all left [right] regular bands is strongly embeddable in the class of all regular bands, we show that the class of all semigroups satisfying the identity $axy = axay[yxa = yaxa]$ has the special amalgamation property.

Keywords : epimorphism; special semigroup amalgam; left [right] regular band; regular band; zigzag equations.
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1 Introduction

In [1], Scheiblich has shown that the class of normal bands is closed. In [2], the authors have generalized this result and have shown that the class of all left [right] regular bands is closed. In this paper, we further extend this result and show, by using zigzag manipulations, that the class of all left [right] regular bands is closed within the class of all regular bands. However, it is not known whether the class of all regular bands is closed.

In [3, Theorem 2.2], the authors have shown that the class of all left [right] quasinormal bands has the special amalgamation property. Now, we generalize this result, by showing that the class of all semigroups satisfying the identity

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$axy = axay[yxa = yaxa]$ has special amalgamation property. Notice that this class of semigroups contains the class of all left [right] quasinormal bands.

2 Preliminaries

Let U, S be semigroups with $U \subseteq S$. Following Isbell [4], we say that U *dominates* an element d of S if for every semigroup T and for all homomorphisms $\beta, \gamma : S \to T$, $u\beta = u\gamma$ for all $u \in U$ implies $d\beta = d\gamma$. The set of all elements of S dominated by U is called the *dominion* of U in S, and we denote it by $Dom(U, S)$. It may be easily seen that $Dom(U, S)$ is a subsemigroup of S containing U. A semigroup U is said to be $\mathcal{C}$-*closed* if for all $S \in \mathcal{C}$ such that U is a subsemigroup of S, $Dom(U, S) = U$. Let $\mathcal{B}$ and $\mathcal{C}$ be classes of semigroups such that $\mathcal{B} \subseteq \mathcal{C}$. Then $\mathcal{B}$ is said to be $\mathcal{C}$-*closed* if every member of $\mathcal{B}$ is $\mathcal{C}$-*closed*. A class $\mathcal{C}$ of semigroups is said to be *closed* if for all $U, S \in \mathcal{C}$ with U a subsemigroup of S, $Dom(U, S) = U$.

A morphism $\alpha : A \to B$ in the category $\mathcal{C}$ of all semigroups is called an *epimorphism* (epi for short) if for all $C \in \mathcal{C}$ and for all morphisms $\beta, \gamma : B \to C$, $\alpha\beta = \alpha\gamma$ implies $\beta = \gamma$. It may easily be seen that a morphism $\alpha : S \to T$ is epi if and only if the inclusion mapping $i : S\alpha \to T$ is epi, and an inclusion map $i : U \to S$ is epi if and only if $Dom(U, S) = S$. For more details, one may refer to [5–7].

A most useful characterization of semigroup dominions is provided by Isbell's Zigzag Theorem.

Result 2.1 ([4, Theorem 2.3] or [8, Theorem VII.2.13]). *Let U be a subsemigroup of a semigroup S and let $d \in S$. Then $d \in Dom(U, S)$ if and only if $d \in U$ or there exists a series of factorizations of d as follows:*

$$d = a_0 y_1 = x_1 a_1 y_1 = x_1 a_2 y_2 = x_2 a_3 y_2 = \cdots = x_m a_{2m-1} y_m = x_m a_{2m}, \qquad (2.1)$$

where $m \geq 1$, $a_i \in U$ $(i = 0, 1, \ldots, 2m)$, $x_i, y_i \in S$ $(i = 1, 2, \ldots, m)$; and

$$a_0 = x_1 a_1, \qquad a_{2m-1} y_m = a_{2m},$$
$$a_{2i-1} y_i = a_{2i} y_{i+1}, \qquad x_i a_{2i} = x_{i+1} a_{2i+1} \qquad (1 \leq i \leq m-1).$$

Such a series of factorization is called a zigzag in S over U with value d, length m and spine $a_0, a_1, \ldots, a_{2m}$.

We refer to the equations in Result 2.1, in whatever follows, as *the zigzag equations*.

A *(semigroup)amalgam* $\mathcal{A} = [\{S_i : i \in I\};\ U;\ \{\phi_i : i \in I\}]$ consists of a semigroup U (called the *core* of the amalgam), a family $\{S_i : i \in I\}$ of semigroups disjoint from each other and from U, and a family $\phi_i : U \to S_i (i \in I)$ of monomorphisms. We shall simplify the notation to $\mathcal{U} = [S_i;\ U;\ \phi_i]$ or to $\mathcal{U} = [S_i;\ U]$ when the context allows.

We shall say that the amalgam $\mathcal{A}$ is *embedded* in a semigroup T if there exist a monomorphism $\lambda : U \to T$ and, for each $i \in I$, a monomorphism $\lambda_i : S_i \to T$ such that

(a) $\phi_i\lambda_i = \lambda$ for each $i \in I$;

(b) $S_i\lambda_i \cap S_j\lambda_j = U\lambda$ for all $i, j \in I$ such that $i \neq j$.

A semigroup amalgam $\mathcal{U} = [\{S, S'\};\ U;\ \{i, \alpha \mid U\}]$ consisting of a semigroup S, a subsemigroup U of S, an isomorphic copy S' of S, where $\alpha : S \to S'$ be an isomorphism and i is the inclusion mapping of U into S, is called a *special semigroup amalgam*. A class $\mathcal{C}$ of semigroups is said to have the *special amalgamation property* if every special semigroup amalgam in $\mathcal{C}$ is embeddable in $\mathcal{C}$.

Result 2.2 ([8, Theorem VII.2.3]). *Let U be a subsemigroup of a semigroup S. Let S' be a semigroup disjoint from S and let $\alpha : S \to S'$ be an isomorphism. Let $P = S *_U S'$, be the free product of the amalgam*

$$\mathcal{U} = [\{S, S'\};\ U;\ \{i, \alpha \mid U\}],$$

where i is the inclusion mapping of U into S, and let μ, μ' be the natural monomorphisms from S, S' respectively into P. Then

$$(S\mu \cap S'\mu')\mu^{-1} = Dom(U, S).$$

From the above result, it follows that a special semigroup amalgam $[\{S, S'\};\ U;$ $\{i, \alpha \mid U\}]$ is embeddable in a semigroup if and only if $Dom(U, S) = U$. Therefore, the above amalgam with core U is embeddable in a semigroup if and only if U is closed in S.

Recall that a band B (a semigroup in which every element is an idempotent) is called *left* [*right*] *regular* if it satisfies the identity $axa = ax[axa = xa]$, *left*[*right*] *quasi-normal* if it satisfies the identity $axy = axay[yxa = yaxa]$ and *regular* if it satisfies the identity $axya = axaya$ respectively (see [9]).

We shall be using standard notations and refer the reader to Clifford and Preston [10] and Howie [8] for any unexplained symbols and terminology. Further, in whatever follows, bracketed statements or notions are dual to the other statements or notions.

3 Main Results

Lemma 3.1. *Let U be a left regular band and S be any regular band such that U be a subband of S. If for $d \in Dom(U, S) \setminus U$ and (2.1) be a zigzag in S over U of minimal length m, then*

$$\left(\prod_{i=0}^{m-1} a_{2i}\right) y_m = \left(\prod_{i=0}^{m-1} a_{2i}\right) a_{2m-1}(a_{2m-4}a_{2m-6}\cdots a_2a_0)\left(\prod_{i=0}^{m-1} a_{2i}\right) y_m.$$

Proof. Now

$$
\begin{aligned}
&\left(\prod_{i=0}^{m-1} a_{2i}\right) y_m \\
&= a_0a_2a_4\cdots a_{2m-4}a_{2m-2}y_m \\
&= x_1a_1a_2a_4\cdots a_{2m-4}a_{2m-2}y_m && \text{(by zigzag equations)} \\
&= (x_1a_1a_2)^2a_4\cdots a_{2m-4}a_{2m-2}y_m && \text{(as } S \text{ is a band)} \\
&= x_1a_1(a_2x_1x_1a_1a_2)a_4\cdots a_{2m-4}a_{2m-2}y_m \\
&= (x_1a_1)(a_2x_1a_2x_1a_1a_2)a_4\cdots a_{2m-4}a_{2m-2}y_m && \text{(as } S \text{ is a regular band)} \\
&= (x_1a_1a_2)(x_1a_2x_1a_1a_2a_4)\cdots a_{2m-4}a_{2m-2}y_m \\
&= (x_1a_1a_2)(x_2a_3x_1a_1a_2a_4)\cdots a_{2m-4}a_{2m-2}y_m && \text{(by zigzag equations)} \\
&= (x_1a_1a_2)(x_2a_3x_1a_1a_2a_4)(x_2a_3x_1a_1a_2a_4)\cdots a_{2m-4}a_{2m-2}y_m && \text{(as } S \text{ is a band)} \\
&= (x_1a_1a_2)(x_2a_3x_1a_1a_2a_4^2)(x_2a_3x_1a_1a_2a_4)\cdots a_{2m-4}a_{2m-2}y_m \\
&= (x_1a_1a_2)(x_2a_3x_1a_1a_2a_4a_4x_2)(a_3x_1a_1a_2a_4)\cdots a_{2m-4}a_{2m-2}y_m \\
&= (x_1a_1a_2)(x_2a_3x_1a_1a_2a_4x_2a_4x_2)(a_3x_1a_1a_2a_4) \\
&\qquad \cdots a_{2m-4}a_{2m-2}y_m && \text{(as } S \text{ is a regular band)} \\
&= (x_1a_1a_2)(x_2a_3x_1a_1a_2a_4)(x_3a_5x_2a_3x_1a_1a_2a_4a_6) \\
&\qquad \cdots a_{2m-4}a_{2m-2}y_m && \text{(by zigzag equations)} \\
&= (x_1a_1a_2)(x_2a_3x_1a_1a_2a_4)(x_3a_5x_2a_3x_1a_1a_2a_4a_6)^2 \\
&\qquad \cdots a_{2m-4}a_{2m-2}y_m && \text{(as } S \text{ is a band)} \\
&\;\vdots \\
&= (x_1a_1a_2)(x_2a_3x_1a_1a_2a_4)\cdots(x_{m-1}a_{2m-3}x_{m-2}a_{2m-5}\cdots x_2a_3x_1a_1a_2a_4 \\
&\qquad \cdots a_{2m-2})y_m \\
&= (x_1a_1a_2)(x_2a_3x_1a_1a_2a_4) \\
&\qquad \cdots(x_{m-1}a_{2m-3}\cdots x_2a_3x_1a_1a_2a_4\cdots a_{2m-2})^2y_m && \text{(as } S \text{ is a band)} \\
&= (x_1a_1a_2)\cdots\left(x_{m-1}a_{2m-3}\cdots x_2a_3\left(\prod_{i=0}^{m-1} a_{2i}\right)\right) \\
&\qquad \left(x_{m-1}a_{2m-3}\cdots x_2a_3\left(\prod_{i=0}^{m-1} a_{2i}\right)\right)y_m \\
&= (x_1a_1a_2)\cdots x_{m-1}\left(a_{2m-3}\cdots x_2a_3\left(\prod_{i=0}^{m-2} a_{2i}\right)\right)a_{2m-2}x_{m-1} \\
&\qquad \left(a_{2m-3}\cdots x_2a_3\left(\prod_{i=0}^{m-1} a_{2i}\right)\right)y_m \\
&= (x_1a_1a_2)\cdots(x_{m-1}z_1a_{2m-2}x_{m-1})z_2y_m
\end{aligned}
$$

$$\text{(where } z_1 = a_{2m-3}\cdots x_2a_3(\textstyle\prod_{i=0}^{m-2} a_{2i}) \text{ and } z_2 = a_{2m-3}\cdots x_2a_3(\textstyle\prod_{i=0}^{m-1} a_{2i}))$$

$$= (x_1a_1a_2)\cdots(x_{m-1}z_1x_{m-1}a_{2m-2}x_{m-1})z_2y_m \quad \text{(as } S \text{ is a regular band)}$$

$$= (x_1a_1a_2)\cdots(x_{m-1}z_1x_ma_{2m-1}x_{m-1})z_2y_m \quad \text{(by zigzag equations)}$$

$$= (x_1a_1a_2)\cdots(x_{m-1}z_1x_m(a_{2m-1}^2)x_{m-1})z_2y_m \quad \text{(as } U \text{ is a band)}$$

$$= (x_1a_1a_2)\cdots(x_{m-1}z_1x_{m-1}a_{2m-2}a_{2m-1}x_{m-1})z_2y_m \quad \text{(by zigzag equations)}$$

$$= (x_1a_1a_2)\cdots(x_{m-1}z_1a_{2m-2}a_{2m-1}x_{m-1})z_2y_m \quad \text{(as } S \text{ is a regular band)}$$

$$= (x_1a_1a_2)\cdots(x_{m-1}z_1a_{2m-2}a_{2m-1}x_{m-1})\left(a_{2m-3}\cdots x_2a_3\left(\prod_{i=0}^{m-1} a_{2i}\right)\right)y_m$$

$$\text{(since } z_2 = a_{2m-3}\cdots x_2a_3(\textstyle\prod_{i=0}^{m-1} a_{2i}))$$

$$= (x_1a_1a_2)\cdots x_{m-1}z_1a_{2m-2}a_{2m-1}\left(x_{m-2}a_{2m-4}x_{m-2}a_{2m-5}\cdots x_2a_3\left(\prod_{i=0}^{m-1} a_{2i}\right)\right)y_m$$

$$\text{(by zigzag equations as } x_{m-1}a_{2m-3} = x_{m-2}a_{2m-4})$$

$$= (x_1a_1a_2)\cdots x_{m-1}a_{2m-3}(x_{m-2}z_3a_{2m-1}x_{m-2}a_{2m-4}x_{m-2})z_3y_m$$

$$\text{(as } z_1 = a_{2m-3}\cdots x_2a_3(\textstyle\prod_{i=0}^{m-2} a_{2i}) \text{ and where)}$$

$$(z_3 = a_{2m-5}x_{m-3}a_{2m-7}\cdots x_2a_3(\textstyle\prod_{i=0}^{m-1} a_{2i}))$$

$$= (x_1a_1a_2)\cdots x_{m-1}a_{2m-3}(x_{m-2}z_3a_{2m-1}a_{2m-4}x_{m-2})z_3y_m$$

$$\text{(as } x_{m-2}, z_3a_{2m-1}, a_{2m-4} \in S \text{ and } S \text{ is a regular band)}$$

$$= (x_1a_1a_2)\cdots x_{m-1}a_{2m-3}(x_{m-2}a_{2m-5}x_{m-3}z_4a_{2m-1}a_{2m-4}x_{m-2}a_{2m-5}x_{m-3})z_4y_m$$

$$\text{(where } z_4 = a_{2m-7}\cdots x_2a_3(\textstyle\prod_{i=0}^{m-1} a_{2i}) \text{ and as)}$$

$$(z_3 = a_{2m-5}x_{m-3}a_{2m-7}\cdots x_2a_3(\textstyle\prod_{i=0}^{m-1} a_{2i}))$$

$$= (x_1a_1a_2)\cdots x_{m-1}a_{2m-3}x_{m-3}a_{2m-6}(x_{m-3}z_4a_{2m-1}a_{2m-4}x_{m-3}a_{2m-6}x_{m-3})z_4y_m$$

$$\text{(by zigzag equations)}$$

$$= (x_1a_1a_2)\cdots x_{m-1}a_{2m-3}x_{m-3}a_{2m-6}(x_{m-3}z_4a_{2m-1}a_{2m-4}a_{2m-6}x_{m-3})z_4y_m$$

$$\text{(since } x_{m-3}, z_4a_{2m-1}a_{2m-4}, a_{2m-6} \in S \text{ and as } S \text{ is a regular band)}$$

$$\vdots$$

$$= (x_1a_1a_2)\cdots\left(x_{m-1}a_{2m-3}\cdots x_2a_3\left(\prod_{i=0}^{m-1} a_{2i}\right)\right)a_{2m-1}$$

$$\left(a_{2m-4}a_{2m-6}\cdots a_2a_0\left(\prod_{i=0}^{m-1} a_{2i}\right)\right)y_m$$

$$= (x_1a_1a_2)\cdots z_6a_{2m-1}z_5y_m \quad \text{(where } z_5 = a_{2m-4}a_{2m-6}\cdots a_2a_0(\textstyle\prod_{i=0}^{m-1} a_{2i}))$$

$$\text{(and } z_6 = x_{m-1}a_{2m-3}\cdots x_2a_3(\textstyle\prod_{i=0}^{m-1} a_{2i}))$$

$$= (x_1a_1a_2)(x_2a_3x_1a_1a_2a_4)\cdots z_6a_{2m-1}z_5y_m$$

$$= (x_1a_1a_2x_1a_2x_1)a_1a_2a_4\cdots z_6a_{2m-1}z_5y_m \quad \text{(by zigzag equations)}$$

$$= (x_1a_1a_2a_2x_1)a_1a_2a_4\cdots z_6a_{2m-1}z_5y_m \qquad \text{(as } S \text{ is a regular band)}$$
$$= (x_1a_1a_2)(x_1a_1a_2)a_4\cdots z_6a_{2m-1}z_5y_m \qquad \text{(as } a_2 \in S\text{)}$$
$$= (x_1a_1a_2)a_4\cdots z_6a_{2m-1}z_5y_m \qquad \text{(as } S \text{ is a band)}$$
$$= a_0a_2a_4\cdots z_6a_{2m-1}z_5y_m \qquad \text{(by zigzag equations)}$$
$$= a_0a_2a_4\cdots \left(x_{m-1}a_{2m-3}\cdots x_2a_3\left(\prod_{i=0}^{m-1} a_{2i}\right)\right)a_{2m-1}z_5y_m$$
$$\text{(as } z_6 = x_{m-1}a_{2m-3}\cdots x_2a_3(\textstyle\prod_{i=0}^{m-1} a_{2i})\text{)}$$
$$\vdots$$
$$= (a_0a_2a_4\cdots a_{2m-4}a_{2m-2})a_{2m-1}z_5y_m$$
$$= (a_0a_2a_4\cdots a_{2m-6}a_{2m-4}a_{2m-2})a_{2m-1}(a_{2m-4}a_{2m-6}\cdots a_2a_0(\prod_{i=0}^{m-1} a_{2i}))y_m$$
$$\text{(as } z_5 = a_{2m-4}a_{2m-6}\cdots a_2a_0(\textstyle\prod_{i=0}^{m-1} a_{2i})\text{)}$$
$$= \left(\prod_{i=0}^{m-1} a_{2i}\right)a_{2m-1}(a_{2m-4}a_{2m-6}\cdots a_2a_0)\left(\prod_{i=0}^{m-1} a_{2i}\right)y_m,$$

as required. □

Theorem 3.2. *Let $\mathcal{V}$ be the class of all left regular bands and $\mathcal{C}$ be the class of all regular bands. Then $\mathcal{V}$ is $\mathcal{C}$-closed.*

Proof. Let U and S be a left regular band and a regular band respectively with U a subband of S. Take any $d \in Dom(U,S) \setminus U$. Then, by Result 2.1, we may let (2.1) be a zigzag in S over U with value d of minimal length m. Now

$$d = a_0y_1$$
$$= x_1a_1y_1 \qquad \text{(by zigzag equations)}$$
$$= x_1a_1a_1y_1$$
$$= x_1a_1a_2y_2 \qquad \text{(by zigzag equations)}$$
$$= (x_1a_1a_2)^2y_2 \qquad \text{(as } S \text{ is a band)}$$
$$= (x_1a_1a_2x_1)(a_1a_2y_2)$$
$$= (x_1a_1x_1a_2x_1)a_1a_2y_2 \qquad \text{(as } S \text{ is a regular band)}$$
$$= x_1a_1x_2a_3x_1a_1a_2y_2 \qquad \text{(by zigzag equations)}$$
$$= (x_1a_1x_2a_3^2x_1)a_1a_2y_2$$
$$= (x_1a_1x_1a_2a_3x_1)a_1a_2y_2 \qquad \text{(by zigzag equations)}$$
$$= (x_1a_1a_2a_3x_1)a_1a_2y_2 \qquad \text{(as } S \text{ is a regular band)}$$
$$= a_0a_2a_3a_0a_2y_2 \qquad \text{(by zigzag equations)}$$
$$= a_0a_2a_3y_2 \qquad \text{(as } U \text{ is a left regular band)}$$
$$= x_1a_1a_2a_4y_3 \qquad \text{(by zigzag equations)}$$

$$
\begin{aligned}
&= (x_1a_1a_2)^2a_4y_3 && \text{(as } S \text{ is a band)}\\
&= (x_1a_1a_2x_1)a_1a_2a_4y_3\\
&= (x_1a_1x_1a_2x_1)a_1a_2a_4y_3 && \text{(as } S \text{ is a regular band)}\\
&= x_1a_1x_2a_3x_1a_1a_2a_4y_3 && \text{(by zigzag equations)}\\
&= x_1a_1(x_2a_3x_1a_1a_2a_4)^2y_3 && \text{(as } S \text{ is a band)}\\
&= x_1a_1(x_2a_3x_1a_1a_2a_4x_2)(a_3x_1a_1a_2a_4)y_3\\
&= x_1a_1(x_2a_3x_1a_1a_2x_2a_4x_2)(a_3x_1a_1a_2a_4)y_3 && \text{(as } S \text{ is a regular band)}\\
&= x_1a_1(x_2a_3x_1a_1a_2x_3a_5x_2)a_3x_1a_1a_2a_4y_3 && \text{(by zigzag equations)}\\
&= x_1a_1(x_2a_3x_1a_1a_2x_3a_5^2x_2)a_3x_1a_1a_2a_4y_3 && \text{(as } S \text{ is a band)}\\
&= x_1a_1(x_2a_3x_1a_1a_2x_2a_4a_5x_2)a_3x_1a_1a_2a_4y_3 && \text{(by zigzag equations)}\\
&= x_1a_1(x_2a_3x_1a_1a_2a_4a_5x_2)a_3x_1a_1a_2a_4y_3 && \text{(as } S \text{ is a regular band)}\\
&= x_1a_1x_1a_2(x_1a_1a_2a_4a_5x_1a_2x_1)a_1a_2a_4y_3 && \text{(by zigzag equations)}\\
&= x_1a_1x_1a_2(x_1a_1a_2a_4a_5a_2x_1)a_1a_2a_4y_3 && \text{(as } S \text{ is a regular band)}\\
&= (x_1a_1x_1a_2x_1)a_1a_2a_4a_5a_2a_0a_2a_4y_3 && \text{(by zigzag equations)}\\
&= (x_1a_1a_2x_1)a_1a_2a_4a_5a_2a_0a_2a_4y_3 && \text{(as } S \text{ is a regular band)}\\
&= (a_0a_2)^2a_4a_5a_2a_0a_2a_4y_3 && \text{(by zigzag equations)}\\
&= (a_0a_2)a_0a_2a_4a_5a_0a_2a_4y_3 && \text{(as } U \text{ is a band)}\\
&= a_0a_2a_4a_5a_0a_2a_4y_3 && \text{(as } U \text{ is a left regular band)}\\
&= a_0a_2a_4a_5y_3 && \text{(as } U \text{ is a left regular band)}\\
&\vdots\\
&= a_0a_2a_4\cdots a_{2m-4}a_{2m-3}y_{m-1}\\
&= a_0a_2a_4\cdots a_{2m-4}a_{2m-2}y_m && \text{(by zigzag equations)}\\
&= \left(\prod_{i=0}^{m-1} a_{2i}\right) y_m\\
&= \left(\prod_{i=0}^{m-1} a_{2i}\right)(a_{2m-1}a_{2m-4}a_{2m-6}\cdots a_2a_0)\left(\prod_{i=0}^{m-1} a_{2i}\right) y_m && \text{(by Lemma 2.1)}\\
&= \left(\prod_{i=0}^{m-1} a_{2i}\right) w_1 \left(\prod_{i=0}^{m-1} a_{2i}\right) y_m && \text{(where } w_1 = a_{2m-1}a_{2m-4}a_{2m-6}\cdots a_2a_0)\\
&= \left(\prod_{i=0}^{m-1} a_{2i}\right) w_1y_m && \text{(as } \textstyle\prod_{i=0}^{m-1} a_{2i}, w_1 \in U \text{ and } U \text{ is a left regular band)}\\
&= \left(\prod_{i=0}^{m-3} a_{2i})(a_{2m-4}a_{2m-2}a_{2m-1}a_{2m-4}\right) a_{2m-6}\cdots a_2a_0y_m\\
& && \text{(as } w_1 = a_{2m-1}a_{2m-4}a_{2m-6}\cdots a_2a_0)
\end{aligned}
$$

$$= \left(\prod_{i=0}^{m-3} a_{2i}\right)(a_{2m-4}a_{2m-2}a_{2m-1})a_{2m-6}\cdots a_2a_0y_m$$

(as $a_{2m-4}, a_{2m-2}a_{2m-1} \in U$ and U is a left regular band)

$$= \left(\prod_{i=0}^{m-1} a_{2i}\right)a_{2m-1}a_{2m-6}\cdots a_2a_0y_m$$

$$= \left(\prod_{i=0}^{m-4} a_{2i}\right)(a_{2m-6}a_{2m-4}a_{2m-2}a_{2m-1}a_{2m-6})\cdots a_2a_0y_m$$

$$= \left(\prod_{i=0}^{m-4} a_{2i}\right)(a_{2m-6}a_{2m-4}a_{2m-2}a_{2m-1})a_{2m-8}\cdots a_2a_0y_m$$

(as $a_{2m-6}, a_{2m-4}a_{2m-2}a_{2m-1} \in U$ and U is a left regular band)

$$= \left(\prod_{i=0}^{m-1} a_{2i}\right)a_{2m-1}a_{2m-8}\cdots a_2a_0y_m$$

$$\vdots$$

$$= \left(\prod_{i=0}^{m-1} a_{2i}\right)a_{2m-1}a_2a_0y_m$$

$$= a_0a_2(a_4\cdots a_{2m-4}a_{2m-2}a_{2m-1})a_2a_0y_m$$

$$= a_0a_2(a_4\cdots a_{2m-4}a_{2m-2}a_{2m-1})a_0y_m$$

(as $a_2, (a_4\cdots a_{2m-4}a_{2m-2}a_{2m-1}) \in U$ and U is a left regular band)

$$= a_0(a_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m-1})a_0y_m$$

$$= a_0(a_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m-1})y_m$$

(as $a_0, (a_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m-1}) \in U$ and U is a left regular band)

$$= a_0a_2a_4\cdots a_{2m-4}a_{2m-2}(a_{2m-1}y_m)$$

$$= a_0a_2a_4\cdots a_{2m-4}a_{2m-2}(a_{2m}) \qquad \text{(by zigzag equations)}$$

$$= \prod_{i=0}^{m} a_{2i} \in U$$

$$\Rightarrow d \in U.$$

Hence $Dom(U, S) = U$. □

Dually, we may prove the following:

Theorem 3.3. *Let $\mathcal{V}$ be the variety of all right regular bands and $\mathcal{C}$ be the variety of all regular bands. Then $\mathcal{V}$ is $\mathcal{C}$-closed.*

In [3], the authors have shown that the class of all left [right] quasinormal bands is closed within the class of all left [right] quasinormal bands. We now generalize this result and show that the variety $\mathcal{V}$=$[axy = axay]$ of semigroups is closed.

Theorem 3.4. *The variety $\mathcal{V}$=$[axy = axay]$ of semigroups is closed.*

Proof. Take any $U,\ S \in \mathcal{V}$ with U a subsemigroup of S and let $d \in Dom(U, S) \setminus U$. Then, by Result 2.1, we may let (2.1) be a zigzag in S over U with value d of minimal length m. Now

$$\begin{aligned}
d &= a_0y_1 \\
&= x_1a_1y_1 && \text{(by zigzag equations)}\\
&= x_1a_1x_1y_1 && \text{(since } x_1, a_1, y_1 \in S)\\
&= x_1a_1x_1a_1y_1 && \text{(since } a_1, x_1, y_1 \in S)\\
&= x_1a_1a_1y_1 && \text{(since } x_1, a_1, y_1 \in S)\\
&= x_1a_1a_2y_2 && \text{(by zigzag equations)}\\
&= x_1a_1x_1a_2y_2 && \text{(since } x_1, a_1 \in S)\\
&= x_1a_1x_2a_3y_2 && \text{(by zigzag equations)}\\
&= x_1a_1x_2a_3x_2a_3y_2 && \text{(since } x_2, a_3, y_2 \in S)\\
&= x_1a_1x_2a_3a_3y_2 && \text{(since } x_2, a_3 \in S)\\
&= x_1a_1x_1a_2a_3y_2 && \text{(by zigzag equations)}\\
&= x_1a_1a_2a_3y_2 && \text{(since } x_1, a_1, a_2 \in S)\\
&= a_0a_2a_3y_2 && \text{(by zigzag equations)}\\
&= \left(\prod_{i=0}^{1} a_{2i}\right)(a_3y_2)\\
&\ \ \vdots\\
&= \left(\prod_{i=0}^{m-2} a_{2i}\right)(a_{2m-3}y_{m-1})\\
&= a_0a_2a_4\cdots a_{2m-4}(a_{2m-2}y_m) && \text{(by zigzag equations)}\\
&= (x_1a_1a_2)a_4\cdots a_{2m-4}a_{2m-2}y_m && \text{(by zigzag equations)}\\
&= (x_1a_1x_1a_2)a_4\cdots a_{2m-4}a_{2m-2}y_m && \text{(since } x_1, a_1, a_2 \in S)\\
&= (x_1a_1x_2a_3)a_4\cdots a_{2m-4}a_{2m-2}y_m && \text{(by zigzag equations)}\\
&= x_1a_1(x_2a_3x_2a_4)\cdots a_{2m-4}a_{2m-2}y_m && \text{(since } x_2, a_3, a_4 \in S)\\
&\ \ \vdots\\
&= x_1a_1(x_2a_3x_2a_4)\cdots x_{m-2}a_{2m-4}a_{2m-2}y_m\\
&= x_1a_1x_2a_3x_2a_4\cdots(x_{m-1}a_{2m-3}a_{2m-2})y_m && \text{(by zigzag equations)}\\
&= x_1a_1x_2a_3x_2a_4\cdots(x_{m-1}a_{2m-3}x_{m-1}a_{2m-2})y_m\\
& && \text{(since } x_{m-1}, a_{2m-3}, a_{2m-2} \in S)
\end{aligned}$$

$$
\begin{aligned}
&= x_1a_1x_2a_3x_2a_4\cdots(x_{m-1}a_{2m-3}x_ma_{2m-1})y_m && \text{(by zigzag equations)}\\
&= x_1a_1x_2a_3x_2a_4\cdots x_{m-1}a_{2m-3}(x_ma_{2m-1}x_my_m) && \text{(since } x_m, a_{2m-1}, y_m \in S)\\
&= x_1a_1x_2a_3x_2a_4\cdots x_{m-1}a_{2m-3}x_m(a_{2m-1}x_ma_{2m-1}y_m)\\
& && \text{(since } a_{2m-1}, x_m, y_m \in S)\\
&= x_1a_1x_2a_3x_2a_4\cdots x_{m-1}a_{2m-3}(x_ma_{2m-1}a_{2m-1}y_m)\\
& && \text{(since } x_m, a_{2m-1}, a_{2m-1}y_m \in S)\\
&= x_1a_1x_2a_3x_2a_4\cdots(x_{m-1}a_{2m-3}x_{m-1}a_{2m-2})a_{2m-1}y_m && \text{(by zigzag equations)}\\
&= x_1a_1x_2a_3x_2a_4\cdots(x_{m-1}a_{2m-3}a_{2m-2})a_{2m-1}y_m\\
& && \text{(since } x_{m-1}, a_{2m-3}, a_{2m-2} \in S)\\
&= x_1a_1x_2a_3x_2a_4\cdots x_{m-2}a_{2m-4}a_{2m-2}a_{2m-1}y_m && \text{(by zigzag equations)}\\
&\vdots\\
&= x_1a_1(x_2a_3x_2a_4)\cdots a_{2m-4}a_{2m-2}a_{2m-1}y_m\\
&= x_1a_1(x_2a_3a_4)\cdots a_{2m-4}a_{2m-2}a_{2m-1}y_m && \text{(since } x_2, a_3, a_4 \in S)\\
&= x_1a_1x_2a_3a_4\cdots a_{2m-4}a_{2m-2}a_{2m-1}y_m && \text{(since } x_2, a_3, a_4 \in S)\\
&= (x_1a_1x_1a_2)a_4\cdots a_{2m-4}a_{2m-2}a_{2m} && \text{(by zigzag equations)}\\
&= (x_1a_1a_2)a_4\cdots a_{2m-4}a_{2m-2}a_{2m} && \text{(since } x_1, a_1, a_2 \in S)\\
&= a_0a_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m} && \text{(by zigzag equations)}\\
&= \prod_{i=0}^{m} a_{2i} \in U. && (3.1)\\
&\Rightarrow\ d \in U.
\end{aligned}
$$

Hence $Dom(U, S) = U$. □

Dually, we have the following:

Theorem 3.5. *The variety* $\mathcal{V}$=[$yxa = yaxa$] *of semigroups is closed.*

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