



Linear Independence of Continued Fractions in the Field of Formal Series over a Finite Field

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Abstract : A criterion for linear independence, similar to that established in 2002 by Hančl in the classical case, is derived for continued fractions of elements in the field of formal series over a finite field.

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1 Introduction

A number of papers related to transcendence, irrationality and independence of continued fractions in different settings have appeared, see e.g. [1], [2], [4], [5], [6], [7], [8], [9], [10], [11], [12], [15], [16], [17] and [18]. We are here particularly interested in the result of Hančl ([8]) where a linear independence criterion for classical continued fractions is obtained. This criterion is based on a suitable growth condition of the partial quotients involved. Similar results for algebraic independence can be found, e.g. in [1], [9], [10], [11] and [17].

It is one of our two objectives to derive and indeed to extend such a criterion for continued fractions in the field of formal series over a finite field. In order to do so and make this paper self-contained, we are led to make available certain facts about irrational continued fractions in this setting, and this becomes our second objective. In the next section, we describe the construction of continued fractions in the field of formal series $\mathbb{F}_q((x^{-1}))$, where \mathbb{F}_q is a finite field of q elements. This construction mimics that of the classical simple continued fractions in the real case. We then derive basic properties and show that a continued fraction terminates if and only if it represents a rational element. As to the characterization of quadratic irrationals, such continued fraction is (infinite) periodic if and only if it represents a quadratic irrational. Next we establish the analogous result of Hančl ([8]) for continued fractions in $\mathbb{F}_q((x^{-1}))$ but with weaker conditions. Various interesting examples of some classes of continued fractions are worked out in the last section.

2 Construction

Throughout, \mathbf{F} denotes $\mathbb{F}_q((x^{-1}))$, the field of formal series over a finite field \mathbb{F}_q . We start by recalling the construction and basic properties of continued fractions in the field \mathbf{F} ; more details can be found e.g. in [14]. It is well-known that elements of \mathbf{F} are formal series (in x) written uniquely under the form

$$\xi = a_m x^m + a_{m-1} x^{m-1} + a_{m-2} x^{m-2} + \dots,$$

where the coefficients $a_m, a_{m-1}, a_{m-2}, \dots$ are in \mathbb{F}_q . Thus $\mathbb{F}_q(x)$, the quotient field of $\mathbb{F}_q[x]$, is a subfield of \mathbf{F} . A valuation $|\cdot|$ in \mathbf{F} is defined by putting

$$|0| = 0, \quad |\xi| = q^m \text{ if } \xi = a_m x^m + a_{m-1} x^{m-1} + a_{m-2} x^{m-2} + \dots, \quad a_m \neq 0.$$

The construction of the continued fraction for ξ runs as follows: Define

$$\xi = [\xi] + (\xi),$$

where

$$[\xi] := a_m x^m + a_{m-1} x^{m-1} + a_{m-2} x^{m-2} + \dots + a_1 x + a_0, \quad (\xi) := a_{-1} x^{-1} + a_{-2} x^{-2} + \dots$$

We call $[\xi]$ and (ξ) the *head* and the *tail* parts of ξ , respectively. Clearly, the head and the tail parts of ξ are uniquely determined. Let

$$\beta_0 = [\xi] \in \mathbb{F}_q[x].$$

Then $|\beta_0| = |\xi| \geq 1$, provided $[\xi] \neq 0$.

If $(\xi) = 0$, then the process stops. If $(\xi) \neq 0$, then write

$$\xi = \beta_0 + \xi_1^{-1},$$

where $\xi_1^{-1} = (\xi)$ with $|\xi_1| > 1$. Next write $\xi_1 = [\xi_1] + (\xi_1)$. Let

$$\beta_1 = [\xi_1] \in \mathbb{F}_q[x] \setminus \mathbb{F}_q.$$

Then $|\beta_1| = |\xi_1| > 1$.

If $(\xi_1) = 0$, then the process stops. If $(\xi_1) \neq 0$, then write

$$\xi_1 = \beta_1 + \xi_2^{-1},$$

where $\xi_2^{-1} = (\xi_1)$ with $|\xi_2| > 1$. Let

$$\beta_2 = [\xi_2] \in \mathbb{F}_q[x] \setminus \mathbb{F}_q.$$

Then $|\beta_2| = |\xi_2| > 1$.

Again, if $(\xi_2) = 0$, then the process stops. If $(\xi_2) \neq 0$, then continue in the same manner. By so doing, we obtain the unique representation

$$\xi = [\beta_0, \beta_1, \beta_2, \dots, \beta_{n-1}, \xi_n] := \beta_0 + \frac{1}{\beta_1 + \frac{1}{\beta_2 + \dots + \frac{1}{\beta_{n-1} + \frac{1}{\xi_n}}}}$$

where $\beta_i \in \mathbb{F}_q[x] \setminus \mathbb{F}_q$ ($i \geq 1$), $\xi_n \in \mathbf{F}$, $|\xi_n| > 1$ if exists, and ξ_n is referred to as the n^{th} complete quotient. The sequence (β_n) is uniquely determined and the β_n are called the *partial quotients* of ξ .

In order to establish convergence, we define two sequences (C_n) , (D_n) as follows:

$$\begin{aligned} C_{-1} &= 1, & C_0 &= \beta_0, & C_{n+1} &= \beta_{n+1}C_n + C_{n-1} & (n \geq 0) \\ D_{-1} &= 0, & D_0 &= 1, & D_{n+1} &= \beta_{n+1}D_n + D_{n-1} & (n \geq 0). \end{aligned}$$

The results in the following lemma are easily verified by induction.

Lemma 1 For any $n \geq 0$, $\alpha \in \mathbf{F} \setminus \{0\}$, we have

- (i) $\frac{\alpha C_n + C_{n-1}}{\alpha D_n + D_{n-1}} = [\beta_0, \beta_1, \beta_2, \dots, \beta_n, \alpha]$,
- (ii) $C_n D_{n-1} - C_{n-1} D_n = (-1)^{n-1}$,
- (iii) $|D_n| > |D_{n-1}|$,
- (iv) $|D_n| = |\beta_1 \beta_2 \dots \beta_n|$ ($n \geq 1$),
- (v) $\xi - \frac{C_n}{D_n} = \frac{(-1)^n}{D_n(\xi_{n+1} D_n + D_{n-1})}$ ($n \geq 1$).

From Lemma 1 (i), we have

$$\frac{C_n}{D_n} = \frac{\beta_n C_{n-1} + C_{n-2}}{\beta_n D_{n-1} + D_{n-2}} = [\beta_0, \beta_1, \beta_2, \dots, \beta_n] \quad (n \geq 1),$$

and C_n/D_n is called the n^{th} convergent of the continued fraction of ξ . If $(\xi_n) = 0$ for some n , then $\xi = [\beta_0, \beta_1, \beta_2, \dots, \beta_{n-1}]$, i.e. the continued fraction of ξ terminates. Otherwise, $(\xi_n) \neq 0$ for all n and the continued fraction is infinite and this is the case of interest from now on. Since $|\xi_n| = |\beta_n| \geq q$, Lemma 1 (iii) and (iv) give

$$|D_n(\xi_{n+1} D_n + D_{n-1})| = |D_n|^2 |\beta_{n+1}| \geq q^{2n+1}.$$

Using Lemma 1 (v), we get the approximation

$$\left| \xi - \frac{C_n}{D_n} \right| \leq \frac{1}{q^{2n+1}} \rightarrow 0 \quad (n \rightarrow \infty),$$

which immediately implies that $C_n/D_n \rightarrow \xi$, and enables us to write meaningfully

$$\xi = [\beta_0, \beta_1, \beta_2, \beta_3, \dots],$$

where the right hand side is referred to as the continued fraction of ξ .

3 Rational and Quadratic Irrationals

Although results in this section are already known, see e.g. [14], we provide details for completeness. The first result is that rational elements are precisely those having finite continued fraction expansions. We then look at quadratic irrationals. It is not difficult to see that infinite periodic continued fractions represent quadratic irrationals. We show that each irrational element in \mathbf{F} satisfying a quadratic equation over $\mathbb{F}_q(x)$ has a periodic continued fraction.

Theorem 2 *Let $\xi \in \mathbf{F}$. Then ξ is rational if and only if its continued fraction is finite.*

Proof. It is easy to see that if the continued fraction of $\xi \in \mathbf{F}$ is finite, then ξ is rational. Assume $\xi \in \mathbf{F}$ is rational. Let its continued fraction be $[\beta_0, \beta_1, \beta_2, \dots, \beta_n, \dots]$. Since $\xi = [\beta_0, \beta_1, \beta_2, \dots, \beta_{n-1}, \xi_n]$, and ξ is rational, then each ξ_n is rational and $|\xi_n| = |\beta_n| > 1$ ($n > 1$). Writing ξ_n as fraction, using the above notation,

$$\xi_n = \frac{x_n}{x_{n+1}} = \beta_n + \frac{x_{n+2}}{x_{n+1}}$$

with $x_n, x_{n+1}, x_{n+2} \in \mathbb{F}_q[x]$. We see that $1 \leq |x_{n+1}| < |x_n|$. It follows that (x_n) is a sequence of polynomials in $\mathbb{F}_q[x]$ with strictly decreasing degrees and must then terminate. \square

Theorem 2 can also be proved using the Euclidean algorithm as follows: let $\xi = C(x)/D(x) \in \mathbb{F}_q(x)$. By Euclidean algorithm, there are $Q_1, \dots, Q_{n+1}, R_1, \dots, R_n \in \mathbb{F}_q[x]$, $0 \leq \deg R_n < \deg R_{n-1} < \dots < \deg R_1 < \deg D$ such that

$$\begin{aligned} C(x) &= Q_1(x)D(x) + R_1(x) \\ D(x) &= Q_2(x)R_1(x) + R_2(x) \\ R_1(x) &= Q_3(x)R_2(x) + R_3(x) \\ &\vdots \\ R_{n-2}(x) &= Q_n(x)R_{n-1}(x) + R_n(x) \\ R_{n-1}(x) &= Q_{n+1}R_n(x). \end{aligned}$$

Thus the continued fraction of $\xi = C(x)/D(x)$ is finite of the form $[Q_1, Q_2, \dots, Q_{n+1}]$.

An infinite continued fraction of the shape $[\beta_0, \beta_1, \beta_2, \dots]$ is said to be *periodic* if there are positive integers k, N such that $\beta_n = \beta_{n+k}$ for all $n \geq N$.

The following theorem is easily checked and we omit the proof.

Theorem 3 *If the continued fraction of $\xi \in \mathbf{F}$ is periodic, then ξ is an irrational root of a quadratic equation over $\mathbb{F}_q[x]$.*

For the converse of Theorem 3 we have:

Theorem 4 *If $\xi \in \mathbf{F}$ is an irrational root of a quadratic equation of the form $at^2 + bt + c = 0$ where $a, b, c \in \mathbb{F}_q[x]$, $a \neq 0$, then its continued fraction is periodic.*

Proof. Let $\xi = [\beta_0, \beta_1, \beta_2, \dots] \in \mathbf{F}$. Writing

$$\xi = [\beta_0, \beta_1, \beta_2, \dots, \beta_{n-1}, \xi_n], \quad \xi_n = [\beta_n, \beta_{n+1}, \beta_{n+2}, \dots].$$

Then by Lemma 1 (v), $\xi = (\xi_n C_{n-1} + C_{n-2}) / (\xi_n D_{n-1} + D_{n-2})$, where C_n / D_n is the n^{th} convergent to the continued fraction of ξ . Direct substitution gives

$$R_n \xi_n^2 + S_n \xi_n + T_n = 0$$

where

$$\begin{aligned} R_n &= aC_{n-1}^2 + bC_{n-1}D_{n-1} + cD_{n-1}^2, \\ S_n &= 2aC_{n-1}C_{n-2} + b(C_{n-1}D_{n-2} + D_{n-1}C_{n-2}) + 2cD_{n-1}D_{n-2}, \\ T_n &= aC_{n-2}^2 + bC_{n-2}D_{n-2} + cD_{n-2}^2. \end{aligned}$$

Observe that a, b, c, C_i , and D_i all belong to $\mathbb{F}_q[x]$ and so do all R_n, S_n, T_n . If $R_n = 0$, then ξ_n is rational, contradicting the fact that ξ is irrational. Thus $R_n \neq 0$. Note that

$$S_n^2 - 4R_n T_n = (b^2 - 4ac)(C_{n-1}D_{n-2} - D_{n-1}C_{n-2})^2 = b^2 - 4ac.$$

By Lemma 1 (v),

$$\xi D_{n-1} - C_{n-1} = \frac{(-1)^{n-1} D_{n-1}}{D_{n-1}(\xi_n D_{n-1} + D_{n-2})},$$

and so

$$C_{n-1} = \xi D_{n-1} + \frac{\delta_{n-1}}{D_{n-1}},$$

where $\delta_{n-1} = \frac{(-1)^n D_{n-1}}{\xi_n D_{n-1} + D_{n-2}}$. By Lemma 1 (iii), and $|\xi_n| = |\beta_n| > 1$, we have

$$|\delta_{n-1}| = \frac{|D_{n-1}|}{|\xi_n D_{n-1} + D_{n-2}|} = \frac{|D_{n-1}|}{|\beta_n D_{n-1}|} < 1.$$

Now, for all $n \in \mathbb{N}$,

$$\begin{aligned} R_n &= a \left(\xi D_{n-1} + \frac{\delta_{n-1}}{D_{n-1}} \right)^2 + b D_{n-1} \left(\xi D_{n-1} + \frac{\delta_{n-1}}{D_{n-1}} \right) + c D_{n-1}^2 \\ &= (a\xi^2 + b\xi + c) D_{n-1}^2 + 2a\xi\delta_{n-1} + a \frac{\delta_{n-1}^2}{D_{n-1}^2} + b\delta_{n-1} = 2a\xi\delta_{n-1} + a \frac{\delta_{n-1}^2}{D_{n-1}^2} + b\delta_{n-1}, \end{aligned}$$

which gives

$$|R_n| < \max \{ |2a\xi|, |a|, |b| \} := \ell, \text{ say.}$$

Since $T_n = R_{n-1}$, then $|T_n| = |R_{n-1}| < \ell$. Now

$$|S_n^2| = |4R_n T_n + b^2 - 4ac| < \max\{4\ell^2, |b^2 - 4ac|\}.$$

Hence $|R_n|, |S_n|, |T_n|$ are bounded by a constant independent of n . It follows that, being elements in $\mathbb{F}_q[x]$, there are only a finite number of different triplets (R_n, S_n, T_n) and so we can find a triplet (R, S, T) which occurs at least three times, say $(R_{n_1}, S_{n_1}, T_{n_1}), (R_{n_2}, S_{n_2}, T_{n_2}), (R_{n_3}, S_{n_3}, T_{n_3})$. These $\xi_{n_1}, \xi_{n_2}, \xi_{n_3}$ are the roots of $Ry^2 + Sy + T = 0$ and hence at least two of them must be equal. But if, for example, $\xi_{n_1} = \xi_{n_2}$, then $\beta_{n_2} = \beta_{n_1}, \beta_{n_2+1} = \beta_{n_1+1}, \dots$, i.e., the continued fraction is periodic. \square

4 Linear Independence

In this section, we establish a sufficient condition for linear independence of continued fractions in \mathbf{F} .

Theorem 5 *Let $N \in \mathbb{N}, \{a_{n,j}\}_{n=0}^\infty$ ($j = 1, 2, \dots, N$) be N sequences of non-constant polynomials over \mathbb{F}_q . Assume that there exists a strictly increasing sequence $0 = n_0, n_1, n_2, \dots$ of $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ with the following properties:*

$$|a_{n_k, j+1}| \geq |a_{n_k, j}| \varepsilon_{n_k}, \tag{4.1}$$

$$|a_{n, j+1}| \geq |a_{n, j}| c_n \quad (n_k < n < n_{k+1} ; k \in \mathbb{N}_0), \tag{4.2}$$

$$|a_{n_k+1, 1}| \geq |a_{n_k, N}|^{N-1} \delta_{n_k}, \tag{4.3}$$

$$|a_{n+1, 1}| \geq |a_{n, N}|^{N-1} d_n \quad (n_k < n < n_{k+1} ; k \in \mathbb{N}_0), \tag{4.4}$$

where $\varepsilon_{n_k}, \delta_{n_k}, c_n, d_n$ are positive real numbers subject to the conditions that

$$|c_n| \geq c > 0 \quad (n_k < n < n_{k+1} ; k \in \mathbb{N}_0),$$

$$\prod_{i=0}^\infty (c_{n_{i+1}} \cdots c_{n_{i+1}-1} \varepsilon_{n_{i+1}}) = \infty = \prod_{i=0}^\infty (d_{n_{i+1}} \cdots d_{n_{i+1}-1} \delta_{n_{i+1}}).$$

Then $\alpha_j := [a_{0,j}, a_{1,j}, \dots]$ ($j = 1, 2, \dots, N$) and 1 are linearly independent over $\mathbb{F}_q(x)$.

Proof. We start with the case $N = 1$. Here $\alpha_1 := [a_{0,1}, a_{1,1}, \dots]$ is an infinite continued fraction and so α_1 is irrational, i.e., α_1 and 1 are linearly independent over $\mathbb{F}_q(x)$. Henceforth, take $N \geq 2$. Assume that $1, \alpha_1, \alpha_2, \dots, \alpha_N$ are linearly dependent over $\mathbb{F}_q[x]$. Then there exist $A_1, A_2, \dots, A_N, A_{N+1} \in \mathbb{F}_q[x]$ not all zero such that

$$A_{N+1} = \sum_{j=1}^N A_j \alpha_j. \tag{4.5}$$

Write each continued fraction α_j ($j = 1, 2, \dots, N$) as

$$\alpha_j = \frac{C_{n,j}}{D_{n,j}} + R_{n,j}, \quad (4.6)$$

where $\frac{C_{n,j}}{D_{n,j}} = [a_{0,j}, a_{1,j}, \dots, a_{n,j}]$ is the n^{th} convergence of α_j and $R_{n,j}$ is its remainder. Note that by Lemma 1 (v)

$$|R_{n,j}| = \left| \alpha_j - \frac{C_{n,j}}{D_{n,j}} \right| = \frac{1}{|a_{n+1,j} D_{n,j}^2|} \neq 0. \quad (4.7)$$

Substituting (4.6) into (4.5), we obtain

$$A_{N+1} = \sum_{j=1}^N A_j \left(\frac{C_{n,j}}{D_{n,j}} + R_{n,j} \right).$$

Multiplying both sides of the last equation by $\prod_{j=1}^N D_{n,j}$, we obtain

$$M_n := \left(A_{N+1} - \sum_{j=1}^N A_j \frac{C_{n,j}}{D_{n,j}} \right) \prod_{j=1}^N D_{n,j} = \prod_{j=1}^N D_{n,j} \sum_{j=1}^N A_j R_{n,j} \in \mathbb{F}_q[x]. \quad (4.8)$$

We next show $M_n \neq 0$. For $j \in \{1, 2, \dots, N-1\}$, $k \in \mathbb{N}_0$, by Lemma 1 (iv), (4.1) and (4.2), we have

$$\begin{aligned} & |D_{n_k, j+1}| \\ &= |a_{n_k, j+1} a_{n_k-1, j+1} \cdots a_{1, j+1}| \\ &= \prod_{i=1}^k |a_{n_i, j+1}| |a_{n_k-1, j+1} a_{n_k-2, j+1} \cdots a_{n_{k-1}+1, j+1}| \cdots |a_{n_1-1, j+1} a_{n_1-2, j+1} \cdots a_{n_0+1, j+1}| \\ &\geq \prod_{i=0}^k \varepsilon_{n_i} |a_{n_i, j}| \prod_{i=0}^{k-1} (c_{n_i+1} c_{n_i+2} \cdots c_{n_{i+1}-1}) |a_{n_i+1, j} a_{n_i+2, j} \cdots a_{n_{i+1}-1, j}| \\ &\geq \prod_{i=0}^{k-1} (c_{n_i+1} c_{n_i+2} \cdots c_{n_{i+1}-1} \varepsilon_{n_{i+1}}) |D_{n_k, j}| \\ &> |D_{n_k, j}| \end{aligned}$$

whenever $k \geq N_0$ for some $N_0 \in \mathbb{N}$, because $\prod_{i=0}^{\infty} (c_{n_i+1} c_{n_i+2} \cdots c_{n_{i+1}-1} \varepsilon_{n_{i+1}}) = \infty$. Let l be the least positive integer such that $A_l \neq 0$. For $j \in \{l+1, l+2, \dots, N\}$,

$k \in \mathbb{N}_0$, by (4.2),

$$\begin{aligned} \left| \frac{R_{n_k,l}}{R_{n_k,j}} \right| &= \left| \frac{a_{n_k+1,j} D_{n_k,j}^2}{a_{n_k+1,l} D_{n_k,l}^2} \right| \geq c_{n_k+1}^{j-l} \left| \frac{D_{n_k,j}}{D_{n_k,j-1}} \frac{D_{n_k,j-1}}{D_{n_k,j-2}} \cdots \frac{D_{n_k,l+1}}{D_{n_k,l}} \right|^2 \\ &\geq c_{n_k+1}^{j-l} \left[\prod_{i=0}^{k-1} (c_{n_i+1} c_{n_i+2} \cdots c_{n_{i+1}-1} \varepsilon_{n_{i+1}}) \right]^{2(j-l)} \\ &\geq c_{n_k+1}^{j-l} \left[\prod_{i=0}^{k-1} (c_{n_i+1} c_{n_i+2} \cdots c_{n_{i+1}-1} \varepsilon_{n_{i+1}}) \right]^{2(j-l)}. \end{aligned}$$

Since $\prod_{i=0}^{\infty} (c_{n_i+1} c_{n_i+2} \cdots c_{n_{i+1}-1} \varepsilon_{n_{i+1}}) = \infty$, there exists $N_1 \geq N_0$ such that for all $j \in \{l+1, l+2, \dots, N\}$, and all $k \geq N_1$, we have

$$\left| \frac{R_{n_k,l}}{R_{n_k,j}} \right| > \left| \frac{A_j}{A_l} \right|. \tag{4.9}$$

Then from (4.8) and (4.9), for all $k \geq N_1$,

$$\begin{aligned} |M_{n_k}| &= \left| \sum_{i=l}^N \left(\prod_{j=1}^N D_{n_k,j} \right) A_i R_{n_k,i} \right| = \max_{l \leq i \leq N} \left\{ \prod_{j=1}^N |D_{n_k,j}| |A_i R_{n_k,i}| \right\} \\ &= \prod_{j=1}^N |D_{n_k,j}| |A_l R_{n_k,l}| \neq 0. \end{aligned} \tag{4.10}$$

Now we prove that $|M_{n_k}| < 1$ for k sufficiently large. From (4.10), we obtain

$$\begin{aligned} |M_{n_k}| &= \prod_{j=1}^N |D_{n_k,j}| |A_l R_{n_k,l}| = \frac{|A_l| \prod_{j=1}^N |D_{n_k,j}|}{|a_{n_k+1,l} D_{n_k,l}^2|} \quad (\text{using (4.7)}) \\ &\leq \frac{|A_l| \prod_{j=2}^N |D_{n_k,j}|}{c_{n_k+1}^{l-1} |a_{n_k+1,1}| |D_{n_k,1}|} \leq \frac{|A_l| \prod_{j=2}^N |D_{n_k,N}|}{c_{n_k+1}^{l-1} |a_{n_k+1,1}| |D_{n_k,1}|} \quad (\text{by (4.2)}) \\ &\leq \frac{|A_l|}{c_{n_k+1}^{l-1} |a_{1,1}|} \left| \frac{(a_{n_k,N} a_{n_k-1,N} \cdots a_{1,N})^{N-1}}{a_{n_k+1,1} a_{n_k,1} \cdots a_{2,1}} \right| \\ &= \frac{|A_l|}{c_{n_k+1}^{l-1} |a_{1,1}|} \left| \frac{(a_{n_k,N} a_{n_k-1,N} \cdots a_{n_1,N})^{N-1}}{a_{n_k+1,1} a_{n_k,1} \cdots a_{n_1+1,1}} \right| \left| \frac{(a_{n_k-1,N} a_{n_k-2,N} \cdots a_{n_k-1+1,N})^{N-1}}{a_{n_k,1} a_{n_k-1,1} \cdots a_{n_k-1+2,1}} \right| \cdots \\ &\quad \cdots \left| \frac{(a_{n_1-1,N} a_{n_1-2,N} \cdots a_{n_0+1,N})^{N-1}}{a_{n_1,1} a_{n_1-1,1} \cdots a_{n_0+2,1}} \right| \\ &\leq \frac{|A_l|}{c_{n_k+1}^{l-1} |a_{1,1}|} \prod_{i=0}^{k-1} (d_{n_i+1} d_{n_i+2} \cdots d_{n_{i+1}-1} \delta_{n_{i+1}})^{-1} \quad (\text{by (4.3) and (4.4)}) \\ &\leq \frac{|A_l|}{c_{n_k+1}^{l-1} |a_{1,1}|} \prod_{i=0}^{k-1} (d_{n_i+1} d_{n_i+2} \cdots d_{n_{i+1}-1} \delta_{n_{i+1}})^{-1}. \end{aligned}$$

Since $\prod_{i=0}^{\infty} (d_{n_i+1}d_{n_i+2} \cdots d_{n_{i+1}-1}\delta_{n_{i+1}}) = \infty$, there exists $N_2 \geq N_1$ such that, for all $k \geq N_2$, $|M_{n_k}| < 1$. From this and (4.10), we obtain $0 < |M_{n_k}| < 1$ for $k \geq N_2$, which is not tenable because $M_{n_k} \in \mathbb{F}_q[x] \setminus \{0\}$. \square

The criterion of Hančl follows, in the case of $\mathbb{F}_q((x^{-1}))$, by choosing $c_n = \varepsilon_n = 1 + \varepsilon/(n \log n)$, where ε is a positive real number > 1 and $\delta_n = 1 + 1/n$, which gives:

Corollary 6 *Let $\varepsilon > 1$ be a real number, $N \in \mathbb{N}$, $\{a_{n,j}\}_{n=0}^{\infty}$ ($j = 1, 2, \dots, N$) be N sequences of non-constant polynomials over \mathbb{F}_q such that*

$$|a_{n,j+1}| \geq |a_{n,j}| \left(1 + \frac{\varepsilon}{n \log n}\right) \tag{4.11}$$

$$|a_{n+1,1}| \geq |a_{n,N}|^{N-1} \left(1 + \frac{1}{n}\right) \tag{4.12}$$

hold for every sufficiently large positive integer n and $j = 1, 2, \dots, N - 1$. Then $\alpha_j := [a_{0,j}, a_{1,j}, \dots]$ ($j = 1, 2, \dots, N$) and 1 are linearly independent over $\mathbb{F}_q(x)$.

An immediate consequence of our main result is the following particularly pleasing result which holds for both \mathbb{R} and $\mathbb{F}_q((x^{-1}))$.

Corollary 7 *Let $\alpha_1 = [a_0, a_1, a_2, \dots]$ and $\alpha_2 = [b_0, b_1, b_2, \dots]$ be two continued fractions whose partial quotients are subject to the conditions*

$$q|a_n| \leq |b_n| \leq q^{-1}|a_{n+1}|.$$

Then α_1, α_2 and 1 are linearly independent.

5 Examples

Our first example shows the case where the growth inequality conditions, in Corollary 7, are equalities with linearly independent continued fractions of simple shape.

Example 1. The two continued fractions in \mathbf{F} ,

$$\alpha_1 = [x^1, x^3, x^5, x^7, \dots], \quad \alpha_2 = [x^2, x^4, x^6, x^8, \dots]$$

and 1 are linearly independent over $\mathbb{F}_q(x)$ by Corollary 7.

Our second example is quite interesting. In the case of real numbers, it involves several earlier works, see e.g. [3] and [6], where certain real series are shown to have explicit continued fraction expansions.

Let $\beta > 1$ denote a real irrational number and $g \in \mathbb{F}_q[x] \setminus \mathbb{F}_q$. Define

$$\alpha_g(\beta) := \sum_{j=1}^{\infty} (g-1)g^{-[j\beta]} = \sum_{\nu=1}^{\infty} c_{\nu}g^{-\nu} \in \mathbf{F}, \tag{5.13}$$

where $c_\nu = 0$ if there is no $j \in \mathbb{N}$ with $[j\beta] = \nu$, and $c_\nu = g - 1$ if there is $j \in \mathbb{N}$ with $[j\beta] = \nu$. The following auxiliary result is Lemma 1 of [6].

Lemma 8 *Let $[b_0, b_1, b_2, \dots]$ be the simple (real) continued fraction of $1/\beta$ and p_n/q_n its n th convergent. Then the c_ν 's introduced above are determined as follows:*

- (i) $c_{kb_1} = g - 1$ for $k = 1, 2, \dots, b_2$, and $c_\nu = 0$ for all other $\nu \leq b_1b_2 + 1 = q_2$.
- (ii) If $n \geq 2$ and $q_n < \nu \leq q_{n+1}$, then $c_\nu = c_{r_n(\nu)}$, where $r_n(\nu)$ denotes the smallest positive residue of $\nu \pmod{q_n}$.

Just as in the case of real continued fractions, [6], the continued fraction expansion of the lacunary series $\alpha_g(\beta)$ in \mathbf{F} can also be found explicitly as in the following theorem.

Theorem 9 *Let $\beta > 1$ be a real irrational number. Let $[b_0, b_1, b_2, \dots]$ be the simple (real) continued fraction of $1/\beta$, p_n/q_n its n th convergent and let $g \in \mathbb{F}_q[x] \setminus \mathbb{F}_q$. If*

$$A_0 := gb_0, \quad A_n := g^{q_n-2} \sum_{i=0}^{b_n-1} g^{iq_{n-1}} \quad (n \geq 1), \tag{5.14}$$

then $\alpha_g(\beta) = [A_0, A_1, A_2, \dots]$.

Proof. With A_n defined above, for $n \geq 0$, let

$$P_{-2} = 0, P_{-1} = 1, P_n = A_nP_{n-1} + P_{n-2}; \quad Q_{-2} = 1, Q_{-1} = 0, Q_n = A_nQ_{n-1} + Q_{n-2}. \tag{5.15}$$

For $n \geq 1$, we claim that

$$P_n = \sum_{\nu=1}^{q_n} \frac{c_\nu}{g-1} g^{q_n-\nu}, \quad Q_n = \sum_{\nu=1}^{q_n} g^{q_n-\nu}. \tag{5.16}$$

Since the verification for both P_n and Q_n are the same, we only provide that of the former. When $n = 1$, since $b_0 = 0$, $A_0 = 0$, then $P_0 = A_0P_{-1} + P_{-2} = 0$, and so $P_1 = A_1P_0 + P_{-1} = 1$. Since $q_1 = b_1q_0 + q_{-1} = b_1$, by Lemma 8,

$$\sum_{\nu=1}^{q_1} \frac{c_\nu}{g-1} g^{q_1-\nu} = \sum_{\nu=1}^{b_1} \frac{c_\nu}{g-1} g^{b_1-\nu} = \sum_{\nu=1}^{b_1-1} \frac{c_\nu}{g-1} g^{b_1-\nu} + \frac{c_{b_1}}{g-1} = 0 + \frac{g-1}{g-1} = 1,$$

i.e., the claim holds for $n = 1$. When $n = 2$, we have $q_0 = 1, q_1 = b_1, P_0 = 0$, and $P_1 = 1$. By (5.14), (5.15) and Lemma 8,

$$\begin{aligned} P_2 &= \sum_{i=0}^{b_2-1} g^{iq_1+1} \\ &= g^{b_2b_1+1-b_1} + g^{b_2b_1+1-2b_1} + \dots + g^{b_1+1} + g^1 \\ &= g^{q_2-b_1} + g^{q_2-2b_1} + \dots + g^{q_2-(b_2-1)b_1} + g^{q_2-b_2b_1} \\ &= \sum_{k=1}^{b_2} g^{q_2-kb_1} = \sum_{\nu=1}^{q_2} \frac{c_\nu}{g-1} g^{q_2-\nu}, \end{aligned}$$

which yields (5.16) for $n = 2$. Now let $n \geq 2$ and assume (5.16) is true up to n . Then with (5.15), (5.16) and $q_{n+1} = b_{n+1}q_n + q_{n-1}$, we get

$$\begin{aligned}
 P_{n+1} &= A_{n+1}P_n + P_{n-1} \\
 &= \sum_{\lambda=0}^{b_{n+1}-1} \sum_{\nu=1}^{q_n} \frac{c_\nu}{g-1} g^{(1+\lambda)q_n+q_{n-1}-\nu} + \sum_{\nu=1}^{q_{n-1}} \frac{c_\nu}{g-1} g^{b_{n+1}q_n+q_{n-1}-\nu-b_{n+1}q_n} \\
 &= \sum_{\lambda=0}^{b_{n+1}-1} \sum_{\nu=1}^{q_n} \frac{c_\nu}{g-1} g^{q_{n+1}-(\nu+\lambda q_n)} + \sum_{\nu=1}^{q_{n-1}} \frac{c_\nu}{g-1} g^{q_{n+1}-(\nu+b_{n+1}q_n)} \\
 &= \sum_{\mu=1}^{b_{n+1}q_n} \frac{c_\mu}{g-1} g^{q_{n+1}-\mu} + \sum_{\mu=1+b_{n+1}q_n}^{q_{n+1}} \frac{c_\mu}{g-1} g^{q_{n+1}-\mu} \\
 &= \sum_{\nu=1}^{q_{n+1}} \frac{c_\nu}{g-1} g^{q_{n+1}-\nu},
 \end{aligned}$$

and the claim is verified. Observe that

$$\sum_{\lambda=0}^{b_{n+1}-1} \sum_{\nu=1}^{q_n} \frac{c_\nu}{g-1} g^{q_{n+1}-(\nu+\lambda q_n)} = \sum_{\mu=1}^{b_{n+1}q_n} \frac{c_\mu}{g-1} g^{q_{n+1}-\mu}.$$

On the other hand, since

$$Q_n = 1 + g + \dots + g^{q_n-2} + g^{q_n-1} = \frac{g^{q_n} - 1}{g - 1}$$

and $[0, A_1, A_2, \dots, A_n] = P_n/Q_n$, we have

$$\begin{aligned}
 \left| \alpha_g(\beta) - \frac{P_n}{Q_n} \right| &= \left| \alpha_g(\beta) - \frac{g^{q_n}}{g^{q_n} - 1} \sum_{\nu=1}^{q_n} c_\nu g^{-\nu} \right| \\
 &= \left| \left(1 - \frac{g^{q_n}}{g^{q_n} - 1} \right) \sum_{\nu=1}^{q_n} c_\nu g^{-\nu} + \sum_{\nu=q_n+1}^{\infty} c_\nu g^{-\nu} \right| \\
 &= |g^{-q_n}| |(g-1)g^{-b_1}| \rightarrow 0 \quad (n \rightarrow \infty).
 \end{aligned}$$

□

Example 2. Let $\beta > 1$ be a real irrational number. Let $[b_0, b_1, b_2, \dots]$ and p_n/q_n be the real simple continued fraction of $1/\beta$ and its n th convergent, respectively. Let $g_1, g_2 \in \mathbb{F}_q[x] \setminus \mathbb{F}_q$ be such that $|g_1| > |g_2|$. Assume that the real simple continued fraction of $1/\beta$ has unbounded partial quotients. Let $[A_0, A_1, A_2, \dots, A_n, \dots]$ and $C_n(g_1)/D_n(g_1)$ be the continued fraction of $\alpha_{g_1}(\beta)$ and its n th convergent, respectively. Let $[B_0, B_1, B_2, \dots, B_n, \dots]$ and $C_n(g_2)/D_n(g_2)$ be the continued fraction of $\alpha_{g_2}(\beta)$ and its n th convergent, respectively. By Theorem 9,

$$A_0 = g_1 b_0 = 0, \quad A_n = \frac{g_1^{q_n} - g_1^{q_{n-2}}}{g_1^{q_{n-1}} - 1}, \quad D_n(g_1) = \frac{g_1^{q_n} - 1}{g_1 - 1} \quad (n \geq 1)$$

and

$$B_0 = g_2 b_0 = 0, \quad B_n = \frac{g_2^{q_n} - g_2^{q_{n-2}}}{g_2^{q_{n-1}} - 1}, \quad D_n(g_2) = \frac{g_2^{q_n} - 1}{g_2 - 1} \quad (n \geq 1).$$

Since the continued fraction of $1/\beta$ has unbounded partial quotients, there is a subsequence (n_k) such that $b_{n_{k+1}} \rightarrow \infty$ ($k \rightarrow \infty$). Let $n_0 = 0$. Choose $\varepsilon_{n_0} = 1, \varepsilon_{n_k} = q$ ($k \geq 1$) and for all $k \geq 0$, choose $c_n = q$ ($n_k < n < n_{k+1}$). The conditions (4.1) and (4.2) of Theorem 5 are easily checked. To verify (4.3) and (4.4), choose

$$\delta_{n_0} = 1, \delta_{n_k} = \frac{|B_{n_{k+1}}|}{|A_{n_k}|} \quad (k \geq 1)$$

and for all $k \geq 0, n_k < n < n_{k+1}$, choose

$$d_n = \frac{|B_{n+1}|}{|A_n|}.$$

Then

$$\begin{aligned} \prod_{i=0}^k (d_{n_i+1} d_{n_i+2} \cdots d_{n_{i+1}-1} \delta_{n_{i+1}}) &= \left| \frac{B_2}{A_1} \cdots \frac{B_{n_1}}{A_{n_1-1}} \frac{B_{n_1+1}}{A_{n_1}} \right| \cdot \left| \frac{B_{n_1+2}}{A_{n_1+1}} \cdots \frac{B_{n_2}}{A_{n_2-1}} \frac{B_{n_2+1}}{A_{n_2}} \right| \cdots \\ &\quad \cdots \left| \frac{B_{n_k+2}}{A_{n_k+1}} \cdots \frac{B_{n_{k+1}}}{A_{n_{k+1}-1}} \frac{B_{n_{k+1}+1}}{A_{n_{k+1}}} \right| \\ &= \frac{|D_{n_{k+1}}(g_2)| |B_{n_{k+1}+1}|}{|D_{n_{k+1}}(g_1)| |B_1|} \\ &= \frac{1}{|B_1|} \left| \frac{g_2}{g_1} \right|^{q_{n_{k+1}-1}} |g_2|^{(B_{n_{k+1}+1}-1)q_{n_{k+1}}+q_{n_{k+1}-1}} \\ &\geq \frac{1}{|B_1|} \left| \frac{g_2}{g_1} \right|^{q_{n_{k+1}-1}} |g_2|^{(B_{n_{k+1}+1}-1)(q_{n_{k+1}}-1)} \\ &= \frac{1}{|B_1|} \left| \frac{g_2^{B_{n_{k+1}+1}}}{g_1} \right|^{q_{n_{k+1}-1}} \rightarrow \infty \quad (k \rightarrow \infty). \end{aligned}$$

By Theorem 5, $\alpha_{g_1}(\beta), \alpha_{g_2}(\beta)$ and 1 are linearly independent over $\mathbb{F}_q(x)$.

Our next and last example comes from one of our recent results, Theorem 8 of [13], which displays another type of lacunary series having explicit continued fractions.

Theorem 10 *Let I be a fixed positive integer, $\{k_i\}_{i \geq 1}$ a sequence of positive integers, $\{a_i\}_{i \geq I}$ a sequence of nonzero polynomials over \mathbb{F}_q , subject to the condition that if $I = 1$, then a_1 and those a_i ($i \geq 2$) for which $k_i = 2$ are non-constant polynomials over \mathbb{F}_q . Let the sequence $\{Q_i\}_{i \geq 1}$ be defined by*

$$Q_1 = 1, \quad Q_2, Q_3, \dots, Q_I \in \mathbb{F}_q[x] - \mathbb{F}_q; \quad Q_u = a_{u-1} Q_{u-1}^{k_{u-1}} Q_{u-2}^{k_{u-2}} \cdots Q_{u-I}^{k_{u-I}} \quad (u \geq I+1),$$

and let

$$C(u) = \sum_{i=1}^u \frac{1}{Q_i} \quad (u \in \mathbb{N}).$$

Assume that

1. if $I \geq 2$, then $Q_2 | Q_3 | \dots | Q_I$;
2. $k_i \geq 2$ ($i \geq I$).

If $C(u) = [b_0; b_1, \dots, b_n]$ ($u \geq I + 1$), then there exists $\beta \in \mathbb{F}_q \setminus \{0\}$ such that

$$C(u + 1) = [b_0; b_1, \dots, b_n, \beta s_u, -b_n, -b_{n-1}, \dots, -b_1],$$

where $s_u = \frac{a_u Q_u^{k_u-1}}{a_{u-1} Q_{u-1}^{k_{u-1}}}$. Moreover, if Q_2, Q_3, \dots, Q_I and all polynomials a_i ($i \geq I$) are monic, then $\beta = (-1)^n \{l.c.(b_n b_{n-1} \dots b_1)\}^{-2}$, where $l.c.(p)$ denotes the leading coefficient of $p(x) \in \mathbb{F}_q[x]$.

Example 3. Let $j \in \{1, 2\}$, $E_j \in \mathbb{F}_q[x]$ be monic with $\deg E_j \geq 1$ and $|E_1| > |E_2|$. Let $I = 1$, $k_\nu = \nu + 1$ ($\nu \geq 1$), $a_{1,j} = E_j^2$ and $a_{\nu,j} = 1$ ($\nu \geq 2$). With $Q_{1,j} = 1$, define

$$Q_{u,j} = a_{u-1,j} Q_{u-1,j}^{k_{u-1}} \quad \text{and} \quad s_{u,j} = \frac{a_u Q_u^{k_u-1}}{a_{u-1} Q_{u-1}^{k_{u-1}}} \quad (u \geq 2).$$

Then

$$\begin{array}{ll} Q_{2,j} = E_j^{2!}, & s_{2,j} = E_j^{2!} \\ Q_{3,j} = E_j^{3!}, & s_{3,j} = E_j^{3!2} \\ Q_{4,j} = E_j^{4!}, & s_{4,j} = E_j^{4!3} \\ \vdots & \vdots \\ Q_{u,j} = E_j^{u!}, & s_{u,j} = E_j^{u!(u-1)}. \end{array}$$

Define

$$C_j(1) = 1, \quad C_j(u) = \sum_{i=1}^u \frac{1}{Q_{i,j}}, \quad C_j(\infty) = \sum_{i=1}^{\infty} \frac{1}{Q_{i,j}} \quad (j = 1, 2).$$

Here $C_j(1) = 1 = [1]$, $C_j(2) = 1 + 1/E_j = [1, E_j]$. By Theorem 10, we have

$$\begin{array}{l} C_j(3) = [1, E_j^{2!}, -E_j^{2!}, -E_j^{2!}], \\ C_j(4) = [1, E_j^{2!}, -E_j^{2!}, -E_j^{2!}, -E_j^{3!2}, E_j^{2!}, E_j^{2!}, -E_j^{2!}], \\ C_j(5) = [1, E_j^{2!}, -E_j^{2!}, -E_j^{2!}, -E_j^{3!2}, E_j^{2!}, E_j^{2!}, -E_j^{2!}, -E_j^{4!3}, \\ \quad E_j^{2!}, -E_j^{2!}, -E_j^{2!}, E_j^{3!2}, E_j^{2!}, E_j^{2!}, -E_j^{2!}], \\ \vdots \end{array}$$

Let the sequence $(n_k)_{k=0}^\infty$ be defined by $n_0 = 0, n_1 = 1$, and $n_k = 2n_{k-1} + 1$ ($k \geq 2$). Write

$$\frac{P_{u,j}}{Q_{u,j}} := C_j(u) = \frac{E_j^{u!} + E_j^{u!-2!} + \dots + 1}{E_j^{u!}}.$$

We see that $\gcd(P_{u,j}, Q_{u,j}) = 1$ so that $P_{u,j}/Q_{u,j}$ is just the n_{u-1}^{th} convergent of the continued fraction of $C_j(\infty)$. In general, for

$$C_1(\infty) = [\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_n, \dots], \quad C_2(\infty) = [\beta_0, \beta_1, \beta_2, \dots, \beta_n, \dots],$$

choose $\varepsilon_{n_k} = q = c_n$ for all $k \geq 0$ and $n_k < n < n_{k+1}$. The conditions (4.1) and (4.2) of Theorem 5 follow at once. To verify (4.3) and (4.4), for all $k \geq 0$ and $n_k < n < n_{k+1}$, choose

$$\delta_{n_k} = \frac{|\beta_{n_{k+1}}|}{|\alpha_{n_k}|} \quad \text{and} \quad d_n = \frac{|\beta_{n+1}|}{|\alpha_n|}.$$

Then

$$\begin{aligned} \prod_{i=0}^N (d_{n_{i+1}} d_{n_{i+2}} \cdots d_{n_{i+1}-1} \delta_{n_{i+1}}) &= \left| \frac{\beta_2}{\alpha_1} \right| \cdot \left| \frac{\beta_3 \beta_4}{\alpha_2 \alpha_3} \right| \cdot \left| \frac{\beta_5 \beta_6 \beta_7 \beta_8}{\alpha_4 \alpha_5 \alpha_6 \alpha_7} \right| \cdots \\ &\quad \cdots \left| \frac{\beta_{n_{N+2}} \cdots \beta_{n_{N+1}-1} \beta_{n_{N+1}}}{\alpha_{n_{N+1}} \cdots \alpha_{n_{N+1}-1} \alpha_{n_{N+1}}} \right| \\ &= \frac{|Q_{N+2,2}| |s_{N+2,2}|}{|Q_{N+2,1}| |E_2|^{2^1}} \\ &= \frac{1}{|E_2|^{2^1}} \left(\frac{|E_2|^{(N+3)!-1}}{|E_1|} \right)^{(N+2)!} \rightarrow \infty \quad (N \rightarrow \infty). \end{aligned}$$

Theorem 5 shows then that $C_1(\infty), C_2(\infty)$ and 1 are linearly independent over $\mathbb{F}_q(x)$.

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