Thai Journal of Mathematics
Volume 12 (2014) Number 1 : 113–132

http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Some Topological and Geometric Properties[1] of the Domain of the Generalized Difference Matrix $B(r,s)$ in the Sequence Space $\ell(p)^*$

Cafer Aydın[†] and Feyzi Başar[‡,2]

[†]Department of Mathematics, Faculty of Arts and Sciences
Kahramanmaraş Sütçü İmam University, 46100-Kahramanmaraş, Turkey
e-mail : `caydin61@gmail.com`

[‡]Department of Mathematics, Faculty of Arts and Sciences
Fatih University, Büyükçekmece, 34500–İstanbul, Turkey
e-mail : `fbasar@fatih.edu.tr`;
`feyzibasar@gmail.com`

Abstract : The sequence space $\ell(p)$ was introduced by Maddox [Spaces of strongly summable sequences, Quart. J. Math. Oxford 18 (2) (1967) 345–355]. Quite recently, the domain of the generalized difference matrix $B(r,s)$ in the sequence space ℓ_p has been investigated by Kirişçi and Başar [Some new sequence spaces derived by the domain of generalized difference matrix, Comput. Math. Appl. 60 (5) (2010) 1299–1309]. In the present paper, the sequence space $\widehat{\ell}(p)$ of non-absolute type is studied which is the domain of the generalized difference matrix $B(r,s)$ in the sequence space $\ell(p)$. Furthermore, the alpha-, beta- and gamma-duals of the space $\widehat{\ell}(p)$ are determined, and the Schauder basis is constructed. The classes of matrix transformations from the space $\widehat{\ell}(p)$ to the spaces ℓ_∞, c and c_0 are characterized. Additionally, the characterizations of some other matrix transformations from the space $\widehat{\ell}(p)$ to the Euler, Riesz, difference, etc., sequence spaces are obtained by means of a given lemma. The last two sections of the paper are devoted to some results about the rotundity of the space $\widehat{\ell}(p)$ and conclusion.

---

[1] The main results of this paper were presented in part at the conference *Functional Analysis and Its Applications* to be held June 16–18, 2009 in Niš, Republic of Serbia at the University of Niš.

[2] `Corresponding author.`

Keywords : paranormed sequence space; the alpha-, beta- and gamma-duals; matrix transformations; rotundity of a sequence space.
2010 Mathematics Subject Classification : 46A35; 46A45.

1 Preliminaries, Background and Notation

By w, we denote the space of all complex valued sequences. Any vector subspace of w is called a *sequence space*. We write ℓ_∞, c and c_0 for the spaces of all bounded, convergent and null sequences, respectively. Also by bs, cs, ℓ_1 and ℓ_p; we denote the spaces of all bounded, convergent, absolutely convergent and p-absolutely convergent series, respectively; where $1 < p < \infty$.

A linear topological space X over the real field $\mathbb{R}$ is said to be a paranormed space if there is a subadditive function $g : X \to \mathbb{R}$ such that $g(\theta) = 0, g(x) = g(-x)$ and scalar multiplication is continuous, i.e., $|\alpha_n - \alpha| \to 0$ and $g(x_n - x) \to 0$ imply $g(\alpha_n x_n - \alpha x) \to 0$ for all α's in $\mathbb{R}$ and all x's in X, where θ is the zero vector in the linear space X.

Assume here and after that (p_k) be a bounded sequence of strictly positive real numbers with $\sup p_k = H$ and $L = \max\{1, H\}$. Then, the linear space $\ell(p)$ was defined by Maddox [1] (see also Simons [2] and Nakano [3]) as follows:

$$\ell(p) = \left\{ x = (x_k) \in w : \sum_k |x_k|^{p_k} < \infty \right\},\ (0 < p_k \le H < \infty)$$

which is the complete space paranormed by

$$g(x) = \left(\sum_k |x_k|^{p_k} \right)^{1/L}.$$

For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. We assume throughout that $p_k^{-1} + (p_k')^{-1} = 1$ provided $\inf p_k \le H < \infty$ and denote the collection of all finite subsets of $\mathbb{N} = \{0, 1, 2, \dots\}$ by $\mathcal{F}$.

Let λ, μ be any two sequence spaces and $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, we say that A defines a matrix mapping from λ into μ, and we denote it by writing $A : \lambda \to \mu$, if for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = \{(Ax)_n\}$, the A-transform of x, is in μ; where

$$(Ax)_n = \sum_k a_{nk} x_k \quad \text{for all} \quad n \in \mathbb{N}. \tag{1.1}$$

By $(\lambda : \mu)$, we denote the class of all matrices A such that $A : \lambda \to \mu$. Thus, $A \in (\lambda : \mu)$ if and only if the series on the right side of (1.1) converges for each $n \in \mathbb{N}$ and every $x \in \lambda$, and we have $Ax = \{(Ax)_n\}_{n \in \mathbb{N}} \in \mu$ for all $x \in \lambda$. A sequence x is said to be A-summable to l if Ax converges to l which is called as the A-limit of x.

The main purpose of this paper, which is a continuation of Kirişçi and Başar [4], is to introduce the sequence space $\widehat{\ell}(p)$ of non-absolute type consisting of all sequences whose $B(r,s)$-transforms are in the space $\ell(p)$; where the generalized difference matrix $B(r,s)=\{b_{nk}(r,s)\}$ is defined by

$$b_{nk}(r,s):=\begin{cases} r, & k=n,\\ s, & k=n-1,\\ 0, & 0\leq k<n-1 \text{ or } k>n,\end{cases}$$

for all $k,n\in\mathbb{N}$ with $r,s\in\mathbb{R}\backslash\{0\}$. Furthermore, the basis is constructed and the alpha-, beta- and gamma-duals are computed for the space $\widehat{\ell}(p)$. Besides this, the matrix transformations from the space $\widehat{\ell}(p)$ to some sequence spaces are characterized. Finally, some results related to the rotundity of the space $\widehat{\ell}(p)$ are derived.

The rest of this paper is organized, as follows:

In Section 2, the linear sequence space $\widehat{\ell}(p)$ is defined and proved that it is a complete paranormed space with a Schauder basis. Section 3 is devoted to the determination of α-, β- and γ-duals of the space $\widehat{\ell}(p)$. In Section 4, the classes $(\widehat{\ell}(p):\ell_\infty)$, $(\widehat{\ell}(p):c)$ and $(\widehat{\ell}(p):c_0)$ of infinite matrices are characterized. Additionally, the characterizations of some other classes of matrix transformations from the space $\widehat{\ell}(p)$ to the Euler, Riesz, difference, etc., sequence spaces are obtained by means of a given lemma. In Section 5, some consequences about the rotundity of the space $\widehat{\ell}(p)$ are given. In the final section of the paper; after comparing with the related results in the existing literature, open problems and further suggestions are noted.

2 The Sequence Space $\widehat{\ell}(p)$ of Non-absolute Type

In this section, we introduce the complete paranormed linear sequence space $\widehat{\ell}(p)$.

The *matrix domain* λ_A of an infinite matrix A in a sequence space λ is defined by

$$\lambda_A=\big\{x=(x_k)\in w: Ax\in\lambda\big\}, \tag{2.1}$$

which is a sequence space. Choudhary and Mishra [5] defined the sequence space $\overline{\ell(p)}$ which consists of all sequences such that S-transforms of them are in the space $\ell(p)$, where $S=(s_{nk})$ is defined by

$$s_{nk}=\begin{cases} 1 & , \quad 0\leq k\leq n,\\ 0 & , \quad k>n,\end{cases}$$

for all $k,n\in\mathbb{N}$. Başar and Altay [6] have recently examined the space $bs(p)$ which is formerly defined by Başar in [7] as the set of all series whose sequences of partial sums are in $\ell_\infty(p)$. More recently, Aydın and Başar [8] have studied the space $a^r(u,p)$ which is the domain of the matrix A^r in the sequence space $\ell(p)$, where the matrix $A^r = \{a_{nk}(r)\}$ is defined by

$$a_{nk}(r) = \begin{cases} \frac{1+r^k}{n+1} u_k, & 0 \leq k \leq n, \\ 0, & k > n, \end{cases}$$

for all $k, n \in \mathbb{N}$, (u_k) such that $u_k \neq 0$ for all $k \in \mathbb{N}$ and $0 < r < 1$. Altay and Başar [9] have studied the sequence space $r^t(p)$ which is derived from the sequence space $\ell(p)$ of Maddox by the Riesz means R^t. With the notation of (2.1), the spaces $\overline{\ell(p)}, bs(p), a^r(u,p)$ and $r^t(p)$ can be redefined by

$$\overline{\ell(p)} = [\ell(p)]_S, \ \ bs(p) = [\ell_\infty(p)]_S, \ \ a^r(u,p) = [\ell(p)]_{A^r}, \ \ r^t(p) = [\ell(p)]_{R^t}.$$

Following Choudhary and Mishra [5], Mursaleen [10], Malkowsky et al. [11], Çolak et al. [12], Başar and Altay [6], Altay and Başar [9, 13–15], Aydın and Başar [8, 16], we introduce the sequence space $\widehat{\ell}(p)$ as the set of all sequences whose $B(r,s)$-transforms are in the space $\ell(p)$, that is

$$\widehat{\ell}(p) := \left\{ (x_k) \in w : \sum_k |s x_{k-1} + r x_k|^{p_k} < \infty \right\}, \quad (0 < p_k \leq H < \infty).$$

It is trivial that in the case $p_k = p$ for all $k \in \mathbb{N}$, the sequence space $\widehat{\ell}(p)$ is reduced to the sequence space $\widehat{\ell}_p$ which is introduced by Kirişçi and Başar [4]. With the notation of (2.1), we can redefine the space $\widehat{\ell}(p)$ as follows:

$$\widehat{\ell}(p) := [\ell(p)]_{B(r,s)}.$$

Define the sequence $y = (y_k)$, which will be frequently used, as the $B(r,s)$-transform of a sequence $x = (x_k)$, i.e.,

$$y_k := s x_{k-1} + r x_k \quad \text{for all} \quad k \in \mathbb{N}. \tag{2.2}$$

Since the spaces $\ell(p)$ and $\widehat{\ell}(p)$ are linearly isomorphic one can easily observe that $x = (x_k) \in \widehat{\ell}(p)$ if and only if $y = (y_k) \in \ell(p)$, where the sequences $x = (x_k)$ and $y = (y_k)$ are connected with the relation (2.2).

Now, we may begin with the following theorem which is essential in the text:

Theorem 2.1. *$\widehat{\ell}(p)$ is the complete linear metric space paranormed by g_1 defined by*

$$g_1(x) := \left(\sum_k |s x_{k-1} + r x_k|^{p_k} \right)^{1/L},$$

where $0 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$.

Proof. The linearity of $\widehat{\ell}(p)$ with respect to the coordinatewise addition and scalar multiplication follows from the inequalities which are satisfied for $x = (x_k), z = (z_k) \in \widehat{\ell}(p)$, (see [17, p. 30]) and for any $\alpha \in \mathbb{C}$, the complex field, (see [18]), respectively,

$$\left[\sum_k |s(x_{k-1}+z_{k-1})+r(x_k+z_k)|^{p_k}\right]^{1/L} \leq \left(\sum_k |sx_{k-1}+rx_k|^{p_k}\right)^{1/L} + \left(\sum_k |sz_{k-1}+rz_k|^{p_k}\right)^{1/L} \tag{2.3}$$

and

$$|\alpha|^{p_k} \leq \max\{1, |\alpha|^L\}. \tag{2.4}$$

It is clear that $g_1(\theta) = 0$ and $g_1(x) = g_1(-x)$ for all $x \in \widehat{\ell}(p)$. Additionally, the inequalities (2.3) and (2.4) yield the subadditivity of g_1 and

$$g_1(\alpha x) \leq \max\{1, |\alpha|\} g_1(x).$$

Let $\{x^n\}$ be any sequence of the points $\widehat{\ell}(p)$ such that $g_1(x^n - x) \to 0$ and (α_n) also be any sequence of scalars such that $\alpha_n \to \alpha$, as $n \to \infty$. Then, since the inequality

$$g_1(x^n) \leq g_1(x) + g_1(x^n - x)$$

holds by subadditivity of g_1, $\{g_1(x^n)\}$ is bounded and we thus have

$$\begin{aligned} g_1(\alpha_n x^n - \alpha x) &= \left[\sum_k \left|s(\alpha_n x_{k-1}^n - \alpha x_{k-1}) + r(\alpha_n x_k^n - \alpha x_k)\right|^{p_k}\right]^{1/L} \\ &\leq |\alpha_n - \alpha|\, g_1(x^n) + |\alpha|\, g_1(x^n - x) \end{aligned}$$

which tends to zero as $n \to \infty$. That is to say that the scalar multiplication is continuous. Hence, g_1 is a paranorm on the space $\widehat{\ell}(p)$.

It remains to prove the completeness of the space $\widehat{\ell}(p)$. Let $B = B(r, s)$ and $\{x^i\}$ be any Cauchy sequence in the space $\widehat{\ell}(p)$, where $x^i = \left\{x_0^{(i)}, x_1^{(i)}, x_2^{(i)}, \dots\right\}$. Then, for a given $\varepsilon > 0$ there exists a positive integer $n_0(\varepsilon)$ such that

$$g_1(x^i - x^j) < \varepsilon \tag{2.5}$$

for all $i, j > n_0(\varepsilon)$. Using the definition of g_1, we obtain for each fixed $k \in \mathbb{N}$ that

$$\left|(Bx^i)_k - (Bx^j)_k\right| \leq \left[\sum_k \left|(Bx^i)_k - (Bx^j)_k\right|^{p_k}\right]^{1/L} < \varepsilon$$

for all $i, j \geq n_0(\varepsilon)$ which leads us to the fact that $\{(Bx^0)_k, (Bx^1)_k, (Bx^2)_k, \dots\}$ is a Cauchy sequence of complex numbers for each fixed $k \in \mathbb{N}$. Since $\mathbb{C}$ is complete, it converges, say $(Bx^i)_k \to (Bx)_k$ as $i \to \infty$. Using these infinitely many limits $(Bx)_0, (Bx)_1, (Bx)_2, \dots$, we define the sequence $\{(Bx)_0, (Bx)_1, (Bx)_2, \dots\}$. From (2.5) for each $m \in \mathbb{N}$ and $i, j \geq n_0(\varepsilon)$

$$\sum_{k=0}^{m} \left|(Bx^i)_k - (Bx^j)_k\right|^{p_k} \leq g_1(x^i - x^j)^L < \varepsilon^L. \tag{2.6}$$

Take any $i \geq n_0(\varepsilon)$. First let $j \to \infty$ in (2.6) and after $m \to \infty$, to obtain $g_1(x^i - x) \leq \varepsilon$. Finally, taking $\varepsilon = 1$ in (2.6) and letting $i \geq n_0(1)$ we have by Minkowski's inequality for each $m \in \mathbb{N}$ that

$$\left[\sum_{k=0}^{m} |(Bx)_k|^{p_k}\right]^{1/L} \leq g_1(x^i - x) + g_1(x^i) \leq 1 + g_1(x^i)$$

which implies that $x \in \widehat{\ell}(p)$. Since $g_1(x^i - x) \leq \varepsilon$ for all $i \geq n_0(\varepsilon)$ it follows that $x^i \to x$ as $i \to \infty$ which shows that $\widehat{\ell}(p)$ is complete. □

Therefore, one can easily check that the absolute property does not hold on the space $\widehat{\ell}(p)$, that is $g_1(x) \neq g_1(|x|)$; where $|x| = (|x_k|)$. This says that $\widehat{\ell}(p)$ is the sequence space of non-absolute type.

A sequence space λ with a linear topology is called a *K-space* provided each of the maps $p_i : \lambda \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$. A K-space λ is called an *FK-space* provided λ is complete linear metric space. An FK-space whose topology is normable is called a *BK-space*. Now, we may give the following:

Theorem 2.2. *$\widehat{\ell}_p$ is the linear space under the coordinatewise addition and scalar multiplication which is the BK-space with the norm*

$$\|x\| := \left(\sum_k |sx_{k-1} + rx_k|^p\right)^{1/p}, \quad \text{where} \ \ 1 \leq p < \infty.$$

Proof. Because of the first part of the theorem is a routine verification, we omit the detail. Since ℓ_p is the BK-space with respect to its usual norm (see [17, pp. 217-218]) and B is a normal matrix, Theorem 4.3.2 of Wilansky [19, p. 61] gives the fact that $\widehat{\ell}_p$ is the BK-space, where $1 \leq p < \infty$. □

Let us suppose that $1 < p_k \leq s_k$ for all $k \in \mathbb{N}$. Then, it is known that $\ell(p) \subset \ell(s)$ which leads us to the immediate consequence that $\widehat{\ell}(p) \subset \widehat{\ell}(s)$.

With the notation of (2.2), define the transformation T from $\widehat{\ell}(p)$ to $\ell(p)$ by $x \mapsto y = Tx$. Since T is linear and bijection, we have

Corollary 2.3. *The sequence space $\widehat{\ell}(p)$ of non-absolute type is linearly isomorphic to the space $\ell(p)$, where $0 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$.*

We firstly define the concept of the Schauder basis for a paranormed sequence space and nextly give the basis of the sequence space $\widehat{\ell}(p)$.

Let (λ, g) be a paranormed space. A sequence (b_k) of the elements of λ is called a *basis* for λ if and only if, for each $x \in \lambda$, there exists a unique sequence (α_k) of scalars such that

$$\lim_{n\to\infty} g\left(x - \sum_{k=0}^{n} \alpha_k b_k\right) = 0.$$

The series $\sum_k \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n), and written as $x = \sum_k \alpha_k b_k$. Since, it is known that the matrix domain λ_A of a sequence space λ has a basis if and only if λ has a basis whenever $A = (a_{nk})$ is a triangle (cf. [20, Remark 2.4]), we have:

Corollary 2.4. *Let $0 < p_k \leq H < \infty$ and $\lambda_k = (Bx)_k$ for all $k \in \mathbb{N}$. Define the sequence $b^{(k)}(r,s) = \left\{b_n^{(k)}(r,s)\right\}_{n\in\mathbb{N}}$ of the elements of the space $\widehat{\ell}(p)$ by*

$$b_n^{(k)}(r,s) := \begin{cases} 0, & n < k, \\ \frac{1}{r}\left(\frac{-s}{r}\right)^n, & n \geq k, \end{cases} \tag{2.7}$$

for every fixed $k \in \mathbb{N}$. Then, the sequence $\{b^{(k)}(r,s)\}_{k\in\mathbb{N}}$ given by (2.7) is a basis for the space $\widehat{\ell}(p)$ and any $x \in \widehat{\ell}(p)$ has a unique representation of the form $x := \sum_k \lambda_k b^{(k)}(r,s)$.

3 The Alpha-, Beta- and Gamma-duals of the Space $\widehat{\ell}(p)$

In this section, we state and prove the theorems determining the alpha-, beta- and gamma-duals of the sequence space $\widehat{\ell}(p)$ of non-absolute type.

The set $S(\lambda, \mu)$ defined by

$$S(\lambda,\mu) := \left\{z = (z_k) \in w : xz = (x_k z_k) \in \mu \ \text{ for all } x = (x_k) \in \lambda\right\} \tag{3.1}$$

is called the *multiplier space* of the sequence spaces λ and μ. With the notation of (3.1), the alpha-, beta- and gamma-duals of a sequence space λ, which are respectively denoted by λ^α, λ^β and λ^γ, are defined by

$$\lambda^\alpha := S(\lambda, \ell_1), \quad \lambda^\beta := S(\lambda, cs) \quad \text{and} \quad \lambda^\gamma := S(\lambda, bs).$$

Because of Part (i) can be established in the similar way to the proof of Part (ii), we omit the detail of that part and give the proof only for Part (ii) in Theorems 3.4-3.6, below.

We begin with quoting three lemmas which are needed in proving Theorems 3.4-3.6.

Lemma 3.1 (Lascarides and Maddox [21, (i) and (ii) of Theorem 1])**.** *Let* $A = (a_{nk})$ *be an infinite matrix. Then, the following statements hold:*

(i) Let $0 < p_k \leq 1$ *for all* $k \in \mathbb{N}$*. Then,* $A \in (\ell(p) : \ell_\infty)$ *if and only if*

$$\sup_{n,k \in \mathbb{N}} |a_{nk}|^{p_k} < \infty. \tag{3.2}$$

(ii) Let $1 < p_k \leq H < \infty$ *for all* $k \in \mathbb{N}$*. Then,* $A \in (\ell(p) : \ell_\infty)$ *if and only if there exists an integer* $M > 1$ *such that*

$$\sup_{n \in \mathbb{N}} \sum_k \left|a_{nk} M^{-1}\right|^{p_k'} < \infty. \tag{3.3}$$

Lemma 3.2 (Lascarides and Maddox [21, Corollary for Theorem 1])**.** *Let* $0 < p_k \leq H < \infty$ *for all* $k \in \mathbb{N}$*. Then,* $A = (a_{nk}) \in (\ell(p) : c)$ *if and only if (3.2), (3.3) hold, and*

$$\lim_{n \to \infty} a_{nk} = \beta_k \text{ for all } k \in \mathbb{N}. \tag{3.4}$$

Lemma 3.3 (Grosse-Erdmann [22, Theorem 5.1.0])**.** *Let* $A = (a_{nk})$ *be an infinite matrix. Then, the following statements hold:*

(i) Let $0 < p_k \leq 1$ *for all* $k \in \mathbb{N}$*. Then,* $A \in (\ell(p) : \ell_1)$ *if and only if*

$$\sup_{N \in \mathcal{F}} \sup_{k \in \mathbb{N}} \left| \sum_{n \in N} a_{nk} \right|^{p_k} < \infty. \tag{3.5}$$

(ii) Let $1 < p_k \leq H < \infty$ *for all* $k \in \mathbb{N}$*. Then,* $A \in (\ell(p) : \ell_1)$ *if and only if there exists an integer* $M > 1$ *such that*

$$\sup_{N \in \mathcal{F}} \sum_k \left| \sum_{n \in N} a_{nk} M^{-1} \right|^{p_k'} < \infty. \tag{3.6}$$

Theorem 3.4. *Define the sets* $d_1^{rs}(p)$ *and* $d_2^{rs}(p)$ *by*

$$d_1^{rs}(p) = \left\{ a = (a_k) \in w : \sup_{N \in \mathcal{F}} \sup_{k \in \mathbb{N}} \left| \sum_{n \in N_k^*} \frac{1}{r} \left(\frac{-s}{r} \right)^{n-k} a_n \right|^{p_k} < \infty \right\},$$

$$d_2^{rs}(p) = \bigcup_{M>1} \left\{ a = (a_k) \in w : \sup_{N \in \mathcal{F}} \sum_k \left| \sum_{n \in N_k^*} \frac{1}{r} \left(\frac{-s}{r} \right)^{n-k} a_n M^{-1} \right|^{p_k'} < \infty \right\},$$

where $N_k^* = N \cap \{n \in \mathbb{N} : n \geq k\}$*. Then,*

(i) $\{\widehat{\ell(p)}\}^\alpha := d_1^{rs}(p)$*,* $(0 < p_k \leq 1)$*.*

(ii) $\{\widehat{\ell}(p)\}^{\alpha} := d_2^{rs}(p)$, $(1 < p_k \leq H < \infty)$.

Proof. (ii) Let us take any $a = (a_n) \in w$. We easily derive with (2.2) that

$$a_n x_n = \frac{1}{r} \sum_{k=0}^{n} \left(\frac{-s}{r} \right)^{n-k} a_n y_k = (Cy)_n \quad \text{for all} \quad n \in \mathbb{N}, \tag{3.7}$$

where $C = \{c_{nk}(r,s)\}$ is defined by

$$c_{nk}(r,s) = \begin{cases} \frac{1}{r} \left(\frac{-s}{r} \right)^{n-k} a_n, & 0 \leq k \leq n, \\ 0, & k > n, \end{cases}$$

for all $k, n \in \mathbb{N}$. Thus, we deduce from (3.7) that $ax = (a_n x_n) \in \ell_1$ whenever $x = (x_k) \in \widehat{\ell}(p)$ if and only if $Cy \in \ell_1$ whenever $y = (y_k) \in \ell(p)$. From Lemma 3.3, we obtain the desired result that $\{\widehat{\ell}(p)\}^{\alpha} = d_2^{rs}(p)$. □

Theorem 3.5. *Define the sets $d_3^{rs}(p), d_4^{rs}(p)$ and d_5^{rs} by*

$$d_3^{rs}(p) := \left\{ (a_k) \in w : \sup_{k,n \in \mathbb{N}} \left| \frac{1}{r} \sum_{j=k}^{n} \left(\frac{-s}{r} \right)^{j-k} a_j \right|^{p_k} < \infty \right\},$$

$$d_4^{rs}(p) := \bigcup_{M>1} \left\{ (a_k) \in w : \sup_{n \in \mathbb{N}} \sum_{k=0}^{n} \left| \frac{1}{r} \sum_{j=k}^{n} \left(\frac{-s}{r} \right)^{j-k} a_j M^{-1} \right|^{p_k'} < \infty \right\},$$

$$d_5^{rs} := \left\{ (a_k) \in w : \lim_{n \to \infty} \sum_{j=k}^{n} \left(\frac{-s}{r} \right)^{j-k} a_j \ \textit{exists} \right\}.$$

Then,

(i) $\{\widehat{\ell}(p)\}^{\beta} := d_3^{rs}(p) \cap d_5^{rs}$, $(0 < p_k \leq 1)$.

(ii) $\{\widehat{\ell}(p)\}^{\beta} := d_4^{rs}(p) \cap d_5^{rs}$, $(1 < p_k \leq H < \infty)$.

Proof. (ii) Take any $a = (a_k) \in w$ and consider the equality obtained with (2.2) that

$$\sum_{k=0}^{n} a_k x_k = \sum_{k=0}^{n} \left[\sum_{j=k}^{n} \frac{1}{r} \left(\frac{-s}{r} \right)^{j-k} a_j \right] y_k = (Dy)_n \quad \text{for all} \quad n \in \mathbb{N}, \tag{3.8}$$

where $D = \{d_{nk}(r,s)\}$ is defined by

$$d_{nk}(r,s) = \begin{cases} \sum_{j=k}^{n} \frac{1}{r} \left(\frac{-s}{r} \right)^{j-k} a_j, & 0 \leq k \leq n, \\ 0, & k > n, \end{cases} \tag{3.9}$$

for all $k, n \in \mathbb{N}$. Thus, we deduce from (3.8) that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in \widehat{\ell}(p)$ if and only if $Dy \in c$ whenever $y = (y_k) \in \ell(p)$. Therefore, we derive from Lemma 3.2 that

$$\sum_k \left| \sum_{j=k}^{n} \frac{1}{r} \left(\frac{-s}{r} \right)^{j-k} a_j M^{-1} \right|^{p'_k} < \infty$$

and

$$\lim_{n\to\infty} \sum_{j=k}^{n} \left(\frac{-s}{r} \right)^{j-k} a_j \text{ exists.}$$

This shows that $\{\widehat{\ell}(p)\}^{\beta} = d_4^{rs}(p) \cap d_5^{rs}$. □

Theorem 3.6. *The following statements hold:*

- *(i)* $\{\widehat{\ell}(p)\}^{\gamma} := d_4^{rs}(p)$, $(0 < p_k \le 1)$.
- *(ii)* $\{\widehat{\ell}(p)\}^{\gamma} := d_3^{rs}(p)$, $(1 < p_k \le H < \infty)$.

Proof. (ii) We see from (3.8) that $ax = (a_k x_k) \in bs$ whenever $x = (x_k) \in \widehat{\ell}(p)$ if and only if $Dy \in \ell_\infty$ whenever $y = (y_k) \in \ell(p)$, where $D = \{d_{nk}(r,s)\}$ is defined by (3.9). Therefore, we obtain from Part (ii) of Lemma 3.1 that $\{\widehat{\ell}(p)\}^{\gamma} = d_3^{rs}(p)$ and this completes the proof. □

4 Matrix Transformations on the Sequence Space $\widehat{\ell}(p)$

In this section, we characterize some matrix transformations on the space $\widehat{\ell}(p)$. Theorem 4.1 gives the exact conditions of the general case $0 < p_k \le H < \infty$ by combining the cases $0 < p_k \le 1$ and $1 < p_k \le H < \infty$. We consider only the case $1 < p_k \le H < \infty$ and leave the case $0 < p_k \le 1$ to the reader because of it can be proved in the similar way.

We write for brevity that

$$\tilde{a}_{nk} = \sum_{j=k}^{\infty} \frac{1}{r} \left(\frac{-s}{r} \right)^{j-k} a_{nj} \quad \text{for all} \quad k, n \in \mathbb{N}.$$

Theorem 4.1. *Let $A = (a_{nk})$ be an infinite matrix. Then, the following statements hold:*

(i) Let $0 < p_k \le 1$ for all $k \in \mathbb{N}$. Then, $A \in (\widehat{\ell}(p) : \ell_\infty)$ if and only if

$$\{a_{nk}\}_{k\in\mathbb{N}} \in d_3^{rs}(p) \cap d_5^{rs}(p) \quad \textit{for each fixed} \quad n \in \mathbb{N}, \tag{4.1}$$

$$\sup_{n,k\in\mathbb{N}} |\tilde{a}_{nk}|^{p_k} < \infty. \tag{4.2}$$

(ii) Let $1 < p_k \leq H < \infty$ *for all* $k \in \mathbb{N}$. *Then,* $A \in (\widehat{\ell}(p) : \ell_\infty)$ *if and only if there exists an integer* $M > 1$ *such that*

$$\{a_{nk}\}_{k\in\mathbb{N}} \in d_4^{rs}(p) \cap d_5^{rs}(p) \quad \textit{for each fixed} \quad n \in \mathbb{N}, \tag{4.3}$$

$$C(M) = \sup_{n\in\mathbb{N}} \sum_k \left|\tilde{a}_{nk} M^{-1}\right|^{p'_k} < \infty. \tag{4.4}$$

Proof. (ii) Suppose that the conditions (4.3) and (4.4) hold, and $x \in \widehat{\ell}(p)$. In this situation, since $\{a_{nk}\}_{k\in\mathbb{N}} \in \{\widehat{\ell}(p)\}^\beta$ for every fixed $n \in \mathbb{N}$, the A-transform of x exists. Consider the following equality obtained by using the relation (2.2) that

$$\sum_{k=0}^m a_{nk}x_k = \sum_{k=0}^m \sum_{j=k}^m \frac{1}{r}\left(\frac{-s}{r}\right)^{j-k} a_{nj}y_k \tag{4.5}$$

for all $m, n \in \mathbb{N}$. Taking into account the hypothesis we derive from (4.5) as $m \to \infty$ that

$$\sum_k a_{nk}x_k = \sum_k \tilde{a}_{nk}y_k \quad \text{for each} \quad n \in \mathbb{N}. \tag{4.6}$$

Now, by combining (4.6) with the following inequality which holds for any $M > 0$ and any $a, b \in \mathbb{C}$

$$|ab| \leq M\left(\left|aM^{-1}\right|^{p'} + |b|^p\right),$$

where $p > 1$ and $p^{-1} + p'^{-1} = 1$ (see [21]), one can easily see that

$$\begin{aligned} \sup_{n\in\mathbb{N}}\left|\sum_k a_{nk}x_k\right| &\leq \sup_{n\in\mathbb{N}}\sum_k |\tilde{a}_{nk}|\,|y_k| \\ &\leq M[C(M) + g_1^L(y)] < \infty. \end{aligned}$$

Conversely, suppose that $A \in (\widehat{\ell}(p) : \ell_\infty)$ and $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then Ax exists for every $x \in \widehat{\ell}(p)$ and this implies that $\{a_{nk}\}_{k\in\mathbb{N}} \in \{\widehat{\ell}(p)\}^\beta$ for all $n \in \mathbb{N}$. Now, the necessity of (4.3) is immediate. Besides, we have from (4.6) that the matrix $E = (e_{nk})$ defined by $e_{nk} = \tilde{a}_{nk}$ for all $n, k \in \mathbb{N}$, is in the class $(\ell(p) : \ell_\infty)$. Then, E satisfies the condition (3.3) which is equivalent to (4.4).

This completes the proof. □

Theorem 4.2. *Let* $0 < p_k \leq H < \infty$ *for all* $k \in \mathbb{N}$. *Then,* $A \in (\widehat{\ell}(p) : c)$ *if and only if (4.1)-(4.4) hold and*

$$\lim_{n\to\infty} \tilde{a}_{nk} = \alpha_k \quad \textit{for each fixed} \quad k \in \mathbb{N}. \tag{4.7}$$

Proof. Let $A \in (\widehat{\ell}(p) : c)$ and $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, since the inclusion $c \subset \ell_\infty$ holds, the necessities of (4.3) and (4.4) are immediately obtained from Part (i) of Theorem 4.1.

To prove the necessity of (4.7), consider the sequence $b^{(k)}(r,s)$ defined by (2.7) which is in the space $\widehat{\ell}(p)$ for every fixed $k \in \mathbb{N}$. Because of the A-transform of every $x \in \widehat{\ell}(p)$ exists and is in c by the hypothesis,

$$Ab^{(k)}(r,s) = \left\{ \sum_{j=k}^{\infty} \frac{1}{r} \left(\frac{-s}{r} \right)^{j-k} a_{nj} \right\}_{n\in\mathbb{N}} \in c$$

for every fixed $k \in \mathbb{N}$ which shows the necessity of (4.7).

Conversely suppose that the conditions (4.3), (4.4) and (4.7) hold, and take any $x = (x_k)$ in the space $\widehat{\ell}(p)$. Then, Ax exists. We observe for all $m, n \in \mathbb{N}$ that

$$\sum_{k=0}^{m} \left| \sum_{j=k}^{m} \frac{1}{r} \left(\frac{-s}{r} \right)^{j-k} a_{nj} M^{-1} \right|^{p'_k} \leq \sup_{n\in\mathbb{N}} \sum_{k=0}^{m} \left| \sum_{j=k}^{m} \frac{1}{r} \left(\frac{-s}{r} \right)^{j-k} a_{nj} M^{-1} \right|^{p'_k}$$

which gives the fact by letting $m, n \to \infty$ with (4.4) and (4.7) that

$$\lim_{m,n\to\infty} \sum_{k=0}^{m} \left| \sum_{j=k}^{m} \frac{1}{r} \left(\frac{-s}{r} \right)^{j-k} a_{nj} M^{-1} \right|^{p'_k} \leq \sup_{n\in\mathbb{N}} \sum_{k=0}^{m} \left| \sum_{j=k}^{m} \frac{1}{r} \left(\frac{-s}{r} \right)^{j-k} a_{nj} M^{-1} \right|^{p'_k} < \infty.$$

This shows that $\sum_k |\alpha_k M^{-1}|^{p'_k} < \infty$ and so $(\alpha_k)_{k\in\mathbb{N}} \in \{\widehat{\ell}(p)\}^\beta$ for each $n \in \mathbb{N}$ which implies that the series $\sum_k \alpha_k x_k$ converges for every $x \in \widehat{\ell}(p)$.

Let us now consider the equality obtained from (4.6) with $a_{nk} - \alpha_k$ instead of a_{nk}

$$\sum_k (a_{nk} - \alpha_k) x_k = \sum_k e_{nk} y_k \ \text{ for all } \ n \in \mathbb{N}, \tag{4.8}$$

where $E = (e_{nk})$ defined by $e_{nk} = \tilde{a}_{nk} - \alpha_k$ for all $k, n \in \mathbb{N}$. Therefore, we have at this stage from Lemma 3.2 with $\beta_k = 0$ for all $k \in \mathbb{N}$ that the matrix E belongs to the class $(\ell(p) : c_0)$ of infinite matrices. Thus, we see by (4.8) that

$$\lim_{n\to\infty} \sum_k (a_{nk} - \alpha_k) x_k = 0. \tag{4.9}$$

(4.9) means that $Ax \in c$ whenever $x \in \widehat{\ell}(p)$ and this is what we wished to prove. □

Therefore, we have:

Corollary 4.3. *Let* $0 < p_k \leq H < \infty$ *for all* $k \in \mathbb{N}$. *Then,* $A \in (\widehat{\ell}(p) : c_0)$ *if and only if (4.1)-(4.4) hold, and (4.7) also holds with* $\alpha_k = 0$ *for all* $k \in \mathbb{N}$.

Now, we give the following lemma given by Başar and Altay [23] which is useful for deriving the characterizations of the certain matrix classes via Theorems 4.1, 4.2 and Corollary 4.3:

Lemma 4.4 (Başar and Altay [23, Lemma 5.3])**.** *Let* λ, μ *be any two sequence spaces,* A *be an infinite matrix and* B *also be a triangle matrix. Then,* $A \in (\lambda : \mu_B)$ *if and only if* $BA \in (\lambda : \mu)$*.*

It is trivial that Lemma 4.4 has several consequences. Indeed, combining Lemma 4.4 with Theorems 4.1, 4.2 and Corollary 4.3, one can derive the following results:

Corollary 4.5. *Let* $A = (a_{nk})$ *be an infinite matrix and define the matrix* $C = (c_{nk})$ *by*

$$c_{nk} = \sum_{j=0}^{n} \binom{n}{j} (1-t)^{n-j} t^j a_{jk} \quad \text{for all} \quad n, k \in \mathbb{N}.$$

Then, the necessary and sufficient conditions in order for A *belongs to anyone of the classes* $(\widehat{\ell}(p) : e_\infty^t)$*,* $(\widehat{\ell}(p) : e_c^t)$ *and* $(\widehat{\ell}(p) : e_0^t)$ *are obtained from the respective ones in Theorems 4.1, 4.2 and Corollary 4.3 by replacing the entries of the matrix* A *by those of the matrix* C*; where* $0 < t < 1$*,* e_∞^t *and* e_c^t*,* e_0^t *respectively denote the spaces of all sequences whose* E^t*-transforms are in the spaces* ℓ_∞ *and* c*,* c_0 *and are recently studied by Altay et al. [24], Altay and Başar [25], where* E^t *denotes the Euler mean of order* t*.*

Corollary 4.6. *Let* $A = (a_{nk})$ *be an infinite matrix and* $t = (t_k)$ *be a sequence of positive numbers and define the matrix* $C = (c_{nk})$ *by*

$$c_{nk} = \frac{1}{T_n} \sum_{j=0}^{n} t_j a_{jk} \quad \text{for all} \quad n, k \in \mathbb{N},$$

where $T_n = \sum_{k=0}^{n} t_k$ *for all* $n \in \mathbb{N}$*. Then, the necessary and sufficient conditions in order for* A *belongs to anyone of the classes* $(\widehat{\ell}(p) : r_\infty^t)$*,* $(\widehat{\ell}(p) : r_c^t)$ *and* $(\widehat{\ell}(p) : r_0^t)$ *are obtained from the respective ones in Theorems 4.1, 4.2 and Corollary 4.3 by replacing the entries of the matrix* A *by those of the matrix* C*; where* r_∞^t*,* r_c^t *and* r_0^t *are defined by Altay and Başar in [26] as the spaces of all sequences whose* R^t*-transforms are respectively in the spaces* ℓ_∞*,* c *and* c_0*, and are derived from the paranormed spaces* $r_\infty^t(p)$*,* $r_c^t(p)$ *and* $r_0^t(p)$ *in the case* $p_k = p$ *for all* $k \in \mathbb{N}$*.*

Since the spaces r_∞^t, r_c^t and r_0^t reduce in the case $t = e$ to the Cesàro sequence spaces X_∞, $\tilde{c}$ and $\tilde{c}_0$ of non-absolute type, respectively, Corollary 4.6 also includes the characterizations of the classes $(\widehat{\ell}(p) : X_\infty)$, $(\widehat{\ell}(p) : \tilde{c})$ and $(\widehat{\ell}(p) : \tilde{c}_0)$, as a special case; where X_∞ and $\tilde{c}$, $\tilde{c}_0$ are the Cesàro spaces of the sequences consisting of C_1-transforms are in the spaces ℓ_∞ and c, c_0, and studied by Ng and Lee [27]; Şengönül and Başar [28], respectively, where C_1 denotes the Cesàro mean of order 1.

Corollary 4.7. *Let* $A = (a_{nk})$ *be an infinite matrix and define the matrix* $C = (c_{nk})$ *by* $c_{nk} = a_{nk} - a_{n+1,k}$ *for all* $n, k \in \mathbb{N}$*. Then, the necessary and sufficient conditions in order for* A *belongs to anyone of the classes* $(\widehat{\ell}(p) : \ell_\infty(\Delta))$, $(\widehat{\ell}(p) : c(\Delta))$ *and* $(\widehat{\ell}(p) : c_0(\Delta))$ *are obtained from the respective ones in Theorems 4.1, 4.2 and Corollary 4.3 by replacing the entries of the matrix* A *by those of the matrix* C*; where* $\ell_\infty(\Delta)$, $c(\Delta)$, $c_0(\Delta)$ *denote the difference spaces of all bounded, convergent, null sequences and are introduced by Kızmaz [29].*

Corollary 4.8. *Let* $A = (a_{nk})$ *be an infinite matrix and define the matrix* $C = (c_{nk})$ *by* $c_{nk} = \sum_{j=0}^{n} a_{jk}$ *for all* $n, k \in \mathbb{N}$*. Then the necessary and sufficient conditions in order for* A *belongs to anyone of the classes* $(\widehat{\ell}(p) : bs)$, $(\widehat{\ell}(p) : cs)$ *and* $(\widehat{\ell}(p) : cs_0)$ *are obtained from the respective ones in Theorems 4.1, 4.2 and Corollary 4.3 by replacing the entries of the matrix* A *by those of the matrix* C*, where* cs_0 *denotes the set of those series converging to zero.*

5 The Rotundity of the Space $\widehat{\ell}(p)$

Among many geometric properties, the rotundity of Banach spaces is one of the most important topics in functional analysis. For details, the reader may refer to [30–32]. In this section, we characterize the rotundity of the space $\widehat{\ell}(p)$ and emphasize some results related to this concept.

By $S(X)$ and $B(X)$, we denote the unit sphere and unit ball of a Banach space X, respectively. A point $x \in S(X)$ is called an *extreme point* if $2x = y + z$ implies $y = z$ for all $y, z \in S(X)$.

A Banach space X is said to be *rotund* (*strictly convex*) if every point of $S(X)$ is an extreme point.

Let X be a real vector space. A functional $\sigma : X \to [0, \infty)$ is called a *modular* if

(i) $\sigma(x) = 0$ if and only if $x = \theta$,

(ii) $\sigma(\alpha x) = \sigma(x)$ for all scalars α with $|\alpha| = 1$;

(iii) $\sigma(\alpha x + \beta y) \le \sigma(x) + \sigma(y)$ for all $x, y \in X$ and $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$.

(iv) The modular σ is called *convex* if $\sigma(\alpha x + \beta y) \le \alpha\sigma(x) + \beta\sigma(y)$ for all $x, y \in X$ and $\alpha, \beta > 0$ with $\alpha + \beta = 1$.

A modular σ on X is called

(a) *Right continuous* if $\lim_{\alpha \to 1^+} \sigma(\alpha x) = \sigma(x)$ for all $x \in X_\sigma$.

(b) *Left continuous* if $\lim_{\alpha \to 1^-} \sigma(\alpha x) = \sigma(x)$ for all $x \in X_\sigma$.

(c) *Continuous* if it is both right and left continuous, where

$$X_\sigma = \left\{ x \in X : \lim_{\alpha \to 0^+} \sigma(\alpha x) = 0 \right\}.$$

For $\widehat{\ell}(p)$, we define $\sigma_p(x) = \sum_k |sx_{k-1} + rx_k|^{p_k}$. If $p_k \geq 1$ for all $k \in \mathbb{N}$, by the convexity of the function $t \mapsto |t|^{p_k}$ for each $k \in \mathbb{N}$, one can easily observe that σ_p is a convex modular on the sequence space $\widehat{\ell}(p)$. We consider the sequence space $\widehat{\ell}(p)$ equipped with the Luxemburg norm given by

$$\|x\| = \inf\left\{\alpha > 0 : \sigma_p\left(\frac{x}{\alpha}\right) \leq 1\right\}. \tag{5.1}$$

It is easy to show that the sequence space $\widehat{\ell}(p)$ is a Banach space with the norm (5.1).

Now, we may emphasize some basic properties for modular σ_p.

Theorem 5.1. *The modular σ_p on the sequence space $\widehat{\ell}(p)$ satisfies the following properties:*

- *(i) If $0 < \alpha \leq 1$, then $\alpha^L \sigma_p(x/\alpha) \leq \sigma_p(x)$ and $\sigma_p(\alpha x) \leq \alpha\sigma_p(x)$.*
- *(ii) If $\alpha \geq 1$, then $\sigma_p(x) \leq \alpha^L \sigma_p(x/\alpha)$.*
- *(iii) If $\alpha \geq 1$, then $\sigma_p(x) \geq \alpha\sigma_p(x/\alpha)$.*
- *(iv) The modular σ_p is continuous on the sequence space $\widehat{\ell}(p)$.*

Proof. (i) We have for any $x \in \widehat{\ell}(p)$ and $\alpha \in (0,1]$ that

$$\begin{aligned}\sigma_p(x) &= \sum_k |sx_{k-1} + rx_k|^{p_k} \\ &= \sum_k \left|\frac{\alpha(sx_{k-1} + rx_k)}{\alpha}\right|^{p_k} \\ &\geq \alpha^L \sum_k \left|\frac{sx_{k-1} + rx_k}{\alpha}\right|^{p_k} = \alpha^L \sigma_p\left(\frac{x}{\alpha}\right).\end{aligned}$$

Since $p_k \geq 1$ for all k and $0 < \alpha \leq 1$, we have $\alpha^{p_k} \leq \alpha$ for all k, hence $\sigma_p(\alpha x) \leq \alpha\sigma_p(x)$.

(ii) If $\alpha \geq 1$, then $1/\alpha \leq 1$. From (i), we have

$$\left(\frac{1}{\alpha}\right)^L \sigma_p(x) = \left(\frac{1}{\alpha}\right)^L \sigma_p\left(\frac{x/\alpha}{1/\alpha}\right) \leq \sigma_p\left(\frac{x}{\alpha}\right)$$

and hence $\sigma_p(x) \leq \alpha^L \sigma_p(x/\alpha)$.

(iii) If we apply the second part of (i) with $\beta = 1/\alpha \leq 1$, then it is immediate that

$$\alpha\sigma_p\left(\frac{x}{\alpha}\right) = \alpha\sigma_p(\beta x) \leq \alpha\beta\sigma_p(x) = \sigma_p(x),$$

as expected.

(iv) By Parts (ii) and (iii) of the present theorem, we have for $\alpha > 1$ that

$$\sigma_p(x) \leq \alpha\sigma_p(x) \leq \sigma_p(\alpha x) \leq \alpha^L \sigma_p(x). \tag{5.2}$$

By passing to limit as $\alpha \to 1^+$ in (5.2), we have $\lim_{\alpha\to 1^+} \sigma_p(\alpha x) = \sigma_p(x)$. Hence, σ_p is right continuous. If $0 < \alpha < 1$, by Part (i) of the present theorem, we have

$$\alpha^L \sigma_p(x) \leq \sigma_p(\alpha x) \leq \alpha\sigma_p(x). \tag{5.3}$$

Also by letting $\alpha \to 1^-$ in (5.3), we observe that $\lim_{\alpha\to 1^-} \sigma_p(\alpha x) = \sigma_p(x)$ and hence σ_p is left continuous. These two consequences give us the desired fact that σ_p is continuous. $\square$

Now, we may give some relationships between the modular σ_p and the Luxemburg norm on the sequence space $\widehat{\ell}(p)$.

Theorem 5.2. *Let $x \in \widehat{\ell}(p)$. Then, the following statements hold:*

(i) If $\|x\| < 1$, then $\sigma_p(x) \leq \|x\|$.

(ii) If $\|x\| > 1$, then $\sigma_p(x) \geq \|x\|$.

(iii) $\|x\| = 1$ if and only if $\sigma_p(x) = 1$.

(iv) $\|x\| < 1$ if and only if $\sigma_p(x) < 1$.

(v) $\|x\| > 1$ if and only if $\sigma_p(x) > 1$.

Proof. (i) Let $\varepsilon > 0$ such that $0 < \varepsilon < 1 - \|x\|$. By the definition of $\|\cdot\|$, there exists an $\alpha > 0$ such that $\|x\| + \varepsilon > \alpha$ and $\sigma_p(x/\alpha) \leq 1$. From Parts (i) and (ii) of Theorem 5.1, we have

$$\sigma_p(x) \leq \sigma_p\left[(\|x\| + \varepsilon)\frac{x}{\alpha}\right] \leq (\|x\| + \varepsilon)\sigma_p\left(\frac{x}{\alpha}\right) \leq \|x\| + \varepsilon.$$

Since ε is arbitrary, we have (i).

(ii) If we choose $\varepsilon > 0$ such that $0 < \varepsilon < 1 - 1/\|x\|$, then $1 < (1-\varepsilon)\|x\| < \|x\|$. Combining the definition of the Luxemburg norm given by (5.1) and Part (i) of Theorem 5.1, we have

$$1 < \sigma_p\left[\frac{x}{(1-\varepsilon)\|x\|}\right] \leq \frac{1}{(1-\varepsilon)\|x\|}\sigma_p(x),$$

so $(1-\varepsilon)\|x\| < \sigma_p(x)$ for all $\varepsilon \in (0, 1 - 1/\|x\|)$. This implies that $\|x\| < \sigma_p(x)$.

Since σ_p is continuous, (iii) directly follows from Theorem 1.4 of [32].

(iv) follows from Parts (i) and (iii).

(v) follows from Parts (ii) and (iii). $\square$

Theorem 5.3. *The space $\widehat{\ell}(p)$ is rotund if and only if $p_k > 1$ for all $k \in \mathbb{N}$.*

Proof. Necessity. Let $\widehat{\ell}(p)$ be rotund and choose $k \in \mathbb{N}$ such that $p_k = 1$ for $k < 3$. Consider the following sequences given by

$$\begin{aligned} x &= \left\{0, \frac{1}{r}, \frac{-s}{r^2}, \frac{(-s)^2}{r^3}, \frac{(-s)^3}{r^4}, \dots\right\}, \\ z &= \left\{0, 0, \frac{1}{r}, \frac{-s}{r^2}, \frac{(-s)^2}{r^3}, \dots\right\}. \end{aligned}$$

Then, it is immediate that $x \neq z$ and

$$\sigma_p(x) = \sigma_p(z) = \sigma_p\left(\frac{x+z}{2}\right) = 1.$$

By Part (iii) of Theorem 5.2; x, z, $(x+z)/2 \in S[\widehat{\ell}(p)]$ which leads us the contradiction that the sequence space $\widehat{\ell}(p)$ is not rotund.

Sufficiency. Let $x \in S[\widehat{\ell}(p)]$ and $v, z \in S[\widehat{\ell}(p)]$ with $x = (v+z)/2$. By convexity of σ_p and Part (iii) of Theorem 5.2, we have

$$1 = \sigma_p(x) \leq \frac{\sigma_p(v) + \sigma_p(z)}{2} \leq \frac{1}{2} + \frac{1}{2} = 1$$

which gives that $\sigma_p(v) = \sigma_p(z) = 1$ and

$$\sigma_p(x) = \frac{\sigma_p(v) + \sigma_p(z)}{2}. \tag{5.4}$$

Further, we have by (5.4) that

$$\sum_k |sx_{k-1} + rx_k|^{p_k} = \frac{1}{2}\left[\sum_k |sv_{k-1} + rv_k|^{p_k}\right] + \frac{1}{2}\left[\sum_k |sz_{k-1} + rz_k|^{p_k}\right].$$

Since $x = (v+z)/2$, we have

$$\begin{aligned} \sum_k \left|\frac{1}{2}\left[s(v_{k-1} + z_{k-1}) + r(v_k + z_k)\right]\right|^{p_k} = \frac{1}{2}&\left(\sum_k |sv_{k-1} + rv_k|^{p_k}\right) \\ &+ \frac{1}{2}\left(\sum_k |sz_{k-1} + rz_k|^{p_k}\right). \end{aligned}$$

This implies that

$$\left|\frac{1}{2}\left[s(v_{k-1} + z_{k-1}) + r(v_k + z_k)\right]\right|^{p_k} = \frac{1}{2}|sv_{k-1} + rv_k|^{p_k} + \frac{1}{2}|sz_{k-1} + rz_k|^{p_k} \tag{5.5}$$

for all $k \in \mathbb{N}$. Since the function $t \mapsto |t|^{p_k}$ is strictly convex for all $k \in \mathbb{N}$, it follows by (5.5) that $v_k = z_k$ for all $k \in \mathbb{N}$. Hence $v = z$, that is the sequence space $\widehat{\ell}(p)$ is rotund. □

Conclusion

The diference spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ were introduced by Kızmaz [29]. We treat more different than Kızmaz and the other authors following him, and use the technique for obtaining a new sequence space by the domain of a triangle matrix. Following this way, the domain of some triangle matrices in the sequence space $\ell(p)$ was recently studied and obtained certain topological and geometric results by Altay and Başar [9, 14]; Choudhary and Mishra [5]; Başar et al. [33]; Aydın and Başar [8]. Although $bv(e,p) = [\ell(p)]_\Delta$ is investigated, since $B(1,-1) \equiv \Delta$, our results are more general than those of Başar, Altay and Mursaleen [33]. Also in case $p_k = p$ for all $k \in \mathbb{N}$ the results of the present study are reduced to the corresponding results of the recent paper of Kirişçi and Başar [4].

Acknowledgements : We would like to thank Professor Bilâl Altay, Department of Mathematical Education, Faculty of Education, İnönü University, 44280 Malatya, Turkey, for his careful reading and constructive criticism of an earlier version of this paper which improved the presentation and its readability. We have benefited a lot from the constructive reports of the anonymous referees. So, we are thankful for their valuable comments on the first draft of this paper which improved the presentation and readability.

References

[1] I.J. Maddox, Spaces of strongly summable sequences, Quart. J. Math. Oxford 18 (2) (1967) 345–355.

[2] S. Simons, The sequence spaces $\ell(p_v)$ and $m(p_v)$, Proc. London Math. Soc. 15 (3) (1965) 422–436.

[3] H. Nakano, Modulared sequence spaces, Proc. Japan Acad. 27 (2) (1951) 508–512.

[4] M. Kirişçi, F. Başar, Some new sequence spaces derived by the domain of generalized difference matrix, Comput. Math. Appl. 60 (5) (2010) 1299–1309.

[5] B. Choudhary, S.K. Mishra, On Köthe-Toeplitz duals of certain sequence spaces and their matrix transformations, Indian J. Pure Appl. Math. 24 (5) (1993) 291–301.

[6] F. Başar, B. Altay, Matrix mappings on the space $bs(p)$ and its α-, β- and γ-duals, Aligarh Bull. Math. 21 (1) (2002) 79–91.

[7] F. Başar, Infinite matrices and almost boundedness, Boll. Un. Mat. Ital. (7) 6 (3) (1992) 395–402.

[8] C. Aydın, F. Başar, Some generalisations of the sequence space a_p^r, Iran. J. Sci. Technol. Trans. A, Sci. 30 (No. A2) (2006) 175–190.

[9] B. Altay, F. Başar, On the Paranormed Riesz sequence spaces of non-absolute type, Southeast Asian Bull. Math. 26 (5) (2002) 701–715.

[10] M. Mursaleen, Generalized spaces of difference sequences, J. Math. Anal. Appl. 203 (3) (1996) 738–745.

[11] E. Malkowsky, M. Mursaleen, S. Suantai, The dual spaces of sets of difference sequences of order m and matrix transformations, Acta Math. Sin. Eng. Ser. 23 (3) (2007) 521–532.

[12] R. Çolak, Y. Altın, Mursaalen, On some sets of difference sequences of fuzzy numbers, Soft Comput. 15 (2011) 787–793.

[13] B. Altay, F. Başar, Some paranormed sequence spaces of non-absolute type derived by weighted mean, J. Math. Anal. Appl. 319 (2) (2006) 494–508.

[14] B. Altay, F. Başar, Generalisation of the sequence space $\ell(p)$ derived by weighted mean, J. Math. Anal. Appl. 330 (1) (2007) 174–185.

[15] B. Altay, F. Başar, Certain topological properties and duals of the matrix domain of a triangle matrix in a sequence space, J. Math. Anal. Appl. 336 (1) (2007) 632–645.

[16] C. Aydın, F. Başar, Some new paranormed sequence spaces, Inform. Sci. 160 (1-4) (2004) 27–40.

[17] I.J. Maddox, *Elements of Functional Analysis*, The University Press, 2^{nd} ed., Cambridge, 1988.

[18] I.J. Maddox, Paranormed sequence spaces generated by infinite matrices, Proc. Camb. Phil. Soc. 64 (1968) 335–340.

[19] A. Wilansky, *Summability through Functional Analysis*, Nort-Holland Mathematics Studies, 85, Amsterdam-New York-Oxford, 1984.

[20] A.M. Jarrah, E. Malkowsky, BK spaces, bases and linear operators, Rendiconti Circ. Mat. Palermo II 52 (1990) 177–191.

[21] C.G. Lascarides, I.J. Maddox, Matrix transformations between some classes of sequences, Proc. Camb. Phil. Soc. 68 (1970) 99–104.

[22] K.-G. Grosse-Erdmann, Matrix transformations between the sequence spaces of Maddox, J. Math. Anal. Appl. 180 (1993) 223–238.

[23] F. Başar, B. Altay, On the space of sequences of p-bounded variation and related matrix mappings, Ukrainian Math. J. 55 (1) (2003) 136–147.

[24] B. Altay, F. Başar, M. Mursaleen, On the Euler sequence spaces which include the spaces ℓ_p and ℓ_∞ I, Inform. Sci. 176 (10) (2006) 1450–1462.

[25] B. Altay, F. Başar, Some Euler sequence spaces of non-absolute type, Ukrainian Math. J. 57 (1) (2005) 1–17.

[26] B. Altay, F. Başar, Some paranormed Riezs sequence spaces of non-absolute type, Southeast Asian Bull. Math. 30 (5) (2006) 591–608.

[27] P.-N. Ng, P.-Y. Lee, Cesàro sequence spaces of non-absolute type, Comment. Math. Prace Mat. 20 (2) (1978) 429–433.

[28] M. Şengönül, F. Başar, Some new Cesàro sequence spaces of non-absolute type which include the spaces c_0 and c, Soochow J. Math. 31 (1) (2005) 107–119.

[29] H. Kızmaz, On certain sequence spaces, Canad. Math. Bull. 24 (2) (1981) 169–176.

[30] S. Chen, Geometry of Orlicz spaces, Dissertationes Math. 356 (1996) 1–224.

[31] J. Diestel, *Geometry of Banach Spaces- Selected Topics*, Springer-Verlag, 1984.

[32] L. Maligranda, *Orlicz Spaces and Interpolation*, Inst. Math. Polish Academy of Sciences, Poznan, 1985.

[33] F. Başar, B. Altay, M. Mursaleen, Some generalizations of the space bv_p of p-bounded variation sequences, Nonlinear Anal. 68 (2) (2008) 273–287.

(Received 22 February 2012)
(Accepted 24 September 2012)