Thai Journal of Mathematics
Volume 12 (2014) Number 1 : 103–112
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

A Common Fixed Point Theorem for Gregus Type Mappings

S.D. Diwan[1] and Rajlaxmi Gupta

Department of Applied Mathematics
Shri Shankaracharya College of Engg. and Tech.
Junwani, Bhilai, 490020, India
e-mail : sdd11@rediffmail.com (S.D. Diwan)
rajlaxmi28476@gmail.com (R. Gupta)

Abstract : In this paper, we establish a common fixed point theorem for Gregus type mappings in metric spaces using the (CLRg) property. Our result generalizes and improves upon, among others the corresponding results of Gregus [1], Fisher and Sessa [2], Huang and Cho [3], Jungck[4], Ciric [5] and Mukherjee and Verma [6].

Keywords : Gregus type mappings; weakly compatible mappings; (CLRg) property; property (E.A); metric space.
2010 Mathematics Subject Classification : 54H25; 47H10.

1 Introduction

Generalizing the concept of commuting mappings, Sessa [7] introduced concept of weakly commuting mappings. Later on, Jungck [8] enlarged the class of non-commuting mappings by compatible mappings. Further generalizations of compatible mappings are given by Jungck et al. [9], Pathak and Khan [10] and Pathak et al. [11], Cho et al. [12]. However, the study of common fixed points of noncompatible mappings are also very interesting. Work along these lines has been initiated by Pant [13, 14].

On the other hand Gregus [1] proved a fixed point theorem in Banach spaces, known as Gregus fixed point theorem and then many authors have obtained fixed

[1]Corresponding author.

point theorems of Gregus type (See [2, 3, 6, 10, 15, 16]).

Huang and Cho [3] proved the following common fixed point theorems as a generalization of Gregus's [1] result for compatible mappings in complete convex metric spaces:

Theorem 1.1 ([3]). *Let X be a complete convex metric space with a convex structure W and C is a nonempty closed convex subset of X. Let f and g be compatible mappings of C into itself satisfying the condition*

$$d^p(fx, fy) \leq ad^p(gx, gy) + b\max\{d^p(fx, gx), d^p(fy, gy)\} + c\max\{d^p(gx, gy), d^p(fx, gx), d^p(fy, gy)\} \tag{B}$$

for all x, y in C, where $a, b, c > 0, p \geq 1,\ a + b + c = 1$ and $\max\left\{\frac{(1-b)^2}{a}b + c\right\} < (2 - 2^{1-p})(2^p - 1)^{-1}$. If g is w-affine and continuous in C and $f(C) \subset g(C)$, then f and g have a unique common fixed point z in C and f is continuous at z.

In 2002, Amari and Moutawakil [17] defined the notion of property (E.A) which contains the class of non-compatible mappings. It is observed that property (E.A) requires the completeness (or closedness) of subspaces for existence of the fixed point. Recently, Sintunavarat and Kuman [18] defined the notion of (CLRg) property. They showed that (CLRg) property never requires completeness (or closedness) of subspaces (also see [19, 20]).

The purpose of this paper is to prove existence of common fixed points for Gregus type mappings in metric spaces using the (CLRg) property. Our result is more general, it extends various known results from Banach spaces to general metric spaces.

2 Preliminaries

Sessa [7] introduced the notion of weak commutativity:

Definition 2.1 ([7]). Two self-mappings f and g of a metric space (X, d) are said to be *weakly commuting* if

$$d(fgx, gfx) \leq d(fx, gx), \quad \text{for all } x \in X.$$

It is clear that two commuting mappings are weakly commuting but the converse is not true as is shown in [7].

Definition 2.2 ([8]). Two self-mappings f and g of a metric space (X, d) are said to be *compatible* if

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t,$$

for some $t \in X$.

Obviously, two weakly commuting mappings are compatible, but the converse is not true as shown in [8].

Definition 2.3 ([21])**.** Two self-mappings f and g of a metric space (X, d) are said to be *weakly compatible* if they commute at their coincidence points, i.e. if $fu = gu$ for some $u \in X$, then $fgu = gfu$.

It is easy to see that two compatible mappings are weakly compatible.

Definition 2.4 ([17])**.** Two self-mappings f and g of a metric space (X, d) are said to satisfy *the property* $(E.A)$ if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t,$$

for some $t \in X$.

Definition 2.5 ([18])**.** Two self-mappings f and g of a metric space (X, d) are said to satisfy *the common limit in the range of g property* if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gu,$$

for some $u \in X$.

In what follows, the common limit in the range of g property will be denoted by the (CLRg) property.

Now, we give examples of mappings f and g which are satisfying the (CLRg) property.

Example 2.6. *Let $X = [0, \infty)$ with the usual metric on X. Define $f, g : X \to X$ by $fx = x/2$ and $gx = 2x$ for all $x \in X$. Consider the sequence $\{x_n\} = \{1/n\}$. Since*

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 0 = g0,$$

therefore f and g satisfy the (CLRg) property.

Example 2.7. *Let $X = [0, \infty)$ with the usual metric on X. Define $f, g : X \to X$ by $fx = x+2$ and $gx = 3x$ for all $x \in X$. Consider the sequence $\{x_n\} = \{1+1/n\}$. Since*

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 3 = g1,$$

therefore f and g satisfy the (CLRg) property.

Remark 2.8. *It is clear from the Jungck's definition [8] that two self-mappings f and g of a metric space (X, d) will be non-compatible if there exists atleast one sequence $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t, \text{ for some } t \in X,$$

but $\lim_{n\to\infty} d(fgx_n, gfx_n)$ is either non-zero or non-existent. Therefore, two non-compatible self-mappings of a metric space (X, d) satisfy the property $(E.A)$.

3 Main Results

Theorem 3.1. *Let f and g be two weakly compatible self-mappings of a metric space (X,d) such that*

(i) f and g satisfy the (CLRg) property,

(ii)

$$\begin{aligned} d^p(fx,fy) \leq ad^p(gx,gy) + b\max\{d^p(fx,gx), d^p(fy,gy)\} \\ + c\max\{d^p(gx,gy), d^p(fx,gx), d^p(fy,gy)\} \end{aligned}$$

for all $x, y \in X$, where $a, b, c > 0, p \geq 1,\ a+b+c = 1$,

then f and g have a unique common fixed point.

Proof. Since f and g satisfy the (CLRg) property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gu, \quad \text{for some } u \text{ in } X.$$

Now we show that $fu = gu$. Suppose that $fu \neq gu$. Then using condition (ii) with $x = x_n$ *and* $y = u$, we get

$$\begin{aligned} d^p(fx_n, fu) \leq ad^p(gx_n, gu) + b\ \max\{d^p(fx_n, gx_n), d^p(fu, gu)\} \\ + c\ \max\{d^p(gx_n, gu), d^p(fx_n, gx_n), d^p(fu, gu)\}. \end{aligned}$$

Making $n \to \infty$ this yields

$$\begin{aligned} d^p(gu, fu) &\leq ad^p(gu, gu) + b\ \max\{d^p(gu, gu), d^p(fu, gu)\} \\ &\quad + c\ \max\{d^p(gu, gu), d^p(gu, gu), d^p(fu, gu)\} \\ &\leq (b+c)d^p(gu, fu), \end{aligned}$$

which is a contradiction. Hence $fu = gu$.

Since f and g are weakly compatible, $fu = gu$ implies $fgu = gfu$ and therefore $ffu = fgu = gfu$.

Finally, we show that fu is a common fixed point of f and g. Suppose that $fu \neq ffu$. Then using condition (ii) with $x = u$ and $y = fu$, we get

$$\begin{aligned} d^p(fu, ffu) &\leq ad^p(gu, gfu) + b\ \max\{d^p(fu, gu), d^p(ffu, gfu)\} \\ &\quad + c\ \max\{d^p(gu, gfu), d^p(fu, gu), d^p(ffu, gfu)\} \\ &\leq ad^p(fu, ffu) + b\ \max\{d^p(fu, fu), d^p(ffu, ffu)\} \\ &\quad + c\ \max\{d^p(fu, ffu), d^p(fu, fu), d^p(ffu, ffu)\} \\ &\leq (a+c)d^p(fu, ffu), \end{aligned}$$

which is a contradiction. Hence $fu = ffu$ and $gfu = ffu = fu$. Thus fu is a common fixed point of mappings f and g.

Let t and v two common fixed points of mappings f and g. Then using condition (ii), we have

$$\begin{aligned} d^p(t,v) &= d^p(ft,fv) \\ &\leq ad^p(gt,gv) + b\max\{d^p(ft,gt),d^p(fv,gv)\} \\ &\quad + c\max\{d^p(gt,gv),d^p(ft,gt),d^p(fv,gv)\} \\ &= ad^p(t,v) + b\max\{d^p(t,t),d^p(v,v)\} + c\max\{d^p(t,v),d^p(t,t),d^p(v,v)\} \\ &= (a+c)d^p(t,v), \end{aligned}$$

which is a contradiction. Hence $t = v$. Thus f and g have a unique common fixed point in X. $\square$

We now give an example to illustrate Theorem 3.1.

Example 3.2. *Let* $X = [0,2]$ *with the usual metric on* X. *Define* $f, g : X \to X$ *as follows:*

$$f(x) = \begin{cases} 1, & 0 \leq x \leq 1, \\ \frac{5}{4}, & 1 < x \leq 2. \end{cases} \qquad g(x) = \begin{cases} 2-x, & 0 \leq x \leq 1, \\ 0, & 1 < x \leq 2. \end{cases}$$

It is clear that f *and* g *satisfy (CLRg) property. To see this let us consider the sequence* $\{x_n\}$ *given by* $x_n = 1 - \frac{1}{n}$. *Then* $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 1 = g1$. *Also* $f1 = g1 \Rightarrow fg1 = gf1$, *which shows that the* f *and* g *are weakly compatible.*

By a simple calculation one can verify that f *and* g *satisfy the condition* (ii).

Thus all the conditions of Theorem 3.1 are satisfied and 1 is the unique common fixed point of f *and* g.

Remark 3.3. *Theorem 3.1 is more general, so it extends various well known results from Banach space to general metric space. Also from result it is asserted that the (CLRg) property never requires Closedness of subspace, continuity of one or more mappings and containment of range of involved mappings.*

If we take $p = 1$ in Theorem 3.1, we obtain:

Corollary 3.4. *Let* f *and* g *be two weakly compatible self-mappings of a metric space* (X,d) *such that*

(i) S *and* T *satisfy the (CLRg) property,*

(ii)

$$\begin{aligned} d(fx,fy) \leq ad(gx,gy) &+ b\max\{d(fx,gx),d(fy,gy)\} \\ &+ c\max\{d(gx,gy),d(fx,gx),d(fy,gy)\} \end{aligned}$$

for all $x, y \in X$, *where* $a, b, c > 0$, $a + b + c = 1$,

then f *and* g *have a unique common fixed point.*

Corollary 3.5. *Let f and g be two weakly compatible self-mappings of a metric space (X, d) such that*

(i) S and T satisfy the (CLRg) property,

(ii)

$$d(fx, fy) \leq ad(gx, gy) + b\max\{d(fx, gx), d(fy, gy)\}$$

for all $x, y \in X$, where $a, b > 0$, $a + b = 1$,

then f and g have a unique common fixed point.

Next we prove existence of unique common fixed points for a pair of weakly compatible mappings using property $(E.A)$ under additional condition closedness of subspaces.

Theorem 3.6. *Let f and g be two weakly compatible self-mappings of a metric space (X, d) such that*

(i) f and g satisfy the property (E.A),

(ii)

$$d^p(fx, fy) \leq ad^p(gx, gy) + b\max\{d^p(fx, gx), d^p(fy, gy)\} \\ + c\max\{d^p(gx, gy), d^p(fx, gx), d^p(fy, gy)\}$$

for all $x, y \in X$, where $a, b, c > 0, p \geq 1$, $a + b + c = 1$.

If range of g is closed subspace of X, then f and g have a unique common fixed point.

Proof. Since f and g satisfy the property $(E.A)$, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t, \quad \text{for some } t \text{ in } X.$$

Since $g(X)$ is a closed subspace of X there exists $u \in X$ such that $t = gu$. Therefore f and g satisfy the (CLRg) property. It follows from Theorem 3.1 that f and g have a unique common fixed point in X. □

Corollary 3.7. *Let f and g be two weakly compatible self-mappings of a metric space (X, d) such that*

(i) f and g satisfy the property $(E.A)$,

(ii)

$$d(fx, fy) \leq ad(gx, gy) + b\max\{d(fx, gx), d(fy, gy)\} \\ + c\max\{d(gx, gy), d(fx, gx), d(fy, gy)\}$$

for all $x, y \in X$, where $a, b, c > 0$, $a + b + c = 1$, *(iii)* $f(X) \subset g(X)$.

If range of g is a closed subspace of X, then f and g have a unique common fixed point.

Corollary 3.8. *Let f and g be two weakly compatible self-mappings of a metric space (X,d) such that*

(i) f *and* g *satisfy the property* $(E.A)$,

(ii)

$$d(fx,fy) \leq ad(gx,gy) + b\max\{d(fx,gx), d(fy,gy)\}$$

for all $x,y \in X$, *where* $a,b > 0,\ a+b=1$,

(iii) $f(X) \subset g(X)$.

If range of g is a closed subspace of X, then f and g have a unique common fixed point.

Since Property $(E.A)$ contains the class of non-compatible mappings. Following results are direct consequences:

Corollary 3.9. *Let f and g be two weakly compatible non-compatible self-mappings of a metric space (X,d) such that*

(i)

$$\begin{aligned} d^p(fx,fy) \leq{}& ad^p(gx,gy) + b\max\{d^p(fx,gx), d^p(fy,gy)\} \\ &+ c\max\{d^p(gx,gy), d^p(fx,gx), d^p(fy,gy)\} \end{aligned}$$

for all $x,y \in X$, *where* $a,b,c > 0, p \geq 1,\ a+b+c=1$,

(ii) $f(X) \subset g(X)$.

If range of g is a closed subspace of X, then f and g have a unique common fixed point.

Corollary 3.10. *Let f and g be two weakly compatible non-compatible self-mappings of a metric space (X,d) such that*

(i)

$$\begin{aligned} d(fx,fy) \leq{}& ad(gx,gy) + b\max\{d(fx,gx), d(fy,gy)\} \\ &+ c\max\{d(gx,gy), d(fx,gx), d(fy,gy)\} \end{aligned}$$

for all $x,y \in X$, *where* $a,b,c > 0,\ a+b+c=1$,

(ii) $f(X) \subset g(X)$.

If range of g is a closed subspace of X, then f and g have a unique common fixed point.

Corollary 3.11. *Let f and g be two weakly compatible non-compatible self-mappings of a metric space (X,d) such that*

(i)

$$d(fx,fy) \leq ad(gx,gy) + b\max\{d(fx,gx), d(fy,gy)\}$$

for all $x, y \in X$, where $a, b > 0$, $a + b = 1$,

(ii) $f(X) \subset g(X)$.

If range of g is a closed subspace of X, then f and g have a unique common fixed point.

Remark 3.12. *Our results improve several known results of Gregus [1], Fisher and Sessa [2], Huang and Cho [3], Jungck [4], Ciric [5] and Mukherjee and Verma [6] concerning Gregus type mappings in the following ways:*

(i) the completeness of space is not required,

(ii) the completeness of subspace is not required (even closedness of subspace is not required in case of (CLRg) property),

(iii) containment of ranges of involved mappings is not necessary in case of (CLRg) property,

(iv) continuity of mappings is not required,

(v) the convexity structure in the results is not required,

(vi) the linearity condition of mapping g is dropped.

Acknowledgements : The authors would like to thank the referee, who have made a number of valuable comments and suggestions which have improved the manuscript greatly.

References

[1] Jr.M. Gregus, A fixed point theorem in Banach space, Bull. Un. Math. Ital. 17-A (5) (1980) 193–198.

[2] B. Fisher, S. Sessa, On a fixed point theorem of Gregus, Internat. J. Math. Math. Sci. 9 (1986) 23–28.

[3] N.J. Huang, Y.J. Cho, Common fixed point theorems of Gregus type in convex metric spaces, Math. Japonica 48 (1) (1998) 83–89.

[4] G. Jungck, On a fixed point theorem of Fisher and Sessa, Internet. J. Math. Math. Sci. 13 (1990) 497–500.

[5] L.B. Ciric, On diviccaro, Fisher and Sessa open questions, Arch. Math. (BRNO) Tomus 29 (1993) 145–152.

[6] R.N. Mukherjee, V. Verma, A note on a fixed point theorem of Gregus, Math. Japan. 3 (1988) 745–749.

[7] S. Sessa, On a weak commutativity condition in fixed point considerations, Publ. Inst. Math. 32 (1982) 144–153.

[8] G. Jungck, Compatible mappings and common fixed points, Internat. J. Math. Math. Sci. 9 (1986) 771–779.

[9] G. Jungck, P.P. Murty, Y.J. Cho, Compatible mappings of type (A) and common fixed points, Math. Japonica 38 (2) (1993) 381–390.

[10] H.K. Pathak, M.S. Khan, Compatible mappings of type (B) and common fixed point theorems of Gregus type, Czechoslovak Math. J. 45 (120) (1995) 685–698.

[11] H.K. Pathak, S.M. Kang, Y.J. Cho, J.S. Jung, Gregus type common fixed point theorems for compatible mappings of type (T) and variational inequalities, Publ. Math. 46 (1995) 285–299.

[12] Y.J. Cho, B.K. Sharma, D.R. Sahu, Semicompatibility and fixed points, Math. Japonica 42 (1) (1995) 91–98.

[13] R.P. Pant, Common fixed points of noncommuting mappings, J. Math. Anal. Appl. 188 (2) (1994) 436–440.

[14] R.P. Pant, Note discontinuity and fixed points, J. Math. Anal. Appl. 240 (1999) 284–289.

[15] W. Sintunavarat, P. Kuman, Gregus-type common fixed point theorems for tangential multivalued mappings of integral type in metric spaces, Internat. J. Math. Math. Sci., Vol. 2011 (2011), Artcal ID 923458, 12 pages.

[16] W. Sintunavarat, P. Kuman, Gregus type fixed points for a tangential multivalued mappings satisfying contractive conditions of integral type, Journal of Inequalities and Applications, 2011, 2011:3.

[17] M. Aamri, D. El Moutawakil, Some new common fixed point theorems under strict contractive conditions, J. Math. Anal. Appl. 270 (2002) 181–188.

[18] W. Sintunavarat, P. Kuman, Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces, Journal of Applied Mathematics, Vol. 2011 (2011), Artcal ID 637958, 14 pages.

[19] W. Sintunavarat, P. Kuman, Common fixed points for R-weakly commuting in fuzzy metric spaces, Annali dell'Università di Ferrara, (in Press).

[20] W. Sintunavarat, P. Kuman, Fixed point theorems for a generalized intuitionistic fuzzy contraction in intuitionistic fuzzy metric spaces, Thai J. Math. 10 (1) (2012) 123–135.

[21] G. Jungck, B.E. Rhoades, Fixed points for set-valued functions without continuity, Indian J. Pure Appl. Math. 29 (3) (1998) 227–238.

(Received 27 February 2012)
(Accepted 21 August 2012)