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Ideal Theory in Quotient Semirings

J. N. Chaudhari[†,1] **and D. R. Bonde**[‡]

[†]Department of Mathematics
M. J. College, Jalgaon-425 002, India
e-mail : jnchaudhari@rediffmail.com

[‡]Department of Mathematics
ACS College, Dharangaon-425 105, India
e-mail : drbonde@rediffmail.com

Abstract : Atani [1] proved some results in quotient semirings using subtractive ideals. We introduce the notion of subtractive extension of an ideal which is a generalization of subtractive ideals. In the results of Atani, we replace the condition of subtractive ideal by subtractive extension of an ideal and hence generalize all the results of Atani. We obtain some equivalent conditions for subtractive extension of an ideal and hence characterize the ideals, prime ideals and primary ideals in quotient semirings.

Keywords : semiring; subtractive ideal; Q-ideal; prime ideal; primary ideal; quotient semiring; subtractive extension of an ideal.
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1 Introduction

For the definition of semiring we refer [2]. All semirings in this paper are commutative with nonzero identity. The concept of ideal, principal ideal, finitely generated ideal in a commutative semiring with an identity element can be defined on the similar lines as in commutative ring with an identity element. An ideal I of a semiring R is called a subtractive ideal ($= k$-ideal) if $a, a+b \in I$, $b \in R$, then $b \in I$. $\mathbb{Z}_0^+$ ($\mathbb{N}$) will denote the set of all non-negative integers (positive integers).

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[1]Corresponding author.

For $a_1, a_2, ..., a_n \in \mathbb{Z}_0^+$, we denote $< a_1, a_2, ..., a_n >$ = the ideal generated by $a_1, a_2, ..., a_n$ in the semiring $(\mathbb{Z}_0^+, +, \cdot)$.

Definition 1.1 ([3, Definition 4])**.** An ideal I of a semiring R is called a *Q-ideal* (= partitioning ideal) if there exists a subset Q of R such that:

1) $R = \cup\{q + I : q \in Q\}$.
2) If $q_1, q_2 \in Q$, then $(q_1 + I) \cap (q_2 + I) \neq \emptyset \Leftrightarrow q_1 = q_2$.

Let I be a Q-ideal of a semiring R. Then $R/I_{(Q)} = \{q + I : q \in Q\}$ forms a semiring under the following addition "$\oplus$"and multiplication "$\odot$", $(q_1 + I) \oplus (q_2 + I) = q_3 + I$ where $q_3 \in Q$ is unique such that $q_1 + q_2 + I \subseteq q_3 + I$, and $(q_1 + I) \odot (q_2 + I) = q_4 + I$ where $q_4 \in Q$ is unique such that $q_1q_2 + I \subseteq q_4 + I$. This semiring $R/I_{(Q)}$ is called the quotient semiring of R by I and denoted by $(R/I_{(Q)}\ , \oplus\ , \odot)$ or just $R/I_{(Q)}$. If q_0 is a unique element such that $q_0 + I = I$, then $q_0 + I$ is the zero element of $R/I_{(Q)}$ ([2, Proposition 8.21]).

A proper ideal P of a semiring R is said to be prime (primary) if $ab \in P$, then either $a \in P$ or $b \in P$ ($a \in P$ or $b^n \in P$ for some $n \in \mathbb{N}$). A proper ideal P of a semiring R is said to be weakly prime (weakly primary) if $0 \neq ab \in P$, then either $a \in P$ or $b \in P$ ($a \in P$ or $b^n \in P$ for some $n \in \mathbb{N}$) ([1, 4]). $(\mathbb{Z}_0^+, +, \cdot)$ is a strongly Euclidean semiring ([5]) and hence by Theorem 1.4 ([5]), we have the following theorem.

Theorem 1.2. *Let I be an ideal in the semiring $(\mathbb{Z}_0^+, +, \cdot)$. Then following statements are equivalent:*

1) I is a principal ideal;

2) I is a Q-ideal;

3) I is a subtractive ideal.

Example 1.3 ([3, Example 6])**.** *If $m \in \mathbb{Z}_0^+$ - $\{0\}$, then the ideal $< m > = \{mn : n \in \mathbb{Z}_0^+\}$ is a Q-ideal where $Q = \{0, 1, ..., m - 1\}$. If $m = 0$, then the ideal $< m >$ is a Q-ideal where $Q = \mathbb{Z}_0^+$.*

Lemma 1.4 ([6, Lemma 1])**.** *Let I be an ideal of a semiring R and let $a, x \in R$ such that $a + I \subseteq x + I$. Then $ra + I \subseteq rx + I, a + r + I \subseteq x + r + I$ for all $r \in R$.*

2 Subtractive extension of an ideal

Atani [1], proved some results in quotient semirings using subtractive ideals. Chaudhari and Bonde [7] proved some results in quotient semimodules using subtractive subsemimodules. We prove some of these results without using subtractive ideal and prove some of these by using subtractive extension of an ideal instead of subtractive ideal.

Denote $\mathbb{Z}_0^+ \times \mathbb{Z}_0^+ = (\mathbb{Z}_0^+, +, \cdot) \times (\mathbb{Z}_0^+, +, \cdot)$, the semiring with pointwise addition and pointwise multiplication.

Lemma 2.1. *I is an ideal in the semiring $\mathbb{Z}_0^+ \times \mathbb{Z}_0^+$ if and only if $I = J_1 \times J_2$ where J_1, J_2 are ideals in the semiring $(\mathbb{Z}_0^+, +, \cdot)$.*

Proof. If I is an ideal in the semiring $\mathbb{Z}_0^+ \times \mathbb{Z}_0^+$, then clearly $J_1 = \{a \in \mathbb{Z}_0^+ : (a, 0) \in I\}$ and $J_2 = \{b \in \mathbb{Z}_0^+ : (0, b) \in I\}$ are ideals in $\mathbb{Z}_0^+$ such that $I = J_1 \times J_2$. Converse is trivial. □

Now the proof of the following lemma is obvious.

Lemma 2.2. *Let I be an ideal in the semiring $\mathbb{Z}_0^+ \times \mathbb{Z}_0^+$. Then*

1) *I is a principal ideal if and only if $I = J_1 \times J_2$ where J_1, J_2 are principal ideals in the semiring $(\mathbb{Z}_0^+, +, \cdot)$.*
2) *I is a Q-ideal if and only if $I = J_1 \times J_2$ where J_1, J_2 are Q_1, Q_2-ideals respectively in the semiring $(\mathbb{Z}_0^+, +, \cdot)$ with $Q = Q_1 \times Q_2$.*
3) *I is a subtractive ideal if and only if $I = J_1 \times J_2$ where J_1, J_2 are subtractive ideals in the semiring $(\mathbb{Z}_0^+, +, \cdot)$.*

Theorem 2.3. *Let I be an ideal in the semiring $R = \mathbb{Z}_0^+ \times \mathbb{Z}_0^+$. Then following statements are equivalent:*

1) *I is a principal ideal;*
2) *I is a Q-ideal;*
3) *I is a subtractive ideal.*

Proof. Follows from Lemma 2.1, Lemma 2.2 and Theorem 1.2. □

Definition 2.4. Let I be an ideal of a semiring R. An ideal A of R with $I \subseteq A$ is said to be *subtractive extension of* I if $x \in I, x + y \in A, y \in R$, then $y \in A$.

Clearly, every subtractive ideal of a semiring R containing an ideal I of R is a subtractive extension of I.

Example 2.5. *Let $I = <6> \times \{0\}$, $A = <3> \times <2, 3>$ be ideals in the semiring $R = \mathbb{Z}_0^+ \times \mathbb{Z}_0^+$. Then $I \subseteq A$ and by Theorem 1.2 and Lemma 2.2(3), A is not a subtractive ideal. Clearly $x \in I, x + y \in A, y \in R \Rightarrow y \in A$. Hence A is a subtractive extension of I.*

Atani ([1, Lemma 2.1]) proved that, "Let R be a semiring. I a Q ideal of R and A a subtractive ideal of R with $I \subseteq A$. Then I is a $Q \cap A$-ideal of A". We replace the condition subtractive ideal by subtractive extension of an ideal and prove the lemma with converse.

Lemma 2.6. *Let $I \subseteq A$ be ideals of a semiring R and I a Q-ideal of R. Then A is a subtractive extension of I if and only if I is a $Q \cap A$-ideal of A.*

Proof. Let A be a subtractive extension of I. Let $a \in A$. Then there exists a unique $q \in Q$ such that $a \in q + I$. So $a = q + i$ for some $i \in I$. Since A is a subtractive extension of I, $q \in A$. Hence $q \in Q \cap A$. If $(q_1 + I) \cap (q_2 + I) \neq \emptyset$ for some $q_1, q_2 \in Q \cap A$, then $q_1 = q_2$ because I is a Q-ideal of R. Thus I is a $Q \cap A$-ideal of A. Conversely suppose that I is a $Q \cap A$-ideal of A and $x \in I$, $x + y \in A$, $y \in R$. Since I is a $Q \cap A$-ideal of A, there exists a unique $q_1 \in Q \cap A$ such that $x + y + I \subseteq q_1 + I$. Also since I is a Q-ideal of R, there exists a unique $q_2 \in Q$ such that $y + I \subseteq q_2 + I$. By using Lemma 1.4, $x + y + I \subseteq x + q_2 + I \subseteq q_2 + I$ as $x \in I$. So $(q_1 + I) \cap (q_2 + I) \neq \emptyset$. Since I is a Q-ideal of R, $q_2 = q_1 \in A$. Now $y \in q_2 + I \subseteq A$. Hence A is a subtractive extension of I. □

Theorem 2.7. *Let $I \subseteq A$ be ideals of a semiring R and I a Q-ideal of R. Then following statements are equivalent:*

1) A is a subtractive extension of I

2) I is a $Q \cap A$-ideal of A

3) $A/I_{(Q\cap A)}$ is an ideal of a semiring $R/I_{(Q)}$

4) $A/I_{(Q\cap A)} \subseteq R/I_{(Q)}$

Proof. (1) $\Leftrightarrow$ (2) Follows from Lemma 2.6.

(2) $\Rightarrow$ (3) As A is an ideal of R, $A/I_{(Q\cap A)}$ is an ideal of semiring $R/I_{(Q)}$.

(3) $\Rightarrow$ (4) Trivial.

(4) $\Rightarrow$ (1) Let $x \in I, x + y \in A, y \in R$. Then $x \in I = q_0 + I$ where $q_0 + I$ is the zero element of $R/I_{(Q)}$. Now by definition of quotient semiring there exists a unique $q_1 + I \in A/I_{(Q\cap A)} \subseteq R/I_{(Q)}$ and a unique $q_2 + I \in R/I_{(Q)}$ such that $x + y \in q_1 + I$ and $y \in q_2 + I$. Here $x + y \in (q_0 + I) \oplus (q_2 + I) = q_2 + I$. So $(q_1 + I) \cap (q_2 + I) \neq \emptyset$. Hence $q_2 = q_1 \in A$. Now $y \in q_2 + I \subseteq A$. □

Corollary 2.8 ([1, Lemma 2.1])**.** *Let R be a semiring, I a Q-ideal of R and A a subtractive ideal of R with $I \subseteq A$. Then I is a $Q \cap A$-ideal of A*

The following example shows that the converse of the above corollary is not true.

Example 2.9. *Let $I = <4> \times \{0\}$, $A = <2> \times <2, 3>$ be ideals in the semiring $R = \mathbb{Z}_0^+ \times \mathbb{Z}_0^+$. By Theorem 1.2, Example 1.3 and Lemma 2.2(2), I is a Q-ideal of R where $Q = \{0, 1, 2, 3\} \times \mathbb{Z}_0^+$ and $I \subseteq A$. Clearly $x \in I, x + y \in A, y \in R \Rightarrow y \in A$. Hence A is a subtractive extension of I. By Lemma 2.6, I is a $Q \cap A$-ideal of A. But by Theorem 1.2 and Lemma 2.2(3), A is not a subtractive ideal of R.*

Atani ([1, Proposition 2.2 and Theorem 2.3]) proved that "Let I be a Q-ideal of a semiring R. Then a subset L of $R/I_{(Q)}$ is a subtractive ideal of $R/I_{(Q)}$ if and only if there exists a subtractive ideal A of R with $I \subseteq A$ and $A/I_{(Q\cap A)} = L$ ". By removing the condition "subtractive ideal" we prove the following:

Theorem 2.10. *Let I be a Q-ideal of a semiring R. Then L is an ideal of $R/I_{(Q)}$ if and only if there exists an ideal A of R such that A is a subtractive extension of I and $A/I_{(Q\cap A)} = L$.*

Proof. Let L be an ideal of a semiring $R/I_{(Q)}$. Denote $A = \{x \in R :$ there exists a unique $q \in Q$ such that $x + I \subseteq q + I \in L\}$. Let $x, y \in A, r \in R$. Then there exist unique $q_1, q_2, q \in Q$ such that $x + I \subseteq q_1 + I \in L, y + I \subseteq q_2 + I \in L$, $r + I \subseteq q + I \in R/I_{(Q)}$. Again there exist unique $q_3, q_4 \in Q$ such that $(q_1 + I) \oplus (q_2 + I) = q_3 + I \in L$ and $(q + I) \odot (q_1 + I) = q_4 + I \in L$ where $q_1 + q_2 + I \subseteq q_3 + I$ and $qq_1 + I \subseteq q_4 + I$. By lemma 1.4, $x + y \in x + y + I \subseteq q_1 + q_2 + I \subseteq q_3 + I$ and $rx \in rx + I \subseteq qq_1 + I \subseteq q_4 + I$. So $x + y, rx \in A$. Hence A is an ideal of R with $I \subseteq A$. Now let $x \in I, x + y \in A, y \in R$. So $x + y \in q + I \in L$. Since I is a Q-ideal of R, there exists a unique $q' \in Q$ such that $y \in q' + I$. Since $x \in I, x + y \in q' + I$. So $(q + I) \cap (q' + I) \neq \emptyset \Rightarrow q = q'$. Now $y \in q' + I = q + I \in L$. Thus $y \in A$. Hence A is a subtractive extension of I. Clearly $A/I_{(Q\cap A)} \subseteq L$. Now if $q + I \in L$, then $q \in A$. So $L \subseteq A/I_{(Q\cap A)}$. Thus $A/I_{(Q\cap A)} = L$. Conversely, suppose that A is a subtractive extension of I and $A/I_{(Q\cap A)} = L$. Then by Theorem 2.7, L is an ideal of $R/I_{(Q)}$. □

Adopting the proof of Theorem 2.10, we have the following theorem.

Theorem 2.11 ([1, Proposition 2.2 and Theorem 2.3])**.** *Let I be a Q-ideal of a semiring R. Then a subset L of $R/I_{(Q)}$ is a subtractive ideal of $R/I_{(Q)}$ if and only if there exists a subtractive ideal A of R with $I \subseteq A$ and $A/I_{(Q\cap A)} = L$.*

Theorem 2.12. *Let R be a semiring, I a Q-ideal of R and P a subtractive extension of I. Then P is a prime ideal of R if and only if $P/I_{(Q\cap P)}$ is a prime ideal of $R/I_{(Q)}$.*

Proof. Let P be a prime ideal of R. Suppose that $q_1 + I, q_2 + I \in R/I_{(Q)}$ are such that $(q_1 + I) \odot (q_2 + I) = q_3 + I \in P/I_{(Q\cap P)}$ where $q_3 \in Q \cap P$ is a unique element such that $q_1q_2 + I \subseteq q_3 + I$. So $q_1q_2 = q_3 + i$ for some $i \in I$. Now $q_1q_2 \in P$ implies $q_1 \in P$ or $q_2 \in P$. Hence $q_1 + I \in P/I_{(Q\cap P)}$ or $q_2 + I \in P/I_{(Q\cap P)}$. Conversely suppose that $P/I_{(Q\cap P)}$ is a prime ideal of $R/I_{(Q)}$. Let $ab \in P$ for some $a, b \in R$. Since I is a Q-ideal of R, there exist unique $q_1, q_2, q_3 \in Q$ such that $a \in q_1 + I, b \in q_2 + I$ and $ab \in (q_1 + I) \odot (q_2 + I) = q_3 + I$. So $ab = q_3 + i'$ for some $i' \in I$. Since P is a subtractive extension of I, $q_3 \in P$. So $(q_1 + I) \odot (q_2 + I) = q_3 + I \in P/I_{(Q\cap P)}$. Since $P/I_{(Q\cap P)}$ is a prime ideal, we may assume $q_1 + I \in P/I_{(Q\cap P)}$. Now $a \in q_1 + I \Rightarrow a = q_1 + i''$ for some $i'' \in I \Rightarrow a \in P$ as $q_1 \in Q \cap P \subseteq P$. □

Corollary 2.13 ([1, Theorem 2.5])**.** *Let R be a semiring, I a Q-ideal of R and P a subtractive ideal of R with $I \subseteq P$. Then P is a prime ideal of R if and only if $P/I_{(Q\cap P)}$ is a prime ideal of $R/I_{(Q)}$.*

Adopting the proof of Theorem 2.12, we have the following theorem.

Theorem 2.14. *Let R be a semiring, I a Q-ideal of R and P a subtractive extension of I.*

1) *If P is a weakly prime (weakly primary) ideal of R, then $P/I_{(Q\cap P)}$ is a weakly prime (weakly primary) ideal of $R/I_{(Q)}$.*
2) *If I and $P/I_{(Q\cap P)}$ is a weakly prime (weakly primary) ideal of R and $R/I_{(Q)}$ respectively, then P is a weakly prime (weakly primary) ideal of R.*

Corollary 2.15 ([1, Theorem 2.8 and Corollary 2.9])**.** *Let R be a semiring, I a Q-ideal of R and P a subtractive ideal of R with $I \subseteq P$. Then*

1) *If P is a weakly prime (weakly primary) ideal of R, then $P/I_{(Q\cap P)}$ is a weakly prime (weakly primary) ideal of $R/I_{(Q)}$.*
2) *If I and $P/I_{(Q\cap P)}$ is a weakly prime (weakly primary) ideal of R and $R/I_{(Q)}$ respectively, then P is a weakly prime (weakly primary) ideal of R.*

Proposition 2.16. *Let I be an Q-ideal of a semiring R and A be an ideal of R with $I \subseteq A$. The subset $\widetilde{A} = \{x \in R :$ there exists $q + I \in R/I_{(Q)}$ such that $x \in q+I$ and $(q+I)\cap A \neq \emptyset\}$ of R is the smallest subtractive extension of I containing A.*

Proof. $q_0 + I = I$ is a zero element of $R/I_{(Q)}$ and $I \subseteq A \Rightarrow q_0 \in (q_0 + I) \cap A \Rightarrow q_0 \in \widetilde{A} \Rightarrow \widetilde{A} \neq \emptyset$. Let $x, y \in \widetilde{A}, r \in R$. There exist unique $q_1, q_2, q_3 \in Q$ such that $x \in q_1 + I, y \in q_2 + I, r \in q_3 + I$ and $(q_1 + I) \cap A \neq \emptyset, (q_2 + I) \cap A \neq \emptyset$. Now $x + y \in (q_1 + I) \oplus (q_2 + I) = q + I$ where q is a unique element of Q such that $q_1 + q_2 + I \subseteq q + I$. Since $(q_1 + I) \cap A \neq \emptyset$ and $(q_2 + I) \cap A \neq \emptyset$, $(q + I) \cap A \neq \emptyset$. So $x + y \in \widetilde{A}$. Similarly $rx \in \widetilde{A}$. If $a \in A$, then there exists a unique $q \in Q$ such that $a \in q + I$. So $(q + I) \cap A \neq \emptyset$. Now $a \in \widetilde{A}$. Thus $\widetilde{A}$ is an ideal of R containing A. Let $x \in I, x + y \in \widetilde{A}, y \in R$. So $x + y \in q' + I$, where $q' + I$ is an unique element of $R/I_{(Q)}$ such that $(q' + I) \cap A \neq \emptyset$. Since I is a Q-ideal of R, there exists a unique $q'' \in Q$ such that $y \in q'' + I$. As $x \in I \Rightarrow x + y \in q'' + I$. So $(q' + I) \cap (q'' + I) \neq \emptyset \Rightarrow q' = q'' \Rightarrow q'' + I = q' + I \Rightarrow (q'' + I) \cap A = (q' + I) \cap A \neq \emptyset$. Hence $y \in A$. Thus A is a subtractive extension of I. Now let B be any subtractive extension of I containing A. Let $a \in \widetilde{A} \Rightarrow a \in q''' + I$ and $(q''' + I) \cap A \neq \emptyset \Rightarrow (q''' + I) \cap B \neq \emptyset$ as $A \subseteq B$. Now suppose $b = q''' + i \in (q''' + I) \cap B$ for some $i \in I$. Since B is a subtractive extension of I, $q''' \in B$. So $a \in q''' + I \subseteq B$. Thus $\widetilde{A}$ is the smallest subtractive extension of I containing A. $\square$

Theorem 2.17. *Let I be a Q-ideal of a semiring R and A, B be ideals of R containing I. Then*

1) *A is a subtractive extension of $I \Leftrightarrow A = \widetilde{A}$;*
2) *$\widetilde{\widetilde{A}} = \widetilde{A}$;*
3) *$A \subseteq B \Rightarrow \widetilde{A} \subseteq \widetilde{B}$;*

4) If $A \cup B$ is an ideal, then $\widetilde{A \cup B} = \widetilde{A} \cup \widetilde{B}$;

5) $\widetilde{A \cap B} \subseteq \widetilde{A} \cap \widetilde{B}$;

6) $A = B \Rightarrow \widetilde{A} = \widetilde{B} \Leftrightarrow \widetilde{A}/I_{(Q\cap\widetilde{A})} = \widetilde{B}/I_{(Q\cap\widetilde{B})}$.

Proof. Straightforward. □

Example 2.18. *Let $I = <4> \times <3>$, $A = <2> \times <2,3>$ be ideals in the semiring $R = \mathbb{Z}_0^+ \times \mathbb{Z}_0^+$. By Theorem 1.2, Example 1.3 and Lemma 2.2(2), I is a Q-ideal of R where $Q = \{0,1,2,3\} \times \{0,1,2\}$ and $I \subseteq A$. Now $(4,3) \in I, (4,3) + (2,1) = (6,4) \in A, (2,1) \in R$ but $(2,1) \notin A$. So A is not a subtractive extension of I. Clearly $\widetilde{A} =<2> \times \mathbb{Z}_0^+$ is a subtractive extension of I.*

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