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# Green's Relations and Partial Orders on Semigroups of Partial Linear Transformations with Restricted Range

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**Abstract :** Let $V$ be any vector space and $P(V)$ the set of all partial linear transformations defined on $V$, that is, all linear transformations $\alpha : S \to T$ where $S, T$ are subspaces of $V$. Then $P(V)$ is a semigroup under composition. Let $W$ be a subspace of $V$. We define $PT(V, W) = \{\alpha \in P(V) : V\alpha \subseteq W\}$. So $PT(V, W)$ is a subsemigroup of $P(V)$. In this paper, we present the largest regular subsemigroup and determine Green's relations on $PT(V, W)$. Furthermore, we study the natural partial order $\leq$ on $PT(V, W)$ in terms of domains and images and find elements of $PT(V, W)$ which are compatible.

**Keywords :** regular elements; Green's relations; partial linear transformation semigroups; natural order; compatibility.
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## 1 Introduction

A partial transformation semigroup is the collection of functions from a subset of $X$ into $X$ with composition which is denoted by $P(X)$. In addition, the

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semigroup $T(X)$ and $I(X)$ are defined by

$$\begin{aligned} T(X) &= \{\alpha \in P(X) : \text{dom } \alpha = X\} \text{ and} \\ I(X) &= \{\alpha \in P(X) : \alpha \text{ is injective}\}. \end{aligned}$$

We note that if we let $\alpha \in P(X)$ and $Z \subseteq X$, the notation $Z\alpha$ means $\{z\alpha : z \in Z \cap \text{dom } \alpha\}$. It is clear that, $X\alpha = \text{im } \alpha$.

In [2], Fernandes and Sanwong introduced the partial transformation semigroup with restricted range. They considered the semigroups $PT(X,Y)$ and $I(X,Y)$ defined by

$$\begin{aligned} PT(X,Y) &= \{\alpha \in P(X) : X\alpha \subseteq Y\} \text{ and} \\ I(X,Y) &= \{\alpha \in I(X) : X\alpha \subseteq Y\} \end{aligned}$$

where $Y$ is a subset of $X$. They proved that $PF = \{\alpha \in PT(X,Y) : X\alpha = Y\alpha\}$ is the largest regular subsemigroup of $PT(X,Y)$. Moreover, they determined Green's relations on $PT(X,Y)$ and $I(X,Y)$.

In 2008, Sanwong and Sommanee [9] studied the subsemigroup $T(X,Y) = T(X) \cap PT(X,Y)$ of $T(X)$ where $Y$ is a subset of $X$. They gave a necessary and sufficient condition for $T(X,Y)$ to be regular. In the case when $T(X,Y)$ is not regular, the largest regular subsemigroup was obtained and this subsemigroup was shown to determine the Green's relations on $T(X,Y)$. Also, a class of maximal inverse subsemigroups of $T(X,Y)$ was obtained.

Analogously to $P(X)$, we can define a partial linear transformation on some vector spaces. Let $V$ be any vector space, $P(V)$ the set of all linear transformations $\alpha : S \to T$ where $S$ and $T$ are subspaces of $V$, that is, every element $\alpha \in P(V)$, the domain and range of $\alpha$ are subspaces of $V$. Then we have $P(V)$ under composition is a semigroup and it is called the partial linear transformation semigroup of $V$. The subsemigroups $T(V)$ and $I(V)$ are defined by

$$\begin{aligned} T(V) &= \{\alpha \in P(V) : \text{dom } \alpha = V\} \text{ and} \\ I(V) &= \{\alpha \in P(V) : \alpha \text{ is injective}\}. \end{aligned}$$

Similarly, the linear transformation semigroups with restricted range can be defined as follows. For any vector space $V$ and a subspace $W$ of $V$,

$$\begin{aligned} PT(V,W) &= \{\alpha \in P(V) : V\alpha \subseteq W\}, \\ T(V,W) &= \{\alpha \in T(V) : V\alpha \subseteq W\} \text{ and} \\ I(V,W) &= \{\alpha \in I(V) : V\alpha \subseteq W\}. \end{aligned}$$

Obviously, $PT(V,V) = P(V)$, $T(V,V) = T(V)$ and $I(V,V) = I(V)$. Hence we may regard $PT(V,W)$, $T(V,W)$ and $I(V,W)$ as generalizations of $P(V)$, $T(V)$ and $I(V)$, respectively.

It is known that Green's relations on $T(V)$ are as follows (see [3], page 63). Let $\alpha, \beta \in T(V)$. Then

$$\alpha\mathcal{L}\beta \text{ if and only if } V\alpha = V\beta;$$
$$\alpha\mathcal{R}\beta \text{ if and only if } \ker\alpha = \ker\beta;$$
$$\alpha\mathcal{D}\beta \text{ if and only if } \dim(V\alpha) = \dim(V\beta);$$
$$\mathcal{D} = \mathcal{J}.$$

In 2007, Droms [1] gave a complete description of Green's relations on $P(V)$ and $I(V)$. We have for $\alpha, \beta \in P(V)$ :

$$\alpha\mathcal{L}\beta \text{ if and only if } V\alpha = V\beta;$$
$$\alpha\mathcal{R}\beta \text{ if and only if } \ker\alpha = \ker\beta \text{ and dom } \alpha = \text{dom } \beta;$$
$$\alpha\mathcal{D}\beta \text{ if and only if } \dim(V\alpha) = \dim(V\beta);$$
$$\mathcal{D} = \mathcal{J}.$$

And for $\alpha, \beta \in I(V)$ :

$$\alpha\mathcal{L}\beta \text{ if and only if } V\alpha = V\beta;$$
$$\alpha\mathcal{R}\beta \text{ if and only if dom } \alpha = \text{dom } \beta;$$
$$\alpha\mathcal{D}\beta \text{ if and only if } \dim(V\alpha) = \dim(V\beta);$$
$$\mathcal{D} = \mathcal{J}.$$

Later in 2008, Sullivan [11] described Green's relations and ideals for the semigroup $T(V, W)$. And its Green's relations are as follows. Let $Q = \{\alpha \in T(V, W) : V\alpha \subseteq W\alpha\}$. For $\alpha, \beta \in T(V, W)$ :

$$\alpha\mathcal{L}\beta \text{ if and only if } \alpha = \beta \text{ or } (\alpha, \beta \in Q \text{ and } V\alpha = V\beta);$$
$$\alpha\mathcal{R}\beta \text{ if and only if } \ker\alpha = \ker\beta;$$
$$\alpha\mathcal{D}\beta \text{ if and only if } \ker\alpha = \ker\beta \text{ or } (\alpha, \beta \in Q \text{ and } \dim(V\alpha) = \dim(V\beta));$$
$$\alpha\mathcal{J}\beta \text{ if and only if } \ker\alpha = \ker\beta \text{ or}$$
$$\dim(V\alpha) = \dim(W\alpha) = \dim(W\beta) = \dim(V\beta).$$

Now, we deal with a natural partial order or Mitsch order [6] on any semigroup $S$ defined by for $a, b \in S$.

$$a \leq b \text{ if and only if } a = xb = by,\ xa = a \text{ for some } x, y \in S^1.$$

In 2005, Sullivan [10] studied the natural partial order $\leq$ on $P(V)$. The author found all elements of $P(V)$ which are compatible with respect to $\leq$.

In 2012, Sangkhanan and Sanwong [7] characterized the natural partial order $\leq$ on $PT(X, Y)$ and found elements of $PT(X, Y)$ which are compatible with $\leq$. Recently, they presented the largest regular subsemigroup of $I(V, W)$ and determined its Green's relations in [8]. Furthermore, the authors studied the natural partial order $\leq$ on $I(V, W)$ in terms of domains and images. Finally, they also found elements of $I(V, W)$ which are compatible.

In this paper, we describe the largest regular subsemigroup of $PT(V, W)$ and characterized its Green's relations. Furthermore, we study the natural partial order $\leq$ on $PT(V, W)$ in terms of domains and images. Moreover, we characterize elements of $PT(V, W)$ which are compatible.

## 2 Regularity and Green's relations on $PT(V,W)$

Since $PT(V,W) = \{\alpha \in P(V) : V\alpha \subseteq W\}$, we have the following simple result on $PT(V,W)$ which will be used throughout the paper.

**Lemma 2.1.** *If $S$ and $T$ are subspaces of $V$ with $S \subseteq T$, then $S\alpha \subseteq T\alpha$ for all $\alpha \in PT(V,W)$.*

For convenience, we adopt the convention: namely, if $\alpha \in P(X)$ then we write

$$\alpha = \begin{pmatrix} X_i \\ a_i \end{pmatrix}.$$

and take as understood that the subscript $i$ belongs to some (unmentioned) index set $I$, the abbreviation $\{a_i\}$ denotes $\{a_i : i \in I\}$, and that $X\alpha = \{a_i\}$ and $a_i\alpha^{-1} = X_i$.

Similarly, we can use this notation for elements in $P(V)$. To construct a map $\alpha \in P(V)$, we first choose a basis $\{e_i\}$ for a subspace of $V$ and a subset $\{a_i\}$ of $V$, and then let $e_i\alpha = a_i$ for each $i \in I$ and extend this map linearly to $V$ . To shorten this process, we simply say, given $\{e_i\}$ and $\{a_i\}$ within the context, then for each $\alpha \in P(V)$, we can write

$$\alpha = \begin{pmatrix} e_i \\ a_i \end{pmatrix}.$$

A subspace $U$ of $V$ generated by a linearly independent subset $\{e_i\}$ of $V$ is denoted by $\langle e_i \rangle$ and when we write $U = \langle e_i \rangle$, we mean that the set $\{e_i\}$ is a basis of $U$, and we have $\dim U = |I|$. For each $\alpha \in P(V)$, the kernel and the range of $\alpha$ denoted by $\ker \alpha$ and $V\alpha$ respectively, and the rank of $\alpha$ is $\dim(V\alpha)$.

Let $V$ be a vector space and $\{u_i\}$ a subset of $V$. The notation $\sum a_i u_i$ means the linear combination:

$$a_{i_1}u_{i_1} + a_{i_2}u_{i_2} + ... + a_{i_n}u_{i_n}$$

for some $n \in \mathbb{N}$, $u_{i_1}, u_{i_2}, ..., u_{i_n} \in \{u_i\}$ and scalars $a_{i_1}, a_{i_2}, ..., a_{i_n}$. Suppose that $\alpha \in PT(V,W)$ and $U$ is a subspace of $V$. If we write $U\alpha = \langle u_i\alpha \rangle$, it means that $u_i \in U \cap \mathrm{dom}\,\alpha$ for all $i$. In addition, we can show that $\{u_i\}$ is linearly independent.

Let $PQ = \{\alpha \in PT(V,W) : V\alpha \subseteq W\alpha\}$. For $\alpha \in PQ$ and $\beta \in PT(V,W)$, we obtain $V\alpha \subseteq W\alpha$ which implies that $V\alpha\beta \subseteq W\alpha\beta$. So $\alpha\beta \in PQ$. Therefore, $PQ$ is a right ideal of $PT(V,W)$.

**Lemma 2.2.** *The set $PQ$ is a right ideal of $PT(V,W)$.*

**Theorem 2.3.** *Let $\alpha \in PT(V,W)$. Then $\alpha$ is regular if and only if $\alpha \in PQ$. Consequently, $PQ$ is the largest regular subsemigroup of $PT(V,W)$.*

*Proof.* From Lemma 2.2, we see that $PQ$ is a subsemigroup of $PT(V,W)$. Let $\alpha \in PQ$. Then $V\alpha \subseteq W\alpha = \langle w_j\alpha \rangle$. So $\{w_j\}$ is linearly independent. If $v \in \mathrm{dom}\,\alpha$, then $v\alpha = \sum x_j(w_j\alpha) = (\sum x_j w_j)\alpha$ for some scalars $x_j$. So $(v - \sum x_j w_j)\alpha = 0$ implies $v - \sum x_j w_j \in \ker\alpha$. Hence $v \in \ker\alpha + \langle w_j\rangle$. Let $u \in \ker\alpha \cap \langle w_j\rangle$. Then $u\alpha = 0$ and $u = \sum y_j w_j$ for some scalars $y_j$, so $0 = u\alpha = \sum y_j w_j\alpha$ implies $y_j = 0$ for all $j$ since $\{w_j\alpha\}$ is linearly independent. Hence $\ker\alpha \cap \langle w_j\rangle = \langle 0\rangle$ which follows that $\operatorname{dom}\alpha = \ker\alpha \oplus \langle w_j\rangle$. If $\ker\alpha = \langle u_i\rangle$ and $W = W\alpha \oplus \langle v_k\rangle$, we can write

$$\alpha = \begin{pmatrix} u_i & w_j \\ 0 & w_j\alpha \end{pmatrix}$$

and define

$$\beta = \begin{pmatrix} v_k & w_j\alpha \\ 0 & w_j \end{pmatrix}.$$

We can see that $V\beta = \langle w_j\rangle \subseteq W$, so $\beta \in PT(V,W)$ and $\alpha = \alpha\beta\alpha$. Hence $\alpha$ is regular. Now, let $\alpha$ be any regular element in $PT(V,W)$. Then $\alpha = \alpha\beta\alpha$ for some $\beta \in PT(V,W)$, so $V\alpha = V\alpha\beta\alpha = (V\alpha\beta)\alpha \subseteq W\alpha$. Therefore, $\alpha \in PQ$. □

By the above theorem, we have the following corollary.

**Corollary 2.4.** *Let $W$ be a non-zero subspace of a vector space $V$. Then $PT(V,W)$ is a regular semigroup if and only if $V = W$.*

*Proof.* It is clear that if $V = W$, then $PT(V,W) = P(V)$ and $PT(V,W)$ is regular. Conversely, if $W$ is a proper subspace of $V$, then we can write $W = \langle w_i\rangle$ and $V = \langle w_i\rangle \oplus \langle v_j\rangle$. Since $W$ is a non-zero subspace, we choose $w_{i_1} \in \{w_i\}$ and $v_{j_1} \in \{v_j\}$. Define

$$\alpha = \begin{pmatrix} w_i & v_{j_1} \\ 0 & w_{i_1} \end{pmatrix}.$$

Hence $W\alpha = \langle 0\rangle \subsetneq \langle w_{i_1}\rangle = V\alpha$ and then $\alpha$ is not regular by Theorem 2.3. □

By the above corollary, we note that if $W$ is a non-zero proper subspace of a vector space $V$, then $PT(V,W)$ is not a regular semigroup. It is concluded that, in this case, $PT(V,W)$ is not isomorphic to $P(U)$ for any vector space $U$ since $P(U)$ is regular. This shows that $PT(V,W)$ is almost never isomorphic to $P(U)$.

**Lemma 2.5.** *Let $\alpha, \beta \in PT(V,W)$. Then $\alpha = \gamma\beta$ for some $\gamma \in PT(V,W)$ if and only if $V\alpha \subseteq W\beta$.*

*Proof.* If $\alpha = \gamma\beta$ for some $\gamma \in PT(V,W)$, then $V\alpha = V\gamma\beta \subseteq W\beta$. Now, assume that $V\alpha \subseteq W\beta$ and write $V\alpha = \langle v_i\alpha\rangle$. Hence $\{v_i\}$ is linearly independent. For each $i$, there is $w_i \in W$ such that $v_i\alpha = w_i\beta$. Thus $\{w_i\beta\}$ is linearly independent. Now, let $V\beta = \langle w_i\beta\rangle \oplus \langle v_j\beta\rangle$, $\ker\alpha = \langle u_r\rangle$ and $\ker\beta = \langle u_s\rangle$. Then $\{u_r\} \cup \{v_i\}$ and $\{u_s\} \cup \{w_i\} \cup \{v_j\}$ are linearly independent. Since $\operatorname{dom}\alpha = \ker\alpha \oplus \langle v_i\rangle$ and $\operatorname{dom}\beta = \ker\beta \oplus \langle w_i\rangle \oplus \langle v_j\rangle$, by the same proof as given for [11, Lemma 2] then is $\gamma \in PT(V,W)$ such that $\alpha = \gamma\beta$, as required. □

By the above lemma, we get the following result immediately.

**Lemma 2.6.** *Let $\alpha, \beta \in PT(V, W)$. If $\beta \in PQ$, then $\alpha = \gamma\beta$ for some $\gamma \in PT(V, W)$ if and only if $V\alpha \subseteq V\beta$.*

**Theorem 2.7.** *Let $\alpha, \beta \in PT(V, W)$. Then $\alpha\mathcal{L}\beta$ if and only if ($\alpha, \beta \in PQ$ and $V\alpha = V\beta$) or ($\alpha, \beta \in PT(V, W) \setminus PQ$ and $\alpha = \beta$).*

*Proof.* Assume that $\alpha\mathcal{L}\beta$. Then $\alpha = \lambda\beta$ and $\beta = \mu\alpha$ for some $\lambda, \mu \in PT(V, W)^1$. Suppose that $\alpha \in PQ$. If $\lambda = 1$ or $\mu = 1$, then $\beta = \alpha \in PQ$ and $V\alpha = V\beta$. On the other hand, if $\lambda, \mu \in PT(V, W)$ then $V\beta = V\mu\alpha = (V\mu\lambda)\beta \subseteq W\beta$ since $V\mu\lambda \subseteq W$. Thus $\beta \in PQ$. From $\alpha = \lambda\beta$ and $\beta = \mu\alpha$, we have $V\alpha = V\beta$ by Lemma 2.6. Now, suppose that $\alpha \in PT(V, W) \setminus PQ$. If $\lambda, \mu \in PT(V, W)$, then $V\alpha = V\lambda\beta = (V\lambda\mu)\alpha \subseteq W\alpha$ which contradicts $\alpha \in PT(V, W) \setminus PQ$. Thus $\lambda = 1$ or $\mu = 1$ and so $\beta = \alpha \in PT(V, W) \setminus PQ$. The converse is a direct consequence of Lemma 2.6 □

**Theorem 2.8.** *If $\alpha, \beta \in PT(V, W)$, then $\alpha = \beta\gamma$ for some $\gamma \in PT(V, W)$ if and only if $\text{dom } \alpha \subseteq \text{dom } \beta$ and $\ker\beta \subseteq \ker\alpha$. Consequently, $\alpha\mathcal{R}\beta$ if and only if $\text{dom } \alpha = \text{dom } \beta$ and $\ker\alpha = \ker\beta$.*

*Proof.* It is clear that if $\alpha = \beta\gamma$ for some $\gamma \in PT(V, W)$, then $\text{dom } \alpha \subseteq \text{dom } \beta$. Let $v \in \ker\beta$. Then $v\beta = 0$ implies $v\alpha = v\beta\gamma = 0$, so $\ker\beta \subseteq \ker\alpha$.

Conversely, suppose $\text{dom } \alpha \subseteq \text{dom } \beta$ and $\ker\beta \subseteq \ker\alpha$. Write $\ker\beta = \langle u_i \rangle$, $\ker\alpha = \langle u_i, u_j \rangle$ and $\text{dom } \alpha = \ker\alpha \oplus \langle v_k \rangle$. Since $\text{dom } \alpha \subseteq \text{dom } \beta$, we have $\text{dom } \beta = \text{dom } \alpha \oplus \langle v_s \rangle$. Then

$$\alpha = \begin{pmatrix} u_i & u_j & v_k \\ 0 & 0 & w'_k \end{pmatrix}, \ \beta = \begin{pmatrix} u_i & u_j & v_k & v_s \\ 0 & w_j & w_k & w_s \end{pmatrix}$$

for some $w'_k, w_j, w_k, w_s \in W$. We can see that $\{w_j, w_k\}$ is linearly independent. Define $\gamma \in PT(V, W)$ by

$$\gamma = \begin{pmatrix} w_j & w_k \\ 0 & w'_k \end{pmatrix}.$$

Then $\alpha = \beta\gamma$, as required. □

**Lemma 2.9.** *Let $\alpha, \beta \in PT(V, W)$. If $\text{dom } \alpha = \text{dom } \beta$ and $\ker\alpha = \ker\beta$ then either both $\alpha$ and $\beta$ are in $PQ$, or neither is in $PQ$. Consequently, $\alpha\mathcal{R}\beta$ if and only if ($\alpha, \beta \in PQ$, $\text{dom } \alpha = \text{dom } \beta$ and $\ker\alpha = \ker\beta$) or ($\alpha, \beta \in PT(V, W) \setminus PQ$, $\text{dom } \alpha = \text{dom } \beta$ and $\ker\alpha = \ker\beta$).*

*Proof.* Assume that $\text{dom } \alpha = \text{dom } \beta$ and $\ker\alpha = \ker\beta$ and suppose that $\alpha, \beta \in PQ$ is false. So one of $\alpha$ or $\beta$ is not in $PQ$, we suppose that $\alpha \notin PQ$. Thus $(V \setminus W)\alpha \nsubseteq W\alpha$, so there is $v_0 \in V \setminus W$ such that $v_0\alpha \neq w\alpha$ for all $w \in W$. Thus $v_0 - w \notin \ker\alpha$ for all $w \in W$. If $\beta \in PQ$, then $V\beta = W\beta$, so $v_0\beta = w\beta$ for some $w \in W$ ($v_0 \in \text{dom } \alpha = \text{dom } \beta$) which implies that $v_0 - w \in \ker\beta = \ker\alpha$ which is a contradiction. Therefore $\beta \notin PQ$. □

As a direct consequence of Theorem 2.7, Theorem 2.8 and Lemma 2.9, we have the following corollary.

**Corollary 2.10.** *Let $\alpha, \beta \in PT(V, W)$. Then $\alpha\mathcal{H}\beta$ if and only if ($\alpha, \beta \in PQ$, $V\alpha = V\beta$,* dom $\alpha$ = dom $\beta$ *and* $\ker\alpha = \ker\beta$*) or ($\alpha, \beta \in PT(V, W) \setminus PQ$ and $\alpha = \beta$).*

**Theorem 2.11.** *Let $\alpha, \beta \in PT(V, W)$. Then $\alpha\mathcal{D}\beta$ if and only if ($\alpha, \beta \in PQ$ and* $\dim(V\alpha) = \dim(V\beta)$*) or ($\alpha, \beta \in PT(V, W) \setminus PQ$,* dom $\alpha$ = dom $\beta$ *and* $\ker\alpha = \ker\beta$*).*

*Proof.* Let $\alpha, \beta \in PT(V, W)$ be such that $\alpha\mathcal{D}\beta$. Then $\alpha\mathcal{L}\gamma$ and $\gamma\mathcal{R}\beta$ for some $\gamma \in PT(V, W)$. If $\alpha \in PQ$, then since $\alpha\mathcal{L}\gamma$, we must have $\gamma \in PQ$ and $V\alpha = V\gamma$. From $\gamma\mathcal{R}\beta$, we get $\beta \in PQ$, dom $\gamma$ = dom $\beta$ and $\ker\gamma = \ker\beta$. So we obtain

$$\begin{aligned}\dim(V\alpha) = \dim(V\gamma) &= \dim(\text{dom } \gamma / \ker\gamma) \\ &= \dim(\text{dom } \beta / \ker\beta) = \dim(V\beta).\end{aligned}$$

If $\alpha \in PT(V, W) \setminus PQ$, then $\gamma = \alpha$ (since $\alpha\mathcal{L}\gamma$) and thus $\alpha\mathcal{R}\beta$ which implies that dom $\alpha$ = dom $\beta$ and $\ker\alpha = \ker\beta$. So by Lemma 2.9, we must have $\beta \in PT(V, W) \setminus PQ$.

Conversely, assume that the conditions hold. Clearly, if $\alpha, \beta \in PT(V, W) \setminus PQ$, dom $\alpha$ = dom $\beta$ and $\ker\alpha = \ker\beta$ then $\alpha\mathcal{R}\beta$, and so $\alpha\mathcal{D}\beta$ (since $\mathcal{R} \subseteq \mathcal{D}$). If $\alpha, \beta \in PQ$ and $\dim(V\alpha) = \dim(V\beta)$, then $V\alpha = W\alpha = \langle w_j\alpha\rangle$ and $V\beta = W\beta = \langle w'_j\beta\rangle$. Let $\ker\alpha = \langle u_r\rangle$ and $\ker\beta = \langle u_s\rangle$, so we can write

$$\alpha = \begin{pmatrix} u_r & w_j \\ 0 & w_j\alpha \end{pmatrix}, \beta = \begin{pmatrix} u_s & w'_j \\ 0 & w'_j\beta \end{pmatrix},$$

where $\langle w_j\rangle, \langle w'_j\rangle \subseteq W$. If $\gamma \in PT(V, W)$ is defined by

$$\gamma = \begin{pmatrix} u_r & w_j \\ 0 & w'_j\beta \end{pmatrix},$$

then dom $\gamma$ = dom $\alpha$, $\ker\gamma = \ker\alpha$, $V\gamma = V\beta$ and $\gamma \in PQ$, so $\alpha\mathcal{R}\gamma\mathcal{L}\beta$. □

**Lemma 2.12.** *Let $\alpha, \beta \in PT(V, W)$. If $\alpha = \lambda\beta\mu$ for some $\lambda \in PT(V, W)$ and $\mu \in PT(V, W)^1$, then* $\dim(V\alpha) \leq \dim(W\beta)$.

*Proof.* Since $V\alpha = (V\lambda)\beta\mu \subseteq W\beta\mu$, we have $\dim(V\alpha) \leq \dim(W\beta\mu)$. Let $W\beta\mu = \langle w_i\mu\rangle$ where $\{w_i\} \subseteq W\beta$ and $\{w_i\}$ is linearly independent. Then $\langle w_i\rangle \subseteq W\beta$ which implies that

$$\dim(W\beta\mu) = \dim\langle w_i\mu\rangle = \dim\langle w_i\rangle \leq \dim(W\beta).$$

Therefore, $\dim(V\alpha) \leq \dim(W\beta)$. □

**Theorem 2.13.** *Let $\alpha, \beta \in PT(V,W)$. Then $\alpha\mathcal{J}\beta$ if and only if (*$\text{dom } \alpha = \text{dom } \beta$ *and* $\ker\alpha = \ker\beta$*) or* $\dim(V\alpha) = \dim(W\alpha) = \dim(W\beta) = \dim(V\beta)$.

*Proof.* Assume that $\alpha\mathcal{J}\beta$. Then $\alpha = \lambda\beta\mu$ and $\beta = \lambda'\alpha\mu'$ for some $\lambda, \lambda', \mu, \mu' \in PT(V,W)^1$. If $\lambda = 1 = \lambda'$, then $\alpha = \beta\mu$ and $\beta = \alpha\mu'$ which imply that $\alpha\mathcal{R}\beta$. Thus $\text{dom } \alpha = \text{dom } \beta$ and $\ker\alpha = \ker\beta$. If either $\lambda$ or $\lambda'$ is in $PT(V,W)$, then we can write $\alpha = \sigma\beta\delta$ and $\beta = \sigma'\alpha\delta'$ for some $\sigma, \sigma' \in PT(V,W)$ and $\delta, \delta' \in PT(V,W)^1$. For example, if $\lambda = 1$ and $\lambda' \in PT(V,W)$, then $\alpha = \beta\mu$ and $\beta = \lambda'\alpha\mu'$ imply $\alpha = \beta\mu = (\lambda'\alpha\mu')\mu = \lambda'\alpha(\mu'\mu) = \lambda'(\beta\mu)\mu'\mu = \lambda'\beta(\mu\mu'\mu)$. Thus, by Lemma 2.12, it follows that

$$\dim(W\beta) \geq \dim(V\alpha) \geq \dim(W\alpha) \geq \dim(V\beta) \geq \dim(W\beta),$$

whence $\dim(V\alpha) = \dim(W\alpha) = \dim(W\beta) = \dim(V\beta)$.

Conversely, assume that the conditions hold. If $\text{dom } \alpha = \text{dom } \beta$ and $\ker\alpha = \ker\beta$, then $\alpha\mathcal{R}\beta$ and so $\alpha\mathcal{J}\beta$. Now, suppose that $\dim(V\alpha) = \dim(W\alpha) = \dim(W\beta) = \dim(V\beta)$. Let $\ker\alpha = \langle u_i\rangle$, $\ker\beta = \langle u_j\rangle$ and $V\alpha = \langle v_k\alpha\rangle$. Then $\text{dom } \alpha = \ker\alpha \oplus \langle v_k\rangle$. Since $\dim(V\alpha) = \dim(W\beta)$, we let $W\beta = \langle w'_k\beta\rangle$ and $\text{dom } \beta = \ker\beta \oplus \langle w'_k\rangle \oplus \langle v_l\rangle$. Then we write

$$\alpha = \begin{pmatrix} u_i & v_k \\ 0 & w_k \end{pmatrix}, \beta = \begin{pmatrix} u_j & w'_k & v_l \\ 0 & w'_k\beta & w_l \end{pmatrix}.$$

Let $V = \langle w'_k\beta\rangle \oplus \langle v_m\rangle$ and define $\lambda, \mu \in PT(V,W)$ by

$$\lambda = \begin{pmatrix} u_i & v_k \\ 0 & w'_k \end{pmatrix}, \mu = \begin{pmatrix} v_m & w'_k\beta \\ 0 & w_k \end{pmatrix}.$$

Then $\alpha = \lambda\beta\mu$, as required. Similarly, we can show that $\beta = \lambda'\alpha\mu'$ for some $\lambda', \mu' \in PT(V,W)$ by using the equality $\dim(V\beta) = \dim(W\alpha)$. □

**Corollary 2.14.** *If $\alpha, \beta \in PQ$, then $\alpha\mathcal{J}\beta$ on $PT(V,W)$ if and only if $\alpha\mathcal{D}\beta$ on $PT(V,W)$.*

*Proof.* In general, we have $\mathcal{D} \subseteq \mathcal{J}$. Let $\alpha, \beta \in PQ$ and $\alpha\mathcal{J}\beta$ on $PT(V,W)$. Then ($\text{dom } \alpha = \text{dom } \beta$ and $\ker\alpha = \ker\beta$) or $\dim(V\alpha) = \dim(W\alpha) = \dim(W\beta) = \dim(V\beta)$. If $\text{dom } \alpha = \text{dom } \beta$ and $\ker\alpha = \ker\beta$, then

$$\dim(V\alpha) = \dim(\text{dom } \alpha/\ker\alpha) = \dim(\text{dom } \beta/\ker\beta) = \dim(V\beta).$$

Thus, both cases imply $\dim(V\alpha) = \dim(V\beta)$ and $\alpha\mathcal{D}\beta$ on $PT(V,W)$ by Theorem 2.11. □

**Theorem 2.15.** *$\mathcal{D} = \mathcal{J}$ on $PT(V,W)$ if and only if $\dim W$ is finite or $V = W$.*

*Proof.* It is clear that if $V = W$, then $PT(V,W) = P(V) = PQ$ which follows that $\mathcal{D} = \mathcal{J}$ by Corollary 2.14. Suppose that $\dim W$ is finite. Let $\alpha, \beta \in PT(V,W)$ with $\alpha\mathcal{J}\beta$. If $\text{dom } \alpha = \text{dom } \beta$ and $\ker\alpha = \ker\beta$, then $\alpha\mathcal{R}\beta$ and hence $\alpha\mathcal{D}\beta$.

Now, assume that $\dim(V\alpha) = \dim(W\alpha) = \dim(W\beta) = \dim(V\beta)$. Since $\dim W$ is finite, we have $\dim(V\alpha), \dim(V\beta)$ are finite which implies that $V\alpha = W\alpha$ and $V\beta = W\beta$. Thus $\alpha, \beta \in PQ$ and $\dim(V\alpha) = \dim(V\beta)$. Therefore, $\alpha\mathcal{D}\beta$.

Conversely, suppose that $\dim W$ is infinite and $W \subsetneq V$. Let $\{v_i\}$ be a basis of $W$ and $\{v_j\}$ a basis of $V$ such that $I \subsetneq J$. Then there is an infinite countable subset $\{u_n\}$ of $\{v_i\}$ where $n \in \mathbb{N}$. Let $v \in \{v_j\} \setminus \{v_i\}$ and define $\alpha, \beta$ by

$$\alpha = \begin{pmatrix} v & u_n \\ u_1 & u_{2n} \end{pmatrix}, \beta = \begin{pmatrix} u_{2n-1} & v & u_{2n} \\ 0 & u_1 & u_{4n} \end{pmatrix}.$$

Then $\alpha, \beta \in PT(V,W) \setminus PQ$ and $\dim(V\alpha) = \dim(W\alpha) = \aleph_0 = \dim(W\beta) = \dim(V\beta)$, so $\alpha\mathcal{J}\beta$. Since $\ker\alpha = \langle 0\rangle \neq \langle u_{2n-1}\rangle = \ker\beta$, we have $\alpha$ and $\beta$ are not $\mathcal{D}$-related on $PT(V,W)$. □

## 3 Partial orders

Recall that the natural partial order on any semigroup $S$ is defined by

$$a \leq b \text{ if and only if } a = xb = by,\ xa = a \text{ for some } x, y \in S^1,$$

or equivalently

$$a \leq b \text{ if and only if } a = wb = bz, az = a \text{ for some } w, z \in S^1. \tag{3.1}$$

In this paper, we use (3.1) to define the partial order on the semigroup $PT(V,W)$, that is for each $\alpha, \beta \in PT(V,W)$

$$\alpha \leq \beta \text{ if and only if } \alpha = \gamma\beta = \beta\mu,\ \alpha = \alpha\mu \text{ for some } \gamma, \mu \in PT(V,W)^1.$$

We note that if $W \subsetneq V$, then $PT(V,W)$ has no identity elements. So, in this case $PT(V,W)^1 \neq PT(V,W)$. In addintion, $\leq$ on $PT(V,W)$ does not coincide with the restriction of $\leq$ on $P(V)$. For example, let $V = \langle v_1, v_2, v_3\rangle$ and $W = \langle v_1, v_2\rangle$. Define

$$\alpha = \begin{pmatrix} v_1 & v_2 & v_3 \\ v_1 & v_1 & v_1 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} v_1 & v_2 & v_3 \\ v_2 & v_2 & v_1 \end{pmatrix}.$$

If we let

$$\gamma = \begin{pmatrix} v_1 & v_2 & v_3 \\ v_3 & v_3 & v_3 \end{pmatrix} \text{ and } \mu = \begin{pmatrix} v_1 & v_2 & v_3 \\ v_1 & v_1 & v_3 \end{pmatrix},$$

then $\alpha = \gamma\beta = \beta\mu,\ \alpha = \alpha\mu$ which implies that $\alpha \leq \beta$ in $P(V)$ but we cannot find $\gamma \in PT(V,W)^1$ such that $\alpha = \gamma\beta$. Hence $\alpha \nleq \beta$ in $PT(V,W)$.

In [4], Kowol and Mitsch characterized $\leq$ on $T(X)$ as follows. If $\alpha, \beta \in T(X)$, then the following statements are equivalent.

(1) $\alpha \leq \beta$.

(2) $X\alpha \subseteq X\beta$ and $\alpha = \beta\mu$ for some idempotent $\mu \in T(X)$.

(3) $\beta\beta^{-1} \subseteq \alpha\alpha^{-1}$ and $\alpha = \lambda\beta$ for some idempotent $\lambda \in T(X)$.

(4) $X\alpha \subseteq X\beta, \beta\beta^{-1} \subseteq \alpha\alpha^{-1}$ and $x\alpha = x\beta$ for all $x \in X$ with $x\beta \in X\alpha$.

In [5], Marques-Smith and Sullivan extended the above result to $P(X)$ as follows. If $\alpha, \beta \in P(X)$, then

$$\alpha \leq \beta \text{ if and only if } X\alpha \subseteq X\beta, \text{dom } \alpha \subseteq \text{dom } \beta, \alpha\beta^{-1} \subseteq \alpha\alpha^{-1} \text{ and } \beta\beta^{-1} \cap (\text{dom } \beta \times \text{dom } \alpha) \subseteq \alpha\alpha^{-1}.$$

Later in [10], Sullivan proved an analogue for $P(V)$ as follows. If $\alpha, \beta \in P(V)$, then

$$\alpha \leq \beta \text{ if and only if } V\alpha \subseteq V\beta, \text{dom } \alpha \subseteq \text{dom } \beta, \ker \beta \subseteq \ker \alpha \text{ and } V\alpha\beta^{-1} \subseteq E(\alpha, \beta)$$

where $E(\alpha, \beta) = \{u \in V : u\alpha = u\beta\}$.

Recently, we extended the result for $P(X)$ to $PT(X, Y)$ (see [7]). For $\alpha, \beta \in PT(X, Y)$, $\alpha \leq \beta$ if and only if $\alpha = \beta$ or the following statements hold.

(1) $X\alpha \subseteq Y\beta$.

(2) dom $\alpha \subseteq$ dom $\beta$ and $\ker \beta \cap (\text{dom } \beta \times \text{dom } \alpha) \subseteq \ker \alpha$ .

(3) For each $x \in \text{dom } \beta$, if $x\beta \in X\alpha$, then $x \in \text{dom } \alpha$ and $x\alpha = x\beta$.

Now, we aim to prove an analogue result for $PT(V, W)$ and this result extends a similar result on $P(V)$.

**Theorem 3.1.** *Let $\alpha, \beta \in PT(V, W)$. Then $\alpha \leq \beta$ if and only if $\alpha = \beta$ or the following statements hold.*

(1) *$V\alpha \subseteq W\beta$.*

(2) *dom $\alpha \subseteq$ dom $\beta$.*

(3) *$V\alpha\beta^{-1} \subseteq E(\alpha, \beta)$.*

*Proof.* Suppose that $\alpha \leq \beta$. Then there exist $\gamma, \mu \in PT(V, W)^1$ such that $\alpha = \gamma\beta = \beta\mu$ and $\alpha = \alpha\mu$. If $\gamma = 1$ or $\mu = 1$, then $\alpha = \beta$. If $\gamma, \mu \in PT(V, W)$, then (1) and (2) hold by Lemma 2.5 and Theorem 2.8. If $v \in V\alpha\beta^{-1}$, then $v\beta \in V\alpha$ which implies that $v\beta = w\alpha$ for some $w \in V$, thus

$$v\beta = w\alpha = w\alpha\mu = v\beta\mu = v\alpha.$$

Hence $v \in \text{dom } \alpha$ and $v\alpha = v\beta$. So $v \in E(\alpha, \beta)$. Conversely, assume that the conditions (1)-(3) hold. To show that $\ker \beta \subseteq \ker \alpha$, let $v \in \ker \beta$. Then $v \in \text{dom } \beta$ and $v\beta = 0 \in V\alpha$ which implies that $v \in V\alpha\beta^{-1} \subseteq E(\alpha, \beta)$. We obtain $v \in \text{dom } \alpha$ and $v\alpha = v\beta = 0$. So, $v \in \ker \alpha$. Again by Lemma 2.5 and Theorem 2.8, there exist $\gamma, \mu \in PT(V, W)$ such that $\alpha = \gamma\beta = \beta\mu$. Now, we prove that $V\alpha \subseteq \text{dom } \mu$, by letting $w \in V\alpha$. Then there is $v \in \text{dom } \alpha$ such that $v\alpha = w$. Since $\alpha = \gamma\beta$, we have $w = v\alpha = v\gamma\beta$ from which it follows that $v\gamma \in V\alpha\beta^{-1} \subseteq E(\alpha, \beta)$. That is $v\gamma \in \text{dom } \alpha$ and $v\gamma\alpha = v\gamma\beta$. Thus $v\gamma\beta = v\gamma\alpha = v\gamma\beta\mu = w\mu$ which implies that $w \in \text{dom } \mu$. So, $V\alpha \subseteq \text{dom } \mu$. Hence

$$\text{dom } \alpha\mu = (\text{im } \alpha \cap \text{dom } \mu)\alpha^{-1} = (\text{im } \alpha)\alpha^{-1} = \text{dom } \alpha.$$

For each $v \in \text{dom } \alpha$, $v\alpha = v\gamma\beta$. We obtain $v\gamma \in V\alpha\beta^{-1} \subseteq E(\alpha, \beta)$ which implies that $v\gamma \in \text{dom } \alpha$ and $v\gamma\alpha = v\gamma\beta$. Thus

$$v\alpha = v\gamma\beta = v\gamma\alpha = v\gamma\beta\mu = v\alpha\mu.$$

Therefore, $\alpha = \alpha\mu$. □

Let $\preceq$ be a partial order on a semigroup $S$. An element $c \in S$ is said to be left [right] compatible if $ca \preceq cb$ [$ac \preceq bc$] for each $a, b \in S$ such that $a \preceq b$. Now, we characterize all elements in $PT(V, W)$ which are compatible with respect to $\leq$. We first prove the following lemma.

We note that a zero partial linear transformation is a zero map having domain as a subspace of $V$.

**Lemma 3.2.** *Let* $\dim W = 1$ *and* $\alpha, \beta \in PT(V, W)$. *If* $\alpha \leq \beta$, *then* $\alpha = \beta$ *or* $\alpha$ *is a zero partial linear transformation.*

*Proof.* Suppose that $\alpha \leq \beta$ and $\alpha$ is not a zero partial linear transformation. So $1 \leq \dim V\alpha \leq \dim W = 1$, and then $V\alpha = W$. For each $v \in \text{dom } \beta$, $v\beta \in V\beta \subseteq W = V\alpha$ which implies that $v \in V\alpha\beta^{-1} \subseteq E(\alpha, \beta)$ by Theorem 3.1(3). Hence $v \in \text{dom } \alpha$ and $v\alpha = v\beta$. Thus $\text{dom } \beta \subseteq \text{dom } \alpha$. Since $\alpha \leq \beta$, we have $\text{dom } \alpha \subseteq \text{dom } \beta$. Therefore, $\text{dom } \alpha = \text{dom } \beta$ and $\alpha = \beta$. □

Theorem 5.2 in [7] showed that if $|Y| > 1$ and $\emptyset \neq \gamma \in PT(X, Y)$, then

(1) $\gamma$ is left compatible with $\leq$ if and only if $Y\gamma = Y$;

(2) $\gamma$ is right compatible with $\leq$ if and only if ($Y \subseteq \text{dom } \gamma$ and $\gamma|_Y$ is injective) or $Y \cap \text{dom } \gamma = \emptyset$.

And Theorem 3.1 in [10] proved that if $\gamma \in P(V)$ has non-zero rank and $\dim V > 1$, then

(1) $\gamma \in P(V)$ is left compatible with $\leq$ if and only if $\gamma$ is surjective;

(2) $\gamma \in P(V)$ is right compatible with $\leq$ if and only if $\gamma \in T(V)$ and $\gamma$ is injective.

For the semigroup $PT(V, W)$, we have the following result.

**Theorem 3.3.** *Let* $\gamma \in PT(V, W)$. *The following statements hold.*

(1) *If* $\dim W = 1$, *then every element in* $PT(V, W)$ *is always left compatible with* $\leq$.

(2) *If* $\dim W > 1$, *then* $\gamma$ *is left compatible with* $\leq$ *if and only if* $W\gamma = W$ *or* $\gamma$ *is a zero partial linear transformation.*

(3) *If* $\dim W \geq 1$, *then* $\gamma \in PT(V, W)$ *is right compatible with* $\leq$ *if and only if* $W \subseteq \text{dom } \gamma$ *and* $\gamma|_W$ *is injective.*

*Proof.* (1) Assume that $\dim W = 1$. Let $\alpha, \beta \in PT(V, W)$ with $\alpha \leq \beta$, and let $\lambda \in PT(V, W)$. By the above lemma, we obtain $\alpha = \beta$ or $\alpha$ is a zero partial linear transformation. If $\alpha = \beta$, then $\lambda\alpha = \lambda\beta$. Now, we consider the case $\alpha$ is a zero partial linear transformation. We obtain $V\lambda\alpha = \langle 0 \rangle \subseteq W\lambda\beta$ and $\text{dom } \lambda\alpha = (\text{im } \lambda \cap \text{dom } \alpha)\lambda^{-1} \subseteq (\text{im } \lambda \cap \text{dom } \beta)\lambda^{-1} = \text{dom } \lambda\beta$. Let $v \in V(\lambda\alpha)(\lambda\beta)^{-1}$. We obtain $v\lambda\beta \in V\lambda\alpha \subseteq V\alpha$ which implies that $v\lambda \in V\alpha\beta^{-1} \subseteq E(\alpha,\beta)$ since $\alpha \leq \beta$. Thus $v\lambda\beta = v\lambda\alpha$. So $v \in E(\lambda\alpha, \lambda\beta)$. Therefore, $\lambda\alpha \leq \lambda\beta$.

(2) Assume that $\dim W > 1$. Suppose that $W\gamma \subsetneq W$ and $\gamma$ is not a zero partial linear transformation. If $W\gamma = \langle 0\rangle$, then there is $v \in V\gamma \setminus W\gamma \subseteq V\gamma$ since $\gamma$ is not a zero partial linear transformation. From $\dim W > 1$, there exists $v \neq w \in W \setminus W\gamma$ where $\{v, w\}$ is linearly independent. If $W\gamma \neq \langle 0\rangle$, there are $0 \neq v \in W\gamma \subseteq V\gamma$ and $w \in W \setminus W\gamma$ where $\{v, w\}$ is linearly independent since $W\gamma \subsetneq W$. It is concluded that we can choose $0 \neq w \in W \setminus W\gamma$ and $0 \neq v \in V\gamma$ where $\{v, w\}$ is linearly independent. Define $\alpha, \beta \in PT(V, W)$ by

$$\alpha = \begin{pmatrix} v & w \\ w & w \end{pmatrix},\ \beta = \begin{pmatrix} v & w \\ v & w \end{pmatrix}.$$

Then $\alpha \leq \beta$. It is clear that $v \in V\gamma\beta$ but $v \notin V\gamma\alpha$, so $\gamma\alpha \neq \gamma\beta$. Since $w \in V\gamma\alpha$ but $w \notin W\gamma\beta$, we conclude that $\gamma\alpha \nleq \gamma\beta$.

Conversely, it is clear that $\gamma\alpha = \gamma$ for each $\alpha \in PT(V, W)$ if $\gamma$ is a zero partial linear transformation. In this case, we obtain $\gamma$ is left compatible. Assume that $W\gamma = W$. Let $\alpha, \beta \in PT(V, W)$ be such that $\alpha \leq \beta$. We have $V\gamma\alpha \subseteq V\alpha \subseteq W\beta = W\gamma\beta$ and

$$\text{dom } \gamma\alpha = (\text{im}\gamma \cap \text{dom } \alpha)\gamma^{-1} \subseteq (\text{im}\gamma \cap \text{dom } \beta)\gamma^{-1} = \text{dom } \gamma\beta.$$

Let $v \in V(\gamma\alpha)(\gamma\beta)^{-1}$. Then $v\gamma\beta \in V\gamma\alpha \subseteq V\alpha$ which implies that $v\gamma \in V\alpha\beta^{-1} \subseteq E(\alpha, \beta)$. Hence $v\gamma \in \text{dom } \alpha$ and $v\gamma\alpha = v\gamma\beta$, so $v \in E(\gamma\alpha, \gamma\beta)$. Therefore, $\gamma\alpha \leq \gamma\beta$.

(3) Suppose that $\dim W \geq 1$. Assume that $W \subseteq \text{dom } \gamma$ and $\gamma|_W$ is injective. Let $\alpha, \beta \in PT(V, W)$ be such that $\alpha \leq \beta$. So $V\alpha \subseteq W\beta$ which implies that $V\alpha\gamma \subseteq W\beta\gamma$. Since $W \subseteq \text{dom } \gamma$, we obtain $\text{dom } \alpha\gamma = (\text{im } \alpha \cap \text{dom } \gamma)\alpha^{-1} \subseteq (W \cap \text{dom } \gamma)\alpha^{-1} = W\alpha^{-1} = \text{dom } \alpha \subseteq \text{dom } \beta = (\text{im } \beta \ \cap \ W)\beta^{-1} \subseteq (\text{im } \beta \ \cap \ \text{dom } \gamma)\beta^{-1} = \text{dom } \beta\gamma$. For each $v \in V(\alpha\gamma)(\beta\gamma)^{-1}$, we have $v\beta\gamma = w\alpha\gamma$ for some $w \in V$. Since $\gamma|_W$ is injective, we have $v\beta = w\alpha \in V\alpha$, thus $v \in \text{dom } \alpha$ and $v\alpha = v\beta$. Hence $v \in \text{dom } \alpha = (\text{im } \alpha \cap W)\alpha^{-1} \subseteq (\text{im } \alpha \cap \text{dom } \gamma)\alpha^{-1} = \text{dom } \alpha\gamma$ and $v\alpha\gamma = v\beta\gamma$ from which it follows that $v \in E(\alpha\gamma, \beta\gamma)$. Therefore, $\alpha\gamma \leq \beta\gamma$.

Conversely, if $\gamma|_W$ is not injective, then $\ker \gamma|_W \neq \langle 0\rangle$. Let $0 \neq w \in \ker \gamma|_W$. Define $\alpha, \beta \in PT(V, W)$ by $\alpha = \binom{0}{0}$ and $\beta = \binom{w}{w}$. So, we obtain $\alpha \leq \beta$ by Theorem 3.1 and $\alpha\gamma \neq \beta\gamma$ since $w \in \text{dom } \beta\gamma$ but $w \notin \text{dom } \alpha\gamma$. We also have $\alpha\gamma \nleq \beta\gamma$ since $w\beta\gamma = w\gamma = 0 \in V\alpha\gamma$ which implies that $w \in V(\alpha\gamma)(\beta\gamma)^{-1}$ but $w \notin \text{dom } \alpha\gamma$. If $W \nsubseteq \text{dom } \gamma$, then there is $w \in W \setminus \text{dom } \gamma$. Define $\alpha, \beta \in PT(V, W)$ by $\alpha = \binom{w}{0}$ and $\beta = \binom{w}{w}$. Thus $\alpha \leq \beta$ and $\alpha\gamma \neq \beta\gamma$. And $\alpha\gamma \nleq \beta\gamma$ since $\text{dom } \alpha\gamma = \langle w\rangle \nsubseteq \langle 0\rangle = \text{dom } \beta\gamma$. Therefore, $\gamma$ is not right compatible. □

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